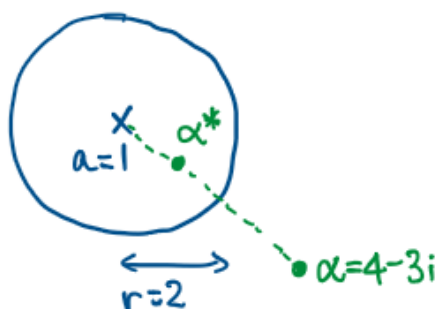


Homework 6 Solutions Mark out of 20

7.4 Q3(c) 1 pt



α is distance $|3-3i|=3\sqrt{2}$ from the center of the circle

Using $\frac{r}{|\alpha-a|} = \frac{|\alpha^*-a|}{r}$, the distance of

α^* from the center is $\frac{2^2}{3\sqrt{2}} = \frac{2}{3}\sqrt{2}$

and $\frac{\alpha^*-a}{\alpha-a} \in \mathbb{R}^+$

$$\text{Then } \alpha^* = 1 + \frac{(3-3i)}{3\sqrt{2}} \cdot \frac{2}{3}\sqrt{2} = 1 + \left(\frac{2}{3} - \frac{2}{3}i\right) = \frac{5}{3} - \frac{2}{3}i$$

(or just use the formula $\alpha^*-a = \frac{r^2}{\bar{\alpha}-\bar{a}}$)

7.4 Q9 2 pts

$$f(1)=\infty, \quad f: \text{real axis} \rightarrow |w|=1$$

1 and -1 are symmetric about the real axis,

so under the Möbius transform f ,

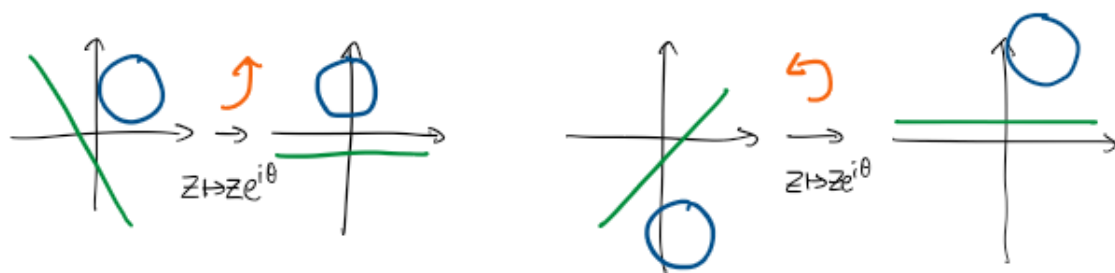
$f(1)$ and $f(-1)$ are symmetric about $|w|=1$

$$\text{If } f(1)=\infty, \text{ then } |\infty-0|=\infty \text{ and } \left| \underset{\substack{\parallel \\ f(-1)}}{\alpha^*-0} \right| = \frac{1^2}{\infty} = 0$$

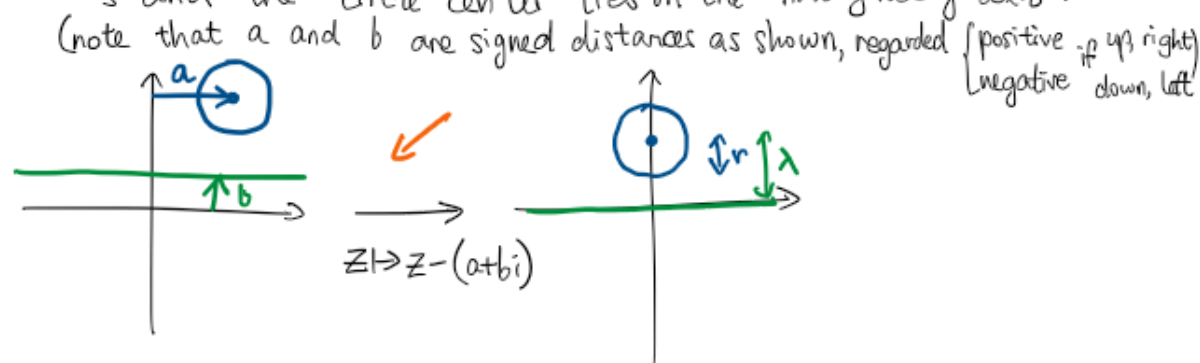
$$\text{So } f(-1)=0$$

7.4 Q10 5 pts total

(a) Non-intersecting line and circle can be rotated to give
 2pts a horizontal line with circle above by some $z \mapsto ze^{i\theta}$, $\theta \in [0, 2\pi)$:



Then translate the picture so that the line lies on the real axis and the circle center lies on the imaginary axis:



Since the original line and circle do not intersect, after rotation and translation they do not intersect. Then the circle is $|w - \lambda i| = r$ with $\lambda \in \mathbb{R}$ and $r < \lambda$.

