Homework 8 Solutions

Mark out of 25

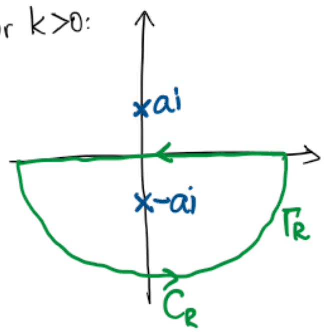
Q1

(a) $f(x) = \dfrac{1}{(x^2+a^2)^2} \quad (a>0)$ **3 pts**

$$\hat{f}(k) = \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2} e^{-ikx}\,dx$$

Use contour integration to obtain this integral

For $k>0$:



$$g(z) = \frac{1}{(z^2+a^2)^2} e^{-ikz}$$

$$\int_{\Gamma_R} = \int_{C_R} - \underbrace{\int_{-\infty}^{\infty}}_{\text{Target Integral}} = 2\pi i \; \mathrm{Res}(g; -ai) \leftarrow \text{Pole order 2}$$

$\int_{C_R} \to 0$ as $R \to \infty$

$$\mathrm{Res} = \frac{d}{dz}\bigg|_{z=-ai} \frac{1}{(z-ai)^2} e^{-ikz}$$

$$= \frac{-ik}{(z-ai)^2} e^{-ikz} - \frac{2}{(z-ai)^3} e^{-ikz}\bigg|_{z=-ai}$$

$$= \frac{-ik}{(-2ai)^2} e^{-ka} - \frac{2}{(-2ai)^3} e^{-ka}$$

$$= \left(\frac{ik}{4a^2} + \frac{i}{4a^3}\right) e^{-ka}$$

$$\Rightarrow \hat{f}(k) = -2\pi i \left(\frac{ik}{4a^2} + \frac{i}{4a^3}\right) e^{-ka} = \frac{\pi(ka+1)}{2a^3} e^{-ka}$$

For $k=0$, may use the same contour and integration techniques same as above

$$g(z) = \frac{1}{(z^2+a^2)^2} \qquad \int_{\Gamma_R} = \underbrace{\int_{C_R}}_{\to 0} - \int_{-\infty}^{\infty} = 2\pi i \; \mathrm{Res}(g; -ai)$$

$$\mathrm{Res} = \frac{d}{dz}\bigg|_{z=-ai} \frac{1}{(z-ai)^2} = \frac{-2}{(z-ai)^3}\bigg|_{z=-ai} = \frac{-2}{(-2ai)^3}$$

$$= \frac{i}{4a^3}$$

$$\hat{f}(k) = -2\pi i \left(\frac{i}{4a^3}\right) = \frac{\pi}{2a^3}$$

For $k<0$, note that $\hat{f}(-k) = \overline{\hat{f}(k)}$ and use the result for $k>0$:

$$\hat{f}(k) = \overline{\hat{f}(-k)} = \frac{\pi(-ka+1)}{2a^3} e^{ka}$$

Therefore, $\hat{f}(k) = \dfrac{\pi(|k|a+1)}{2a^3} e^{-|k|a}$

(b) $\hat{f}(k) = \dfrac{1}{(k^2+a^2)^2}$ 2 pts

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(k^2+a^2)^2} e^{ikx}\, dk$$

By the previous result, $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2} e^{-ikx}\, dx = \frac{\pi(|k|a+1)}{2a^3} e^{-|k|a}$

Then $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(k^2+a^2)^2} e^{ikx}\, dk = \overline{\int_{-\infty}^{\infty} \frac{1}{(k^2+a^2)^2} e^{-ikx}\, dk} = \frac{\pi(|x|a+1)}{2a^3} e^{-|x|a}$

(since above is real)

Then $\displaystyle f(x) = \frac{(|x|a+1)}{4a^3} e^{-|x|a}$

Q2 5 pts

$$iu_t + u_{xx} = 0 \qquad x\in\mathbb{R} \quad t>0 \qquad u(x,0)=f(x)$$
$$f(x) \xrightarrow{x\to\infty} 0$$

