Homework 5 Solutions

Mark out of 35

7.1 Q3 First find a function f with $z \in \mathbb{R} \Rightarrow f(z) \in \mathbb{R}$

4pts and $f(x_1) = f(x_2) = f(x_3) = 0$ and $f(x) \neq 0$ everywhere else on $x \in \mathbb{R}$

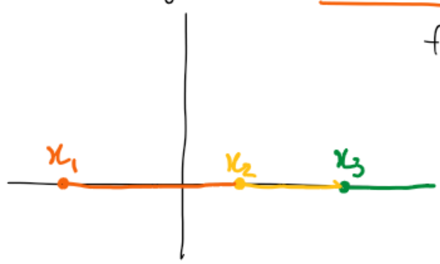
Take $f(z) = (z - x_1)(z - x_2)(z - x_3)$

Now take $g(z) = \log f(z)$, a branch to be determined to be analytic on the upper half plane

A choice would be

$$g(z) = b_0 + \underline{b_1 \operatorname{Log}(z - x_1)} + \underline{b_2 \operatorname{Log}(z - x_2)} + \underline{b_3 \operatorname{Log}(z - x_3)}$$

for $b_0, b_1, b_2, b_3 \in \mathbb{R}$



Branch cuts for each individual function. Ensures analyticity on the upper half plane.

Then $g(z)$ is analytic on the upper half plane

Now take $\varphi(x,y) = \operatorname{Im}(g(\underbrace{x+iy}_{z}))$ which is harmonic

$$= b_0 + b_1 \operatorname{Arg}(z - x_1) + b_2 \operatorname{Arg}(z - x_2) + b_3 \operatorname{Arg}(z - x_3)$$

We want
$$\varphi(x, 0^+) = \begin{cases} a_1 = b_0 + \pi b_1 + \pi b_2 + \pi b_3 & x < x_1 \\ a_2 = b_0 + \pi b_2 + \pi b_3 & x_1 < x < x_2 \\ a_3 = b_0 + \pi b_3 & x_2 < x < x_3 \\ a_4 = b_0 & x > x_3 \end{cases}$$

Then $b_0 = a_4$, $b_3 = \dfrac{a_3 - b_0}{\pi} = \dfrac{a_3 - a_4}{\pi}$

$$b_2 = \frac{a_2 - b_0 - \pi b_3}{\pi} = \frac{a_2 - a_4 - (a_3 - a_4)}{\pi} = \frac{a_2 - a_3}{\pi}$$

$$b_1 = \frac{a_1 - b_0 - \pi b_2 - \pi b_3}{\pi} = \frac{a_1 - a_4 - (a_2 - a_3) - (a_3 - a_4)}{\pi}$$

$$= \frac{a_1 - a_2}{\pi}$$

Then a solution to the problem is, setting $z = x + iy$

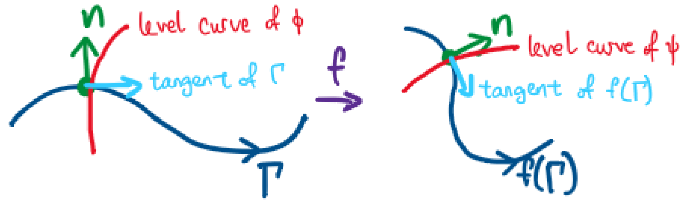
$$\varphi(x,y) = (a_1 - a_2)\frac{\operatorname{Arg}(z - x_1)}{\pi} + (a_2 - a_3)\frac{\operatorname{Arg}(z - x_2)}{\pi} + (a_3 - a_4)\frac{\operatorname{Arg}(z - x_3)}{\pi} + a_4$$

7.1 Q6
2pts

If $\frac{\partial \phi}{\partial n} = 0$ on Γ, then $\frac{\partial \phi}{\partial x} + i\frac{\partial \phi}{\partial y}$ is orthogonal to the normal and the normal is tangent to a level curve of ϕ.

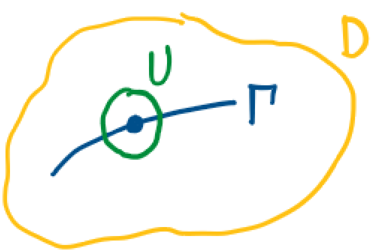
Since the map f is conformal, the tangent and normal remain orthogonal under the map f, so the normal to $f(\Gamma)$ is tangent to a level curve of ψ.