Thus there exists a Möbius transformation formed by rotation and translation mapping $L \rightarrow$ real axis, $C \rightarrow$ circle above real axis, centered on imaginary axis

(b) If w_1 and w_2 are symmetric with respect to both $L^* = \text{real axis}$ and $C^* = f(C) = \{ |w - \lambda i| = r \}$, then $w_1 = \overline{w_2}$, and $w_1, w_2, \lambda i$ must be collinear. This implies that $w_1, w_2 \in \mathbb{R}i$ (purely imaginary) and in fact $w_1 = ai, w_2 = -ai$ for some $a \in \mathbb{R}$. To make symmetric with C^* , we need

$$\frac{|ai - \lambda i|}{r} = \frac{r}{|-ai - \lambda i|} \Rightarrow |(a - \lambda)(a + \lambda)| = r^2$$

$$\text{and } \frac{ai - \lambda i}{-ai - \lambda i} = \frac{\lambda - a}{\lambda + a} \in \mathbb{R}^+ \Rightarrow \text{sgn}(\lambda - a) = \text{sgn}(\lambda + a) \\ \Rightarrow (\lambda - a)(\lambda + a) = \lambda^2 - a^2 = r^2$$

$$\Rightarrow a = \sqrt{\lambda^2 - r^2}$$

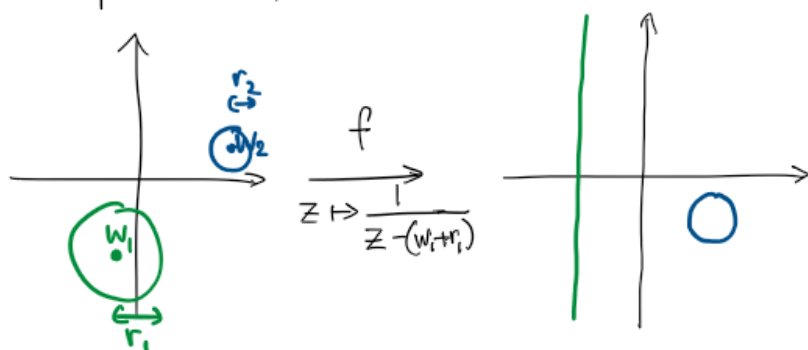
There exist a unique pair of points symmetric about both the circle and line:

$$\sqrt{\lambda^2 - r^2} i, -\sqrt{\lambda^2 - r^2} i$$

(c) Taking the inverse Möbius transform of the transform obtained in (a), by the symmetry principle, the two points found in (b) under this transformation are symmetric about both L and C , and this pair of points is unique.

7.4 Q11 3 pts

Two non-intersecting circles C_1, C_2 can be inverted about a point on C_1 to map C_1 into a line and C_2 into another circle:



Because C_1 and C_2 do not intersect, $f(C_1)$ and $f(C_2)$ also do not intersect.

By the result of the previous question, there exist two points $f(z_1)$ and $f(z_2)$ symmetric about both the line and the circle $f(C_1)$ and $f(C_2)$.

Using the symmetry principle, by applying f^{-1} to this picture, z_1 and z_2 are symmetric about both C_1 and C_2 .

7.4 Q15 3 pts

If $|\alpha| < 1$, and $|z| = 1$, then

$$\left| \frac{z - \alpha}{\bar{\alpha}z - 1} \right| = \frac{|z - \alpha|}{|\bar{\alpha}z - 1|} = \frac{|1 - \alpha\bar{z}|}{|1 - \bar{\alpha}z|} = 1 \quad (\text{by conjugation})$$