Let $\hat{u}(k,t) = \mathcal{F}[u(x,t)]$ (Fourier transform in x)

By the properties of Fourier transforms with derivatives,

$$\mathcal{F}[u_{xx}(x,t)] = (ik)^2\,\hat{u}(k,t)$$

By Fourier transforming the PDE, we get the ODE

$$i\hat{u}_t - k^2\hat{u} = 0 \qquad \hat{u}(k,0) = \hat{f}(k) := \mathcal{F}[f(x)]$$

$$\hat{u}_t = -ik^2\hat{u}$$

$$\hat{u} = \hat{f}(k)e^{-ik^2t}$$

To obtain u, use convolution:

$$u = f * \mathcal{F}^{-1}[e^{-ik^2t}]$$

$$\mathcal{F}^{-1}[e^{-ik^2t}] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ik^2t}\,e^{ikx}\,dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(k^2t-kx)}\,dk$$

$$= \frac{1}{2\pi}e^{\frac{ix^2}{4t}}\int_{-\infty}^{\infty} e^{-i(k^2t-kx+\frac{x^2}{4t})}\,dk \quad \text{(completing the square)}$$

$$= \frac{1}{2\pi}e^{\frac{ix^2}{4t}}\int_{-\infty}^{\infty} e^{-it(k-\frac{x}{2t})^2}\,dk$$

$$= \frac{1}{2\pi}e^{\frac{ix^2}{4t}}\int_{-\infty}^{\infty} e^{-itk^2}\,dk \quad * \quad \text{(shifting in } k)$$

$$= \frac{1}{2\pi}e^{\frac{ix^2}{4t}}\sqrt{\frac{\pi}{it}} \qquad \text{(Gaussian integral)}$$

$$= \frac{1}{2\sqrt{i\pi t}}\,e^{\frac{ix^2}{4t}}$$

$$\Rightarrow\ u = \frac{1}{2\sqrt{i\pi t}}\int_{-\infty}^{\infty} f(x')\,e^{\frac{i(x-x')^2}{4t}}\,dx'$$

Q3 5 pts

$$u_t = D(u_{xx}+u_{yy}) \qquad x,y\in\mathbb{R} \quad t>0$$
$$u(x,y,0)=f(x,y) \qquad f\to 0,\ u\to 0 \text{ as } |x|,|y|\to\infty$$

Fourier transform this equation in x and y:

Let $\hat{u}(\hat{x},\hat{y},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y,t)\, e^{-i(x\hat{x}+y\hat{y})}\, dx\, dy$

Then:

$$\hat{u}_t = D\left((i\hat{x})^2\hat{u} + (i\hat{y})^2\hat{u}\right) = -D\left(\hat{x}^2+\hat{y}^2\right)\hat{u} \quad \text{with } \hat{u}(\hat{x},\hat{y},0)=\hat{f}(\hat{x},\hat{y})$$

$$\Rightarrow \hat{u} = \hat{f}(\hat{x},\hat{y})\, e^{-D(\hat{x}^2+\hat{y}^2)t}$$

To obtain u, use convolution:

$$u = f(x,y) * \mathcal{F}^{-1}\left(e^{-D(\hat{x}^2+\hat{y}^2)t}\right)$$

$$\mathcal{F}^{-1}\left(e^{-D(\hat{x}^2+\hat{y}^2)t}\right) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-D(\hat{x}^2+\hat{y}^2)t}\, e^{i(x\hat{x}+y\hat{y})}\, d\hat{x}\, d\hat{y}$$

$$= \frac{1}{4\pi^2}\underbrace{\left(\int_{-\infty}^{\infty} e^{-D\hat{x}^2t+ix\hat{x}}\, d\hat{x}\right)}_{①}\underbrace{\left(\int_{-\infty}^{\infty} e^{-D\hat{y}^2t+iy\hat{y}}\, d\hat{y}\right)}_{②} \quad \text{(Separate integrals)}$$