Thus $\frac{\partial \psi}{\partial n} = 0$. $\square$



Alternative Perspective:

Each point z on Γ can be placed inside a neighbourhood (disk) $U \subset D$ centered at z. Since ϕ is harmonic in D, there exists



an analytic function $g(z = x+iy)$ with $\operatorname{Re} g(z) = \phi(x,y)$ inside this disk. Assuming $\frac{\partial \phi}{\partial n} = 0$, this means the directional derivative $\frac{\partial g}{\partial n}$ is purely imaginary. The directional derivative $\frac{\partial g}{\partial \Gamma}$ along the curve Γ is then purely real, which implies that $\operatorname{Im} g$ is constant on Γ.

Transforming the picture under the analytic one-to-one map f gives another analytic function $h = g \circ f^{-1}$ in the image of the disk, $f(U)$. $\operatorname{Im} h$ is constant along the image curve $f(\Gamma)$, so $\frac{\partial h}{\partial \Gamma}$ is purely real, and $\frac{\partial h}{\partial n}$ is purely imaginary. We also have the harmonic function $\psi = \operatorname{Re} h$, and this makes $\frac{\partial \psi}{\partial n} = 0$. $\square$

7.2 Q7 If $|z| = \rho$ for $\rho \neq 1$, then $z = \rho e^{it}$ for some $t \in [0, 2\pi)$.

2pts

$$W = z + \frac{1}{z} = \rho e^{it} + \frac{1}{\rho} e^{-it} = \rho \cos t + i\rho \sin t + \frac{1}{\rho}\cos t - \frac{i}{\rho} \sin t$$

$$= \underbrace{\left(\rho + \frac{1}{\rho}\right)\cos t}_{u} + i \underbrace{\left(\rho - \frac{1}{\rho}\right)\sin t}_{v}$$

$$\Rightarrow \left(\frac{u}{\rho + \frac{1}{\rho}}\right)^2 + \left(\frac{v}{\rho - \frac{1}{\rho}}\right)^2 = \cos^2 t + \sin^2 t = 1$$

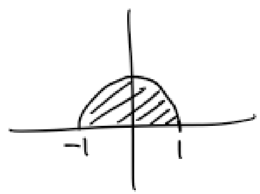
$\square$

7.2 Q11 (c) $\operatorname{Re} z < 0$, $0 < \operatorname{Im} z < \pi$

1pt

$\Rightarrow |e^z| < 1$, e^z has +ve imaginary

$\Rightarrow$ Image =



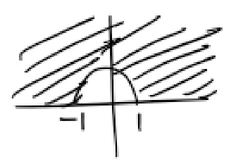
$\{|w| < 1, \ \operatorname{Im} w > 0\}$

