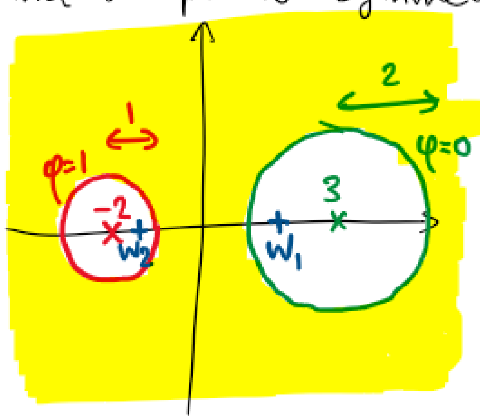
Homework 7 Solutions Mark out of 24

7.6 Q8 4 pts

Find two points symmetric to both circles simultaneously:



We want:

$$w_1 - 3 = \frac{2^2}{\overline{w_2} - 3}$$

$$w_1 + 2 = \frac{1^2}{\overline{w_2} + 2}$$

$$\frac{1}{\overline{w_2}+2} - 5 = \frac{4}{\overline{w_2}-3}$$

$$\overline{w_2} - 3 - 5(\overline{w_2}+2)(\overline{w_2}-3) = 4(\overline{w_2}+2)$$

$$\overline{w_2} - 3 - 5\overline{w_2}^2 + 5\overline{w_2} + 30 = 4\overline{w_2} + 8$$

$$-5\overline{w_2}^2 + 2\overline{w_2} + 19 = 0$$

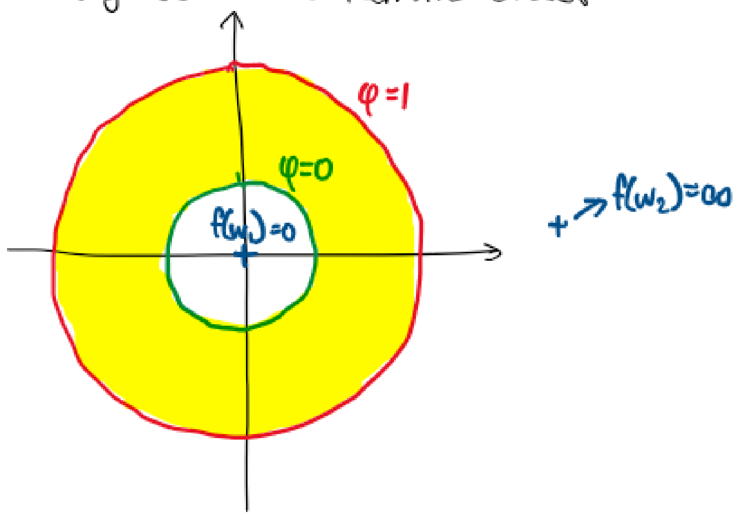
$$\overline{w_2} = \frac{-2 \pm \sqrt{2^2 + 380}}{-10} = \frac{1}{5} \pm \frac{\sqrt{96}}{5}$$

$$= \frac{1}{5} \pm \frac{4\sqrt{6}}{5}$$

Then set $w_1 = \dfrac{1+4\sqrt{6}}{5}$ and $w_2 = \dfrac{1-4\sqrt{6}}{5}$

Transform the picture by $f(z) = \dfrac{z - w_1}{z - w_2}$ $f(w_1) = 0$, $f(w_2) = \infty$

This gives two concentric circles :



Inner circle has radius $\dfrac{7-2\sqrt{6}}{5}$

Outer circle has radius $\dfrac{11+4\sqrt{6}}{5}$

Next find a harmonic function satisfying this. One such function is $a+b\log|w|$

On inner circle, $0=a+b\log\left(\dfrac{7-2\sqrt{6}}{5}\right)$

outer circle, $1=a+b\log\left(\dfrac{11+4\sqrt{6}}{5}\right)$

$$1=b\log\Bigg(\underbrace{\frac{11+4\sqrt{6}}{7-2\sqrt{6}}}_{5+2\sqrt{6}}\Bigg) \Rightarrow b=\frac{1}{\log(5+2\sqrt{6})}$$