$$① = e^{-\frac{x^2}{4Dt}}\int_{-\infty}^{\infty} e^{-D\hat{x}^2t+ix\hat{x}+\frac{x^2}{4Dt}}\, d\hat{x}$$
$$= e^{-\frac{x^2}{4Dt}}\int_{-\infty}^{\infty} e^{-Dt\left(\hat{x}-\frac{ix}{2Dt}\right)^2}\, d\hat{x}$$
$$= e^{-\frac{x^2}{4Dt}}\int_{-\infty}^{\infty} e^{-Dt\hat{x}^2}\, d\hat{x} \quad \text{(shifting) **}$$
$$= e^{-\frac{x^2}{4Dt}}\sqrt{\frac{\pi}{Dt}} \quad \text{(Gaussian integral)}$$

$$② = \text{(same integral as ①)} = e^{-\frac{y^2}{4Dt}}\sqrt{\frac{\pi}{Dt}}$$

$$= \frac{1}{4\pi Dt}\, e^{-\frac{x^2+y^2}{4Dt}}$$

$$\therefore\ u(x,y,t) = \frac{1}{4\pi Dt}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y',t)\, e^{-\frac{(x-x')^2+(y-y')^2}{4Dt}}\, dx'\, dy'$$

Q4 5 pts

Fourier Transform the equation :

$$\hat{u}_t = D_0 (ik)^2 \hat{u} + D_1 (ik)^4 \hat{u} \qquad \hat{u}(k,0) = \hat{f}(k)$$
$$= -D_0 k^2 \hat{u} + D_1 k^4 \hat{u}$$

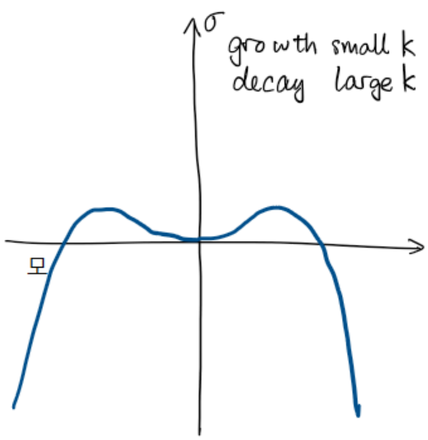
Dispersion Relation : $V = e^{ikx + \sigma t} \Rightarrow \sigma = -D_0 k^2 + D_1 k^4$

$$V = e^{ikx + (-D_0 k^2 + D_1 k^4) t}$$

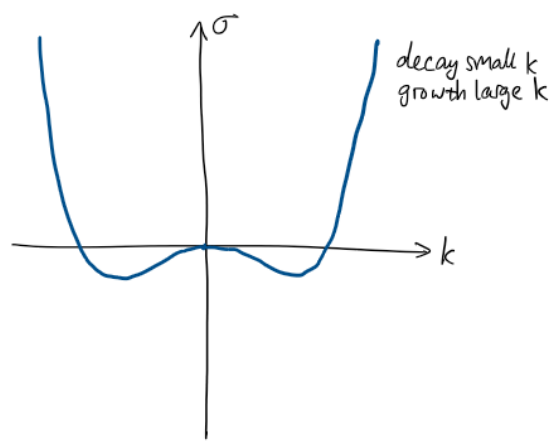
Then
$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx + (-D_0 k^2 + D_1 k^4) t} \, dk$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x') e^{ik(x - x') + (-D_0 k^2 + D_1 k^4) t} \, dk \, dx'$$

Dispersion plots :