7.2 Q11 (d) $\operatorname{Re} z > 0$, $0 < \operatorname{Im} z < \pi$

1pt

$\Rightarrow |e^z| > 1$, e^z has +ve imaginary

$\Rightarrow$ Image =



$\{|w| > 1, \ \operatorname{Im} w > 0\}$

7.2 Q13 (b) $0 < \operatorname{Re} z < \frac{\pi}{2}$, $\operatorname{Im} z > 0$

2pts

$\cos z = \cos x \cosh y - i \sin x \sinh y$

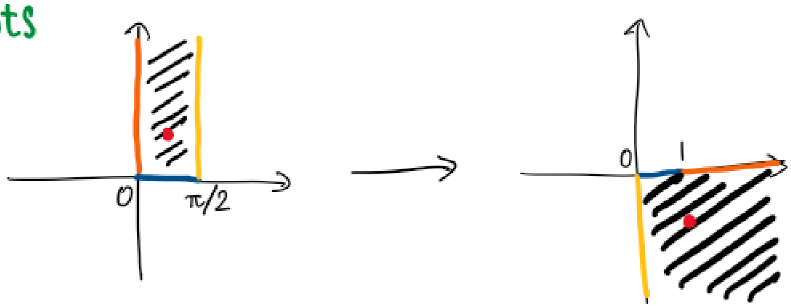
$y = 0 \Rightarrow \cos z = \cos x$

$x = 0 \Rightarrow \cos z = \cosh y$

$x = \frac{\pi}{2} \Rightarrow \cos z = -i \sinh y$

on $y > 0$, $\cosh y > 1$

$\sinh y > 0$

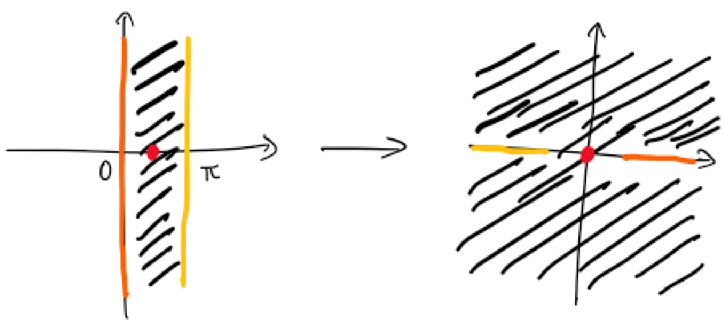


Interior point: $\cos\left(\frac{\pi}{4} + i\right) = \underline{\frac{1}{\sqrt{2}} \cosh 1 - i \frac{1}{\sqrt{2}} \sinh 1}$

Image $= \{\operatorname{Re} w > 0, \operatorname{Im} w < 0\}$

7.2 Q13 (c) $0 < \operatorname{Re} z < \pi$

2pts



$x = 0 \Rightarrow \cos z = \cosh y > 1$

$x = \pi \Rightarrow \cos z = -\cosh y < -1$

Interior point: $\cos\left(\frac{\pi}{2}\right) = 0$

Image $= \mathbb{C} \setminus \{\operatorname{Im} w = 0, |\operatorname{Re} w| \geq 1\}$

7.3 Q3(d) 3pts

This is a Möbius transformation that transforms the circle $|z-2|=1$ into either a circle or a line. One of the points, $z=3$, maps to infinity, so it becomes a line. We need to find two points on the line. Set $z=1$ and $z=2+i$, then

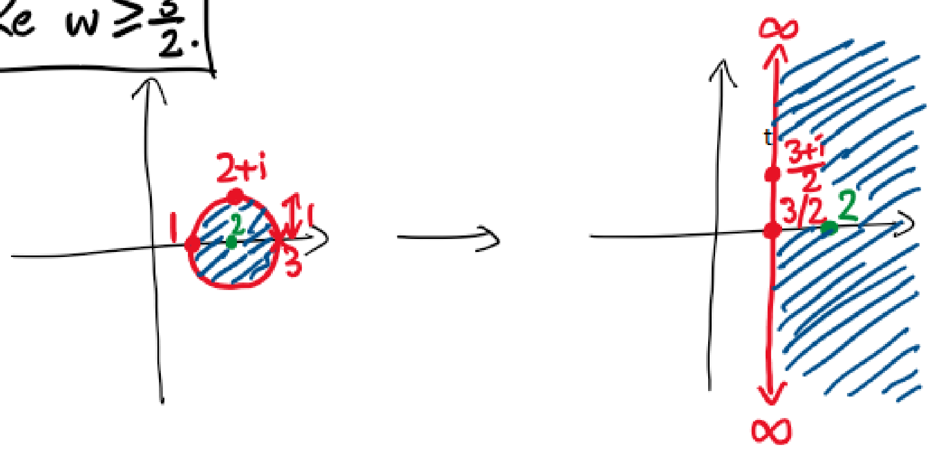
$$w=\frac{1-4}{1-3}=\frac{-3}{-2}=\frac{3}{2} \quad \text{and} \quad w=\frac{2+i-4}{2+i-3}=\frac{-2+i}{-1+i}=\frac{(-2+i)(-1-i)}{2}$$

$$=\frac{3+i}{2}$$

Then the line is a vertical line with $\text{Re}\, w=\frac{3}{2}$.