Inverse: $\frac{z - \alpha}{\bar{\alpha}z - 1} = w$

$$\frac{z - 1/\bar{\alpha} - \alpha}{\bar{\alpha}z - 1} = \frac{1}{\bar{\alpha}} - \frac{\alpha}{\bar{\alpha}z - 1} = w - \frac{1/\bar{\alpha}}{\bar{\alpha}z - 1}$$

$$\frac{1/\bar{\alpha} - \alpha}{\bar{\alpha}z - 1} = w - \frac{1}{\bar{\alpha}}$$

$$\bar{\alpha}z - 1 = \frac{1/\bar{\alpha} - \alpha}{w - 1/\bar{\alpha}} = \frac{1 - |\alpha|^2}{\bar{\alpha}w - 1}$$

$$z = \frac{\frac{1 - |\alpha|^2}{\bar{\alpha}w - 1} + 1}{\bar{\alpha}} = \frac{\frac{\bar{\alpha}w - |\alpha|^2}{\bar{\alpha}w - 1}}{\bar{\alpha}} = \frac{w - \alpha}{\bar{\alpha}w - 1}$$

\therefore The inverse of this map is itself

Thus this map maps $|z| = 1$ to $|w| = 1$

Looking at the interiors now, if $z = 0$

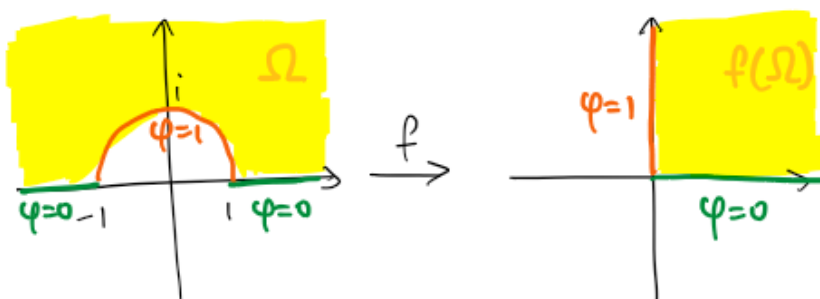
$$w = \alpha \quad \text{and} \quad |w| < 1.$$

Then if $|z| < 1$, $|w| < 1$ since this map is conformal

7.6 Q2 3 pts

Need to find a harmonic function satisfying the boundary.

Consider the map $f(z) = \frac{z-1}{z+1}$



A harmonic function on the domain $f(\Omega)$ can be obtained from:

$$h(w) = a \operatorname{Arg} w + b$$

$$\text{For } w \in \mathbb{R}^+ : h(w) = b = 0 \quad \Rightarrow \quad b = 0$$

$$w \in \mathbb{R}^+ i : h(w) = a\pi/2 + b = 1 \quad \Rightarrow \quad a = \frac{2}{\pi}$$

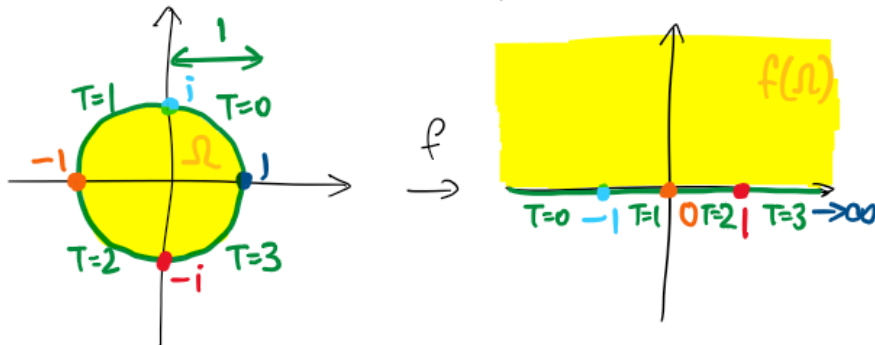
$$\text{Then } h(w) = \frac{2}{\pi} \operatorname{Arg} w$$

Now transform to the original domain Ω via f :

$$\varphi(z) = h \circ f(z) = \frac{2}{\pi} \operatorname{Arg} \left(\frac{z-1}{z+1} \right)$$

7.6 Q4 3pts

Use the map $f(z) = i\left(\frac{1+z}{1-z}\right)$ to map :



Using previous results from 7.1 Q3, a harmonic function on $f(\Omega)$ satisfying the conditions is:

$$h(w) = -\frac{1}{\pi} \left(\text{Arg}(w+1) + \text{Arg}(w) + \text{Arg}(w-1) \right) + 3$$

Finally, to obtain a solution on Ω :

$$T(z) = h \circ f(z)$$

$$= -\frac{1}{\pi} \left(\text{Arg}\left(i \cdot \frac{1+z}{1-z} + 1\right) + \text{Arg}\left(i \cdot \frac{1+z}{1-z}\right) + \text{Arg}\left(i \cdot \frac{1+z}{1-z} - 1\right) \right) + 3$$