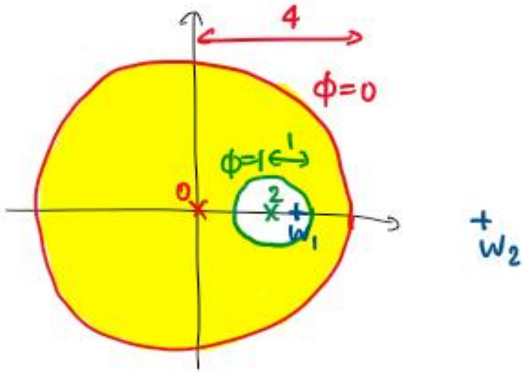
$$a=-\frac{\log\left(\frac{7-2\sqrt{6}}{5}\right)}{\log(5+2\sqrt{6})} \Rightarrow \phi(w)=\frac{\log|w|-\log\left(\frac{7-2\sqrt{6}}{5}\right)}{\log(5+2\sqrt{6})}$$

Finally, find a harmonic function in the original picture by substituting:

$$\boxed{\phi(z)=\frac{\log\left|\dfrac{z-\frac{1+4\sqrt{6}}{5}}{z-\frac{1-4\sqrt{6}}{5}}\right|-\log\left(\dfrac{7-2\sqrt{6}}{5}\right)}{\log(5+2\sqrt{6})}}$$

7.6 Q9   4 pts

Find two points symmetric to both circles simultaneously:



$$w_1 - 0 = \frac{4^2}{\overline{w_2} - 0}$$

$$w_1 - 2 = \frac{1^2}{\overline{w_2} - 2}$$

$$\frac{1}{\overline{w_2} - 2} = \frac{16}{\overline{w_2}} - 2$$

$$\overline{w_2} = 16(\overline{w_2} - 2) - 2\overline{w_2}(\overline{w_2} - 2)$$

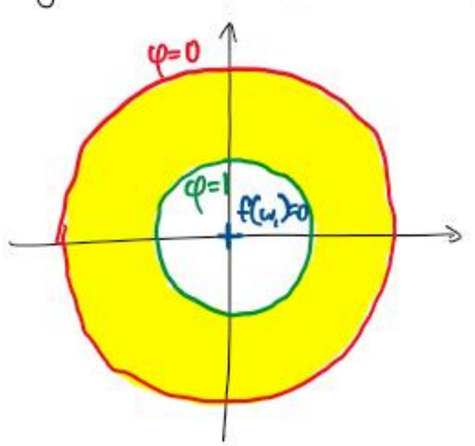
$$= -2\overline{w_2}^2 + 16\overline{w_2} + 4\overline{w_2} - 32$$

$$2\overline{w_2}^2 - 19\overline{w_2} + 32 = 0$$

$$\overline{w_2} = \frac{19 \pm \sqrt{361 - 256}}{4} = \frac{19 \pm \sqrt{105}}{4}$$

Set $w_1 = \frac{19 - \sqrt{105}}{4}$ and $w_2 = \frac{19 + \sqrt{105}}{4}$

Transform by $f(z) = \frac{z - w_1}{z - w_2}$ to map $\begin{matrix} w_1 \mapsto 0 \\ w_2 \mapsto \infty \end{matrix}$ and give two concentric circles:



Inner circle has radius $\frac{11 - \sqrt{105}}{4}$

$+ \rightarrow f(w_2) = \infty$ Outer circle has radius $\frac{19 - \sqrt{105}}{16}$

A harmonic function in this domain is of the form

$$a + b \log|w|$$

$$1 = a + b \log\left(\frac{11-\sqrt{105}}{4}\right)$$

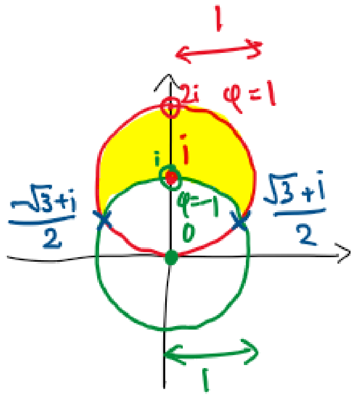
$$0 = a + b \log\left(\frac{19-\sqrt{105}}{16}\right)$$

$$1 = \quad b \log\left(\frac{44-4\sqrt{105}}{19-\sqrt{105}}\right) \Rightarrow b = \frac{1}{\log\left(\frac{44-4\sqrt{105}}{19-\sqrt{105}}\right)}$$

$$a = -\frac{\log\left(\frac{19-\sqrt{105}}{16}\right)}{\log\left(\frac{44-4\sqrt{105}}{19-\sqrt{105}}\right)}$$

$$\Rightarrow \varphi(z) = \frac{\log\left|\frac{z-\frac{19-\sqrt{105}}{4}}{z-\frac{19+\sqrt{105}}{4}}\right| - \log\left(\frac{19-\sqrt{105}}{16}\right)}{\log\left(\frac{44-4\sqrt{105}}{19-\sqrt{105}}\right)}$$