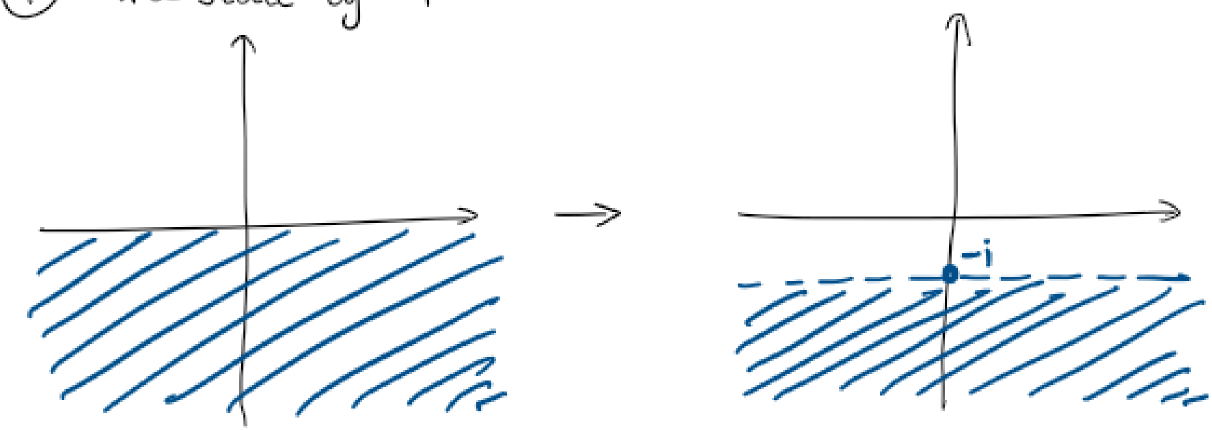
To find the image of the interior, $z=2$ maps to $w=\frac{2-4}{2-3}=2$ which is right of $\text{Re}\, w=\frac{3}{2}$. Then the interior maps to the right half plane

$$\boxed{\text{Re}\, w \geq \frac{3}{2}.}$$



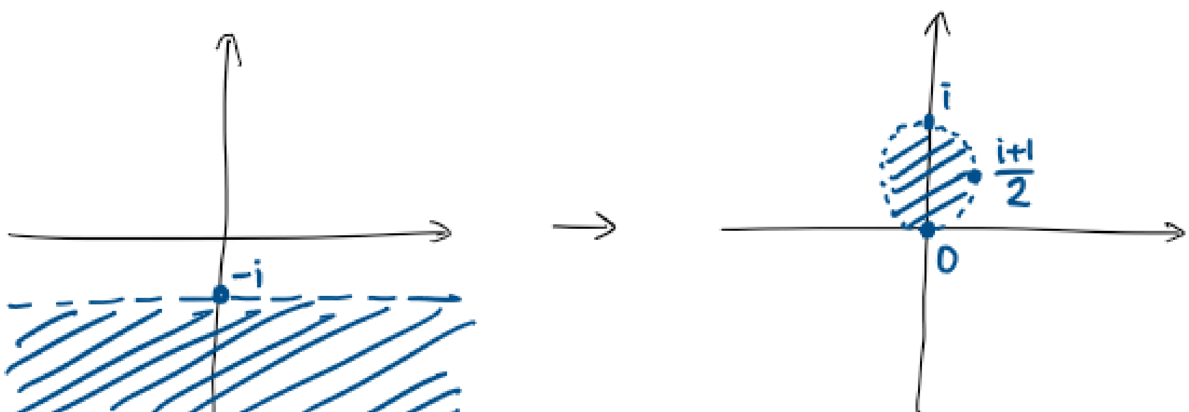
7.3 Q4 3pts

① Translate by $-i$:



$w_0 = z$ $\qquad\qquad$ $w_1 = z - i$

② Invert about zero:



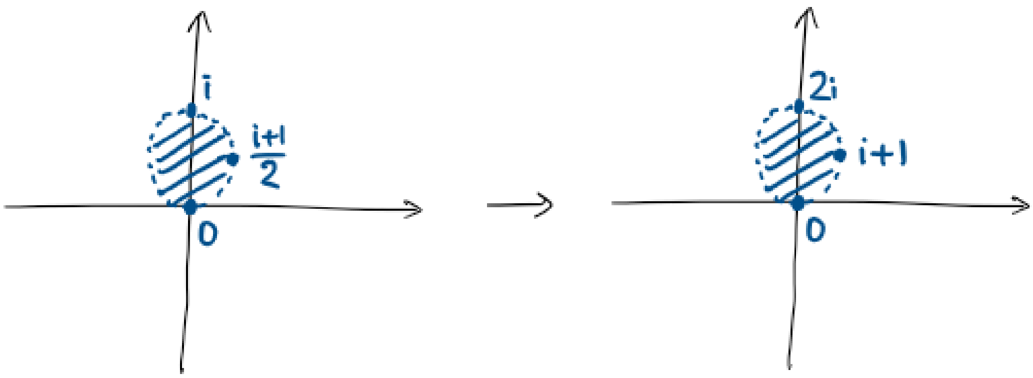
$w_1 = z - i$ $\qquad\qquad$ $w_2 = \dfrac{1}{z-i}$