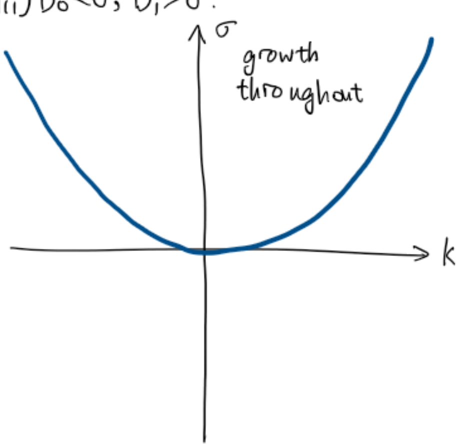
(i) $D_0 < 0, D_1 < 0$:



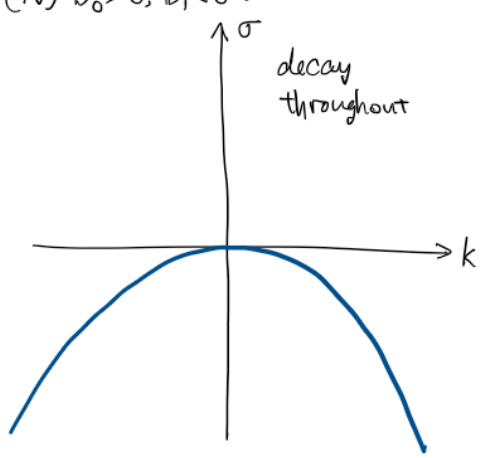
(ii) $D_0 > 0, D_1 > 0$:



(iii) $D_0 < 0, D_1 > 0$:



(iv) $D_0 > 0, D_1 < 0$:



Q5 5 pts

By convolution theorem,

$$|\hat{f}(k)|^2 = \hat{f}(k)\cdot\overline{\hat{f}(k)}$$

$$\mathcal{F}^{-1}\left[|\hat{f}(k)|^2\right] = f(x) * \mathcal{F}^{-1}\left[\overline{\hat{f}(k)}\right]$$

$$\mathcal{F}^{-1}\left[\overline{\hat{f}(k)}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{f}(k)}\, e^{ikx} dk = \overline{\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k) e^{-ikx}\, dk}$$

$$= \overline{f(-x)}$$

$$= f(x) * \overline{f(-x)}$$

$$= \int_{-\infty}^{\infty} f(x')\overline{f(x'-x)}\, dx'$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(k)|^2\, dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(f(x) * \overline{f(-x)}\right) e^{-ikx}\, dx\, dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x')\,\overline{f(x'-x)}\, e^{-ikx}\, dx'\, dx\, dk$$

$$\int_{-\infty}^{\infty} \overline{f(x'-x)} e^{-ikx}\, dx = \int_{-\infty}^{\infty} \overline{f(x'-x) e^{ikx}}\, dx = e^{-ikx'} \overline{\int_{-\infty}^{\infty} f(x'-x) e^{-ik(x'-x)}}\, dx = e^{-ikx'}\overline{\hat{f}(k)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x')\,\overline{\hat{f}(k)}\, e^{-ikx'}\, dx'\, dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x')\,\overline{\hat{f}(k) e^{ikx'}}\, dx'\, dk$$

$$\int_{-\infty}^{\infty} f(x')\,\overline{f(x')}\, dx'$$

$$= \int_{-\infty}^{\infty} |f(x)|^2\, dx$$

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□

Alternative solution:

$$\mathcal{F}(f * g) = \hat{f}\,\hat{g}$$

$$f * g = \mathcal{F}^{-1}(\hat{f}\,\hat{g})$$

$$\int_{-\infty}^{\infty} f(x')\,g(x-x')\,dx' = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\,\hat{g}(k)\,e^{-ikx}\,dk$$

Substitute $x=0$:

$$\int_{-\infty}^{\infty} f(x')\,g(-x')\,dx'$$

$$\parallel$$

$$\int_{-\infty}^{\infty} f(x)\,g(-x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\,\hat{g}(k)\,dk$$