7.6 Q10 4 pts



Take the transformation

$$f(z) = \frac{z - \frac{-\sqrt{3}+i}{2}}{z - \frac{\sqrt{3}+i}{2}}$$

Maps the intersection points

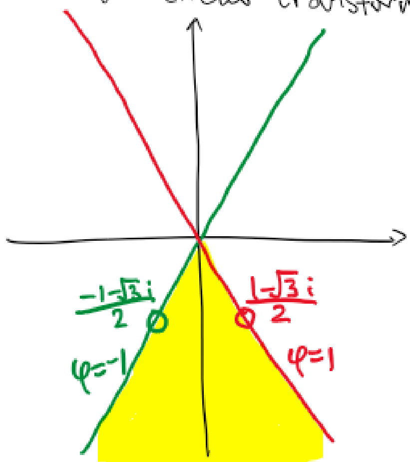
$$\frac{\sqrt{3}+i}{2} \mapsto \infty, \quad \frac{-\sqrt{3}+i}{2} \mapsto 0$$

So both circles map to lines passing through 0

Also,
$$i \mapsto \frac{i - \frac{-\sqrt{3}+i}{2}}{i - \frac{\sqrt{3}+i}{2}} = \frac{-1-\sqrt{3}\,i}{2}$$

$$2i \mapsto \frac{2i - \frac{-\sqrt{3}+i}{2}}{2i - \frac{\sqrt{3}+i}{2}} = \frac{1-\sqrt{3}\,i}{2}$$

The two circles transform to:



A harmonic function in this region is of the form

$\varphi(w) = a + b \operatorname{Arg} w$

For $\operatorname{Arg} w = -\frac{\pi}{3}$, $\quad a - \frac{\pi}{3} b = 1$

For $\operatorname{Arg} w = -\frac{2\pi}{3}$, $\quad a - \frac{2\pi}{3} b = -1$

$$\frac{\pi}{3} b = 2 \Rightarrow b = \frac{6}{\pi}$$

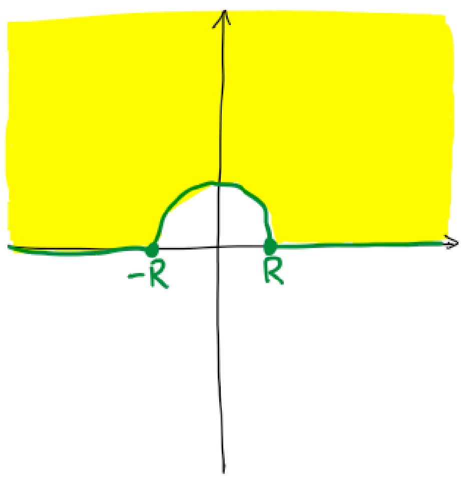
$$a - 2 = 1 \Rightarrow a = 3$$

$$\Rightarrow \varphi(w) = 3 + \frac{6}{\pi} \operatorname{Arg} w$$

Then substituting gives

$$\varphi(z) = 3 + \frac{6}{\pi} \operatorname{Arg}\left(\frac{z - \frac{-\sqrt{3}+i}{2}}{z - \frac{\sqrt{3}+i}{2}} \right)$$

7.7 Q4 4 pts



To map $R \mapsto \infty$, $-R \mapsto 0$, use the Möbius transformation $z \mapsto \frac{z+R}{z-R}$

This transform maps:

$\infty \mapsto 1$

$Ri \mapsto \frac{Ri+R}{Ri-R} = -i$

Then the line $[R,\infty]$ becomes $[1,\infty]$ and $[-\infty,-R]$ becomes $[0,1]$, and the arc becomes the vertical line from $0 \to -\infty i$. This maps to the 4^{th} quadrant.

(Re +ve, Im −ve)

Now multiply by i to rotate counterclockwise by $\pi/2$ and give the first quadrant. Mapping the first quadrant by squaring gives the upper half plane. The overall transformation becomes:

$$z \mapsto \left(i\,\frac{z+R}{z-R}\right)^2 = -\left(\frac{z+R}{z-R}\right)^2$$