$$\text{3 points}\begin{cases} w_1 = -i \Rightarrow w_2 = i \\ w_1 = \infty \Rightarrow w_2 = 0 \\ w_1 = -i+1 \Rightarrow w_2 = \dfrac{1}{-i+1} = \dfrac{i+1}{2} \end{cases}$$

$w_1 = 0$ outside $\Rightarrow$ $w_2 = \infty$ outside

$$\left|w_2 - \frac{i}{2}\right| < \frac{1}{2}$$

③ Scale by a factor of 2 about zero:

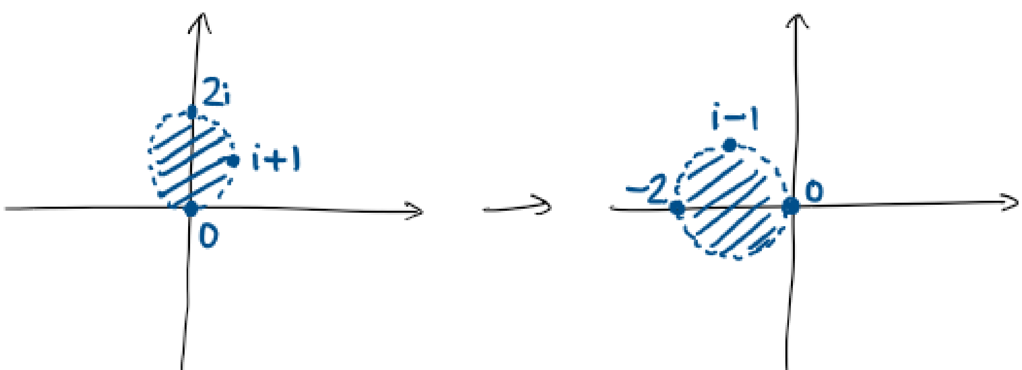


$w_2 = \frac{1}{z-i}$

$w_3 = \frac{2}{z-i}$

$|w_3 - i| < 1$

④ Rotate counterclockwise by $\pi/2$:



$w_3 = \frac{2}{z-i}$

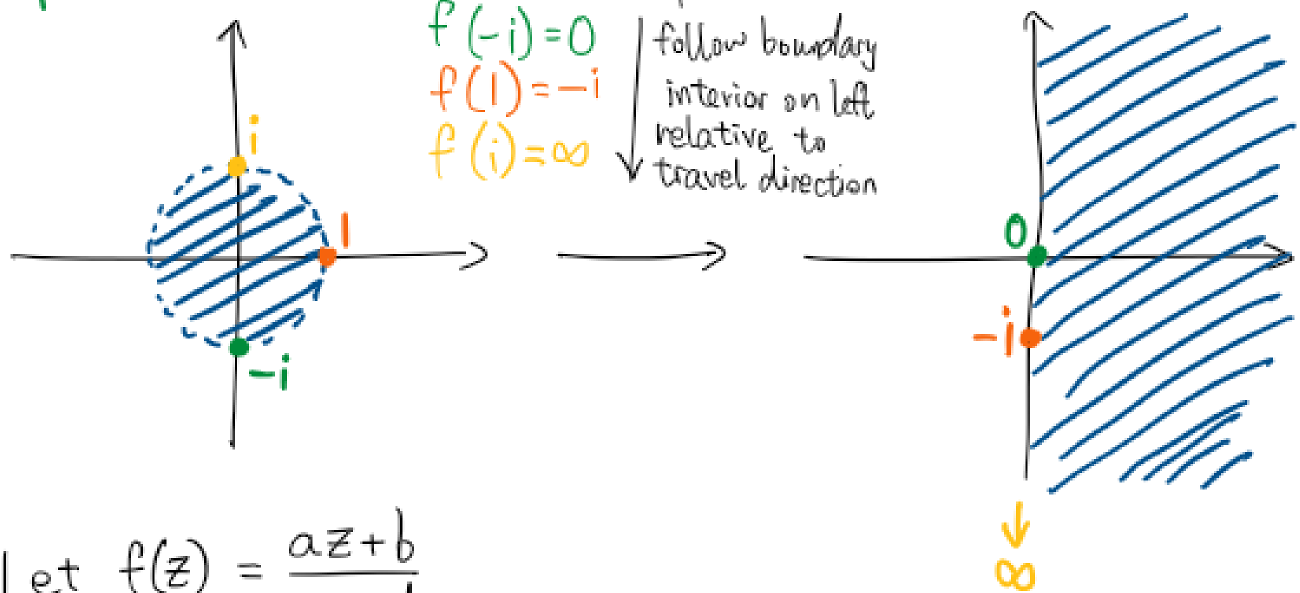
$$\boxed{w = w_4 = \frac{2i}{z-i}}$$ (other solutions possible)