Let $g(x) = \overline{f(-x)}$. Then

$$\begin{aligned}
\hat{g}(k) &= \int_{-\infty}^{\infty} \overline{f(-x)}\,e^{-ikx}\,dx \\
&= \int_{-\infty}^{\infty} \overline{f(-x)\,e^{ikx}}\,dx \\
&= \overline{\int_{-\infty}^{\infty} f(x)\,e^{-ikx}}\,dx \\
&= \overline{\hat{f}(k)}
\end{aligned}$$

Then substituting above, we get

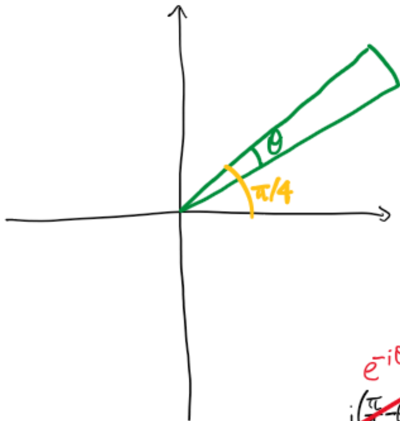
$$\int_{-\infty}^{\infty} f(x)\,\overline{f(x)}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\,\overline{\hat{f}(k)}\,dk$$

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(k)|\,dk$$

$\square$

Notes:

(*) The integral $\int_{-\infty}^{\infty} e^{-itk^2}\,dk$ for t real, $t>0$ is not absolutely integrable. To obtain this integral, consider the contour integral of the function e^{-tz^2} over the sector contour:



On the arc, we have $\operatorname{Re}(-tz^2)$

$$= \operatorname{Re}\left(-t\rho^2 e^{(\frac{\pi}{2}-2\varphi)i}\right) \quad [0 \le \varphi \le \theta]$$

$$= \operatorname{Im}\left(t\rho^2 e^{-2\varphi i}\right) \le 0$$

$$\Rightarrow \left|e^{-tz^2}\right| \le 1$$

so integral bounded by $\theta\rho$

Taking $\theta = \frac{1}{\rho^2}$ and $\rho \to \infty$, for example,

$$\cancel{e^{i(\frac{\pi}{4}-\theta)}}\ e^{-i\theta} \int_0^{\rho} e^{-tk^2 e^{i(\frac{\pi}{2}-2\theta)}}\,dk \to \cancel{e^{i\frac{\pi}{4}}} \int_0^{\infty} e^{-itk^2}\,dk$$

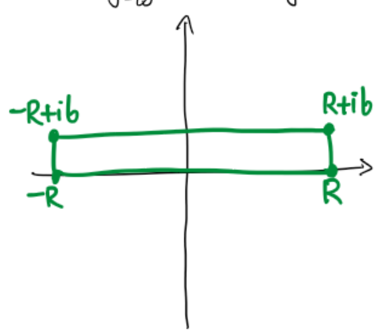
We know the Gaussian integral:

$$\int_0^{\infty} e^{-tk^2 e^{i(\frac{\pi}{2}-2\theta)}}\,dk = \int_0^{\infty} e^{-tk^2(\underbrace{\sin(2\theta)}_{>0} - i\cos(2\theta))}\,dk$$

$$= \sqrt{\frac{\pi}{k^2 e^{i(\frac{\pi}{2}-2\theta)}}} \quad \text{(converges absolutely)}$$

and by taking $\theta \to 0$, $\int_0^{\infty} e^{-itk^2}\,dk = \sqrt{\frac{\pi}{ik^2}}$.

(**) An integral of the form $\int_{-\infty}^{\infty} e^{-a(x+ib)^2}\,dx$ can be equated to $\int_{-\infty}^{\infty} e^{-ax^2}$ by a rectangular contour integrating e^{-az^2}:



By taking $R \to \infty$, the integrals on the vertical edges $\to 0$, and by Residue Theorem,

$$\int_{-\infty}^{\infty} e^{-ax^2}\,dx = \int_{-\infty}^{\infty} e^{-a(x+ib)^2}\,dx$$

as there are no singularities $\square$

(***) If f is square-integrable, Fubini's theorem permits the interchanging of multiple integrals in the way shown.