Streamlines become level curves of

$$\operatorname{Im} w = \operatorname{Im} -\left(\frac{z+R}{z-R}\right)^2 = \operatorname{Im} -\left(\frac{x+R+yi}{x-R+yi}\right)^2$$

$$= \operatorname{Im} -\left(\frac{(x+R+yi)(x-R-yi)}{(x-R)^2+y^2}\right)^2$$

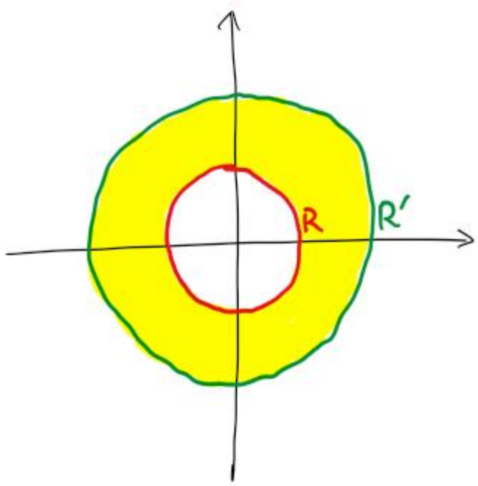
$$= \operatorname{Im} -\left(\frac{x^2-(R+yi)^2}{(x-R)^2+y^2}\right)^2$$

$$= \operatorname{Im} \frac{-(x^2-R^2+y^2-2Ryi)^2}{((x-R)^2+y^2)^2}$$

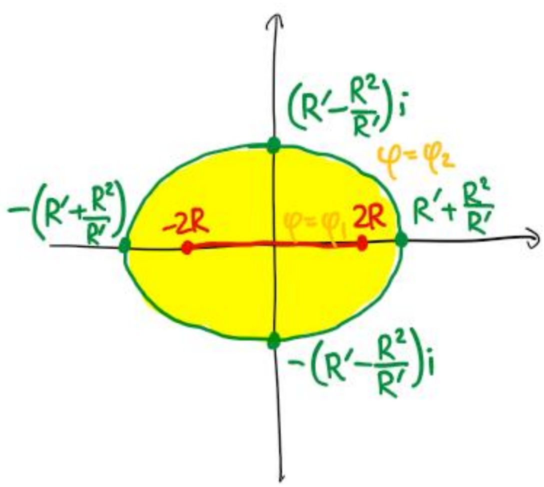
$$= \frac{2(x^2+y^2-R^2)(2Ry)}{((x-R)^2+y^2)^2}$$

$$= \frac{4yR(x^2+y^2-R^2)}{((x-R)^2+y^2)^2} \qquad \square$$

7.7 Q6 4 pts



$$\downarrow z \mapsto z + \frac{R^2}{z}$$



The mapping $f(z) = z + \frac{R^2}{z}$ maps:

Smaller circle (radius R):

$$Re^{i\theta} \mapsto Re^{i\theta} + \frac{R^2}{Re^{i\theta}}$$

$$= Re^{i\theta} + Re^{-i\theta}$$

$$= 2R\cos\theta$$

$\therefore$ maps to $[-2R, 2R]$

Larger circle (radius $R' > R$):

$$R'e^{i\theta} \mapsto R'e^{i\theta} + \frac{R^2}{R'e^{i\theta}}$$

$$= R'(\cos\theta + i\sin\theta) + \frac{R^2}{R'}(\cos\theta - i\sin\theta)$$

$$= \left(R' + \frac{R^2}{R'}\right)\cos\theta + \left(R' - \frac{R^2}{R'}\right) i \sin\theta$$

$\therefore$ maps to an ellipse:

$$\left(\frac{x}{R' + \frac{R^2}{R'}}\right)^2 + \left(\frac{y}{R' - \frac{R^2}{R'}}\right)^2 = 1$$

Now suppose this strip and ellipse are charged with potential φ_1 and φ_2 respectively. Let $\varphi(w)$ be the potential at point w here. By transforming this picture with the inverse of the transform used here, we get a harmonic function between two circles:

$$\varphi(w) = a + b \log|w|$$

(If $\varphi_1 = \varphi_2$, then φ is constant and the solution is trivial, so assume here $\varphi_1 \neq \varphi_2$)

$$\varphi(R) = a + b \log R = \varphi_1$$

$$\varphi(R') = a + b \log R' = \varphi_2$$

$$b \log\left(\frac{R'}{R}\right) = \varphi_2 - \varphi_1 \Rightarrow b = \frac{\varphi_2 - \varphi_1}{\log\left(\frac{R'}{R}\right)}$$

$$a = \varphi_1 - \frac{(\varphi_2 - \varphi_1)\log R}{\log\left(\frac{R'}{R}\right)}$$

$$a = \frac{\varphi_1(\log R' - \log R) - \varphi_2 \log R + \varphi_1 \log R}{\log\left(\frac{R'}{R}\right)}$$

$$a = \frac{\varphi_1 \log R' - \varphi_2 \log R}{\log\left(\frac{R'}{R}\right)}$$

$$\Rightarrow \quad \varphi(z) = \frac{\varphi_1 \log R' - \varphi_2 \log R + (\varphi_2 - \varphi_1)\log|w|}{\log\left(\frac{R'}{R}\right)}$$