$|w+1| < 1$

7.3 Q5 Transformation $w = f(z)$ should satisfy

3 pts

A suitable map:



Let $f(z) = \dfrac{az+b}{cz+d}$

Then $0 = \dfrac{-ai+b}{-ci+d} \Rightarrow -ai+b=0 \Rightarrow \dfrac{b}{a} = i$

$\infty = \dfrac{ai+b}{ci+d} \Rightarrow ci+d=0 \Rightarrow \dfrac{d}{c} = -i$

$-i = \dfrac{a+b}{c+d} \Rightarrow a+b = -i(c+d)$

$a+ia = -ic-c \Rightarrow a=-c$

By setting $a=1$, let $b=i$ $\qquad \dfrac{b}{d}=1$, $\mathrm{Re}\left(\dfrac{b}{d}\right)>0$

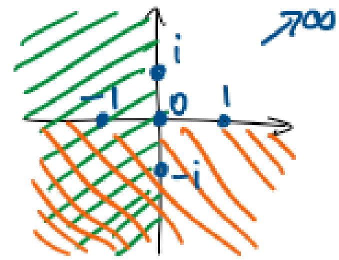
$c=-1$ $\qquad d=i$ $\qquad [f(0)=1]$

$\Rightarrow \boxed{f(z) = \dfrac{z+i}{-z+i}}$ (other solutions possible)

7.3 Q8 3pts

The third quadrant is the intersection of the half-planes

$\{\operatorname{Re} z < 0\} \cap \{\operatorname{Im} z < 0\}$



The map $w = \frac{z+i}{z-i}$ maps each plane to:

① $0 \mapsto -1$

② $i \mapsto \infty$

③ $-i \mapsto 0$

④ $-1 \mapsto \frac{-1+i}{-1-i} = -i$

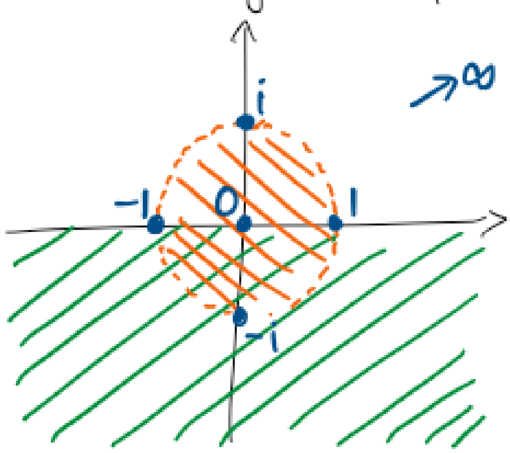
⑤ $1 \mapsto \frac{1+i}{1-i} = i$

⑥ $\infty \mapsto 1$

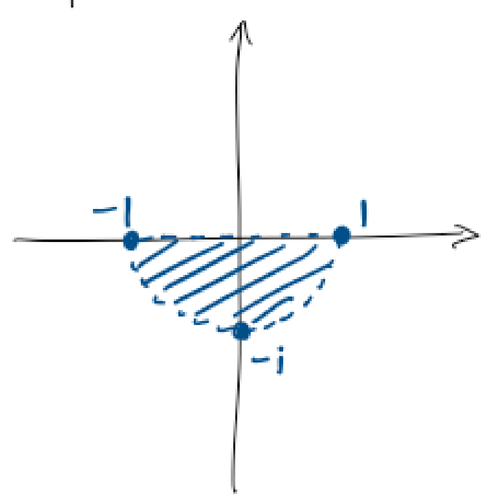
①, ②, ③, ⑥ are on the boundary of (green)
④ is inside (green) ⇒ bottom half plane

①, ④, ⑤, ⑥ are on the boundary of (orange)
③ is inside (orange) ⇒ disk $|w| < 1$

Then each region maps to:



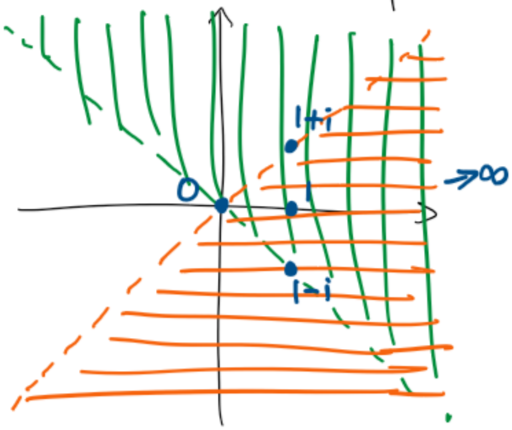
Thus the third quadrant maps to the semicircle:



$\{\operatorname{Im} w < 0\} \cap \{|w| < 1\}$

7.3 Q9 The sector $-\pi/4 < \operatorname{Arg} z < \pi/4$ is the intersection of two half-planes:

3pts



$w = \dfrac{z}{z-1}$ maps the following selected points:

① $0 \mapsto 0$

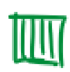
② $\infty \mapsto 1$

③ $1+i \mapsto \dfrac{1+i}{i} = 1-i$

④ $1-i \mapsto \dfrac{1-i}{-i} = 1+i$

⑤ $1 \mapsto \infty$

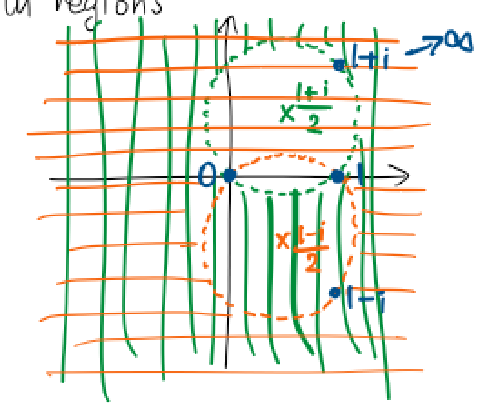
①, ②, ③ are on the boundary of 

①, ②, ④ are on the boundary of 

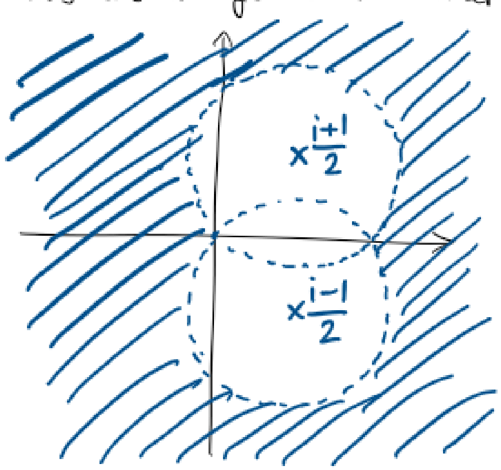
⑤ is in the intersection of both regions

$\Rightarrow$  maps to $\left|w - \frac{1-i}{2}\right| > \frac{1}{\sqrt{2}}$

 maps to $\left|w - \frac{1+i}{2}\right| > \frac{1}{\sqrt{2}}$



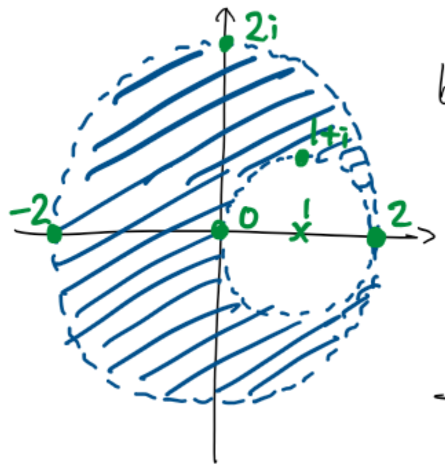
Thus the image is the intersection



$$\left\{\left|w - \frac{1-i}{2}\right| > \frac{1}{\sqrt{2}}\right\} \cap \left\{\left|w - \frac{1+i}{2}\right| > \frac{1}{\sqrt{2}}\right\}$$

7.3 Q11 3pts

Choose an initial transformation $w_1 = \frac{1}{z-2}$



Selected boundary points:

$2 \mapsto \infty$

$0 \mapsto -\frac{1}{2}$

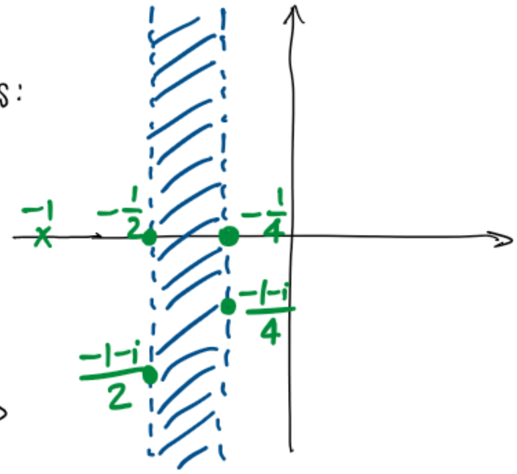
$-2 \mapsto -\frac{1}{4}$

$1+i \mapsto \frac{-1-i}{2}$

$2i \mapsto \frac{-1-i}{4}$

$1 \mapsto -1$ outside the region

Möbius transform maps circles to lines



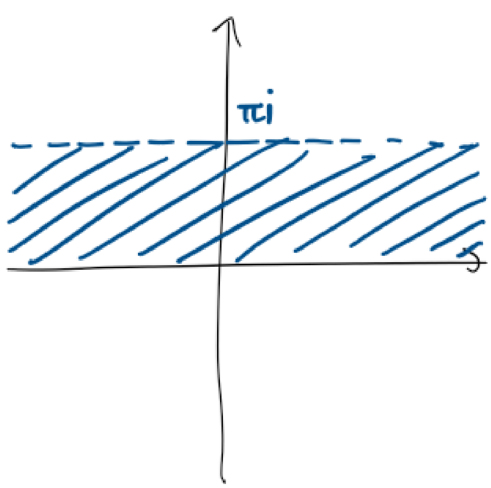
$\left\{ -\frac{1}{2} < \operatorname{Re} w_1 < -\frac{1}{4} \right\}$

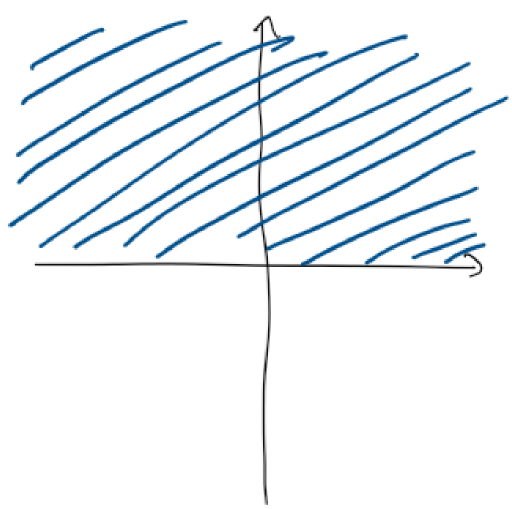
Next, set $w_2 = -4\pi i \left(w_1 + \frac{1}{4}\right) = -4\pi i \left(\frac{\frac{z}{4} + \frac{1}{2}}{z-2} \right) = -\pi i \left(\frac{z+2}{z-2} \right)$

to transform the strip to $\{ 0 < \operatorname{Im} w_2 < \pi \}$

then apply the exponential map $w_3 = e^{w_2}$ to give