To obtain the potential on the transformed picture, substitute $f^{-1}(z)$ above. $f^{-1}(z)$ is obtained by:

$$z = w + \frac{R^2}{w} \qquad zw = w^2 + R^2$$

$$w^2 - zw + R^2 = 0 \Rightarrow f^{-1}(z) = w = \frac{z + (z^2 - 4R^2)^{1/2}}{2}$$

taken as the principal branch with respect to $(\cdot)^{1/2}$

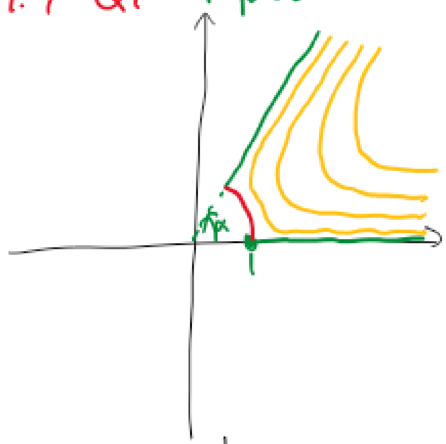
Thus the potential function is

$$\varphi(z) = \frac{\varphi_1 \log R' - \varphi_2 \log R + (\varphi_2 - \varphi_1) \log \left| \frac{z + (z^2 - 4R^2)^{1/2}}{2} \right|}{\log \left(\frac{R'}{R} \right)}$$

with equipotentials along ellipses

$$\left(\frac{x}{r + \frac{R^2}{r}} \right)^2 + \left(\frac{y}{r - \frac{R^2}{r}} \right)^2 = 1 \quad \text{for} \quad R < r < R'$$

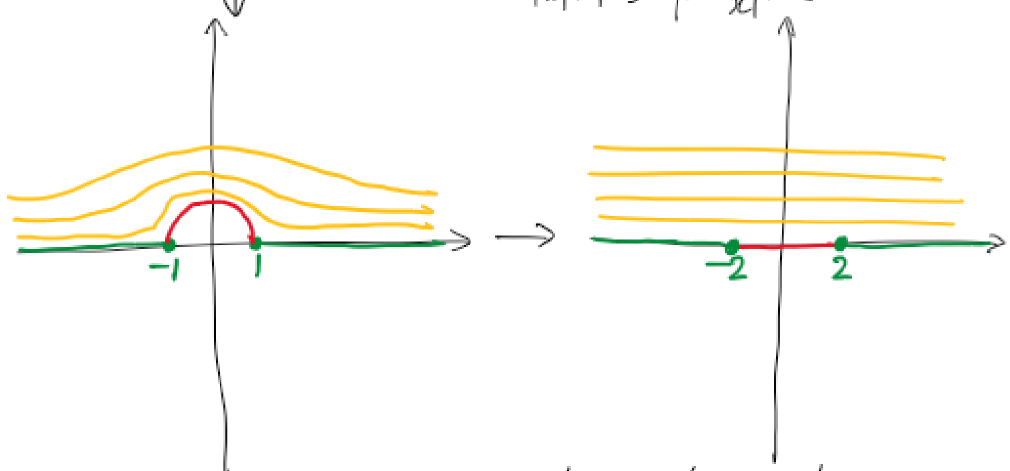
7.7 Q7 4 pts



Take the power transform $z \mapsto z^{\pi/\alpha}$ to transform into the upper half plane outside the unit circle, then use transform $z \mapsto z+\frac{1}{z}$ to transform in to the upper half plane

$e^{i\theta} \mapsto e^{i\theta}+e^{-i\theta} = \sin\theta$

$|x|>1 \Rightarrow \left|x+\frac{1}{x}\right|>2$



Overall transformation is $z \mapsto z^{\pi/\alpha} \mapsto z^{\pi/\alpha}+z^{-\pi/\alpha}=w$

To find stream lines of original picture, these are level curves of a harmonic function in the region with boundary values = 0

We have $\varphi(w) = \operatorname{Im} w$ in the transformed picture

To get a harmonic function in the original picture, substitute $w = z^{\pi/\alpha}+z^{-\pi/\alpha}$ to get

$$\varphi(z) = \operatorname{Im}\left(z^{\pi/\alpha}+z^{-\pi/\alpha}\right)$$

The stream lines are the level curves $\operatorname{Im}\left(z^{\pi/\alpha}+z^{-\pi/\alpha}\right)=k$ for each $k>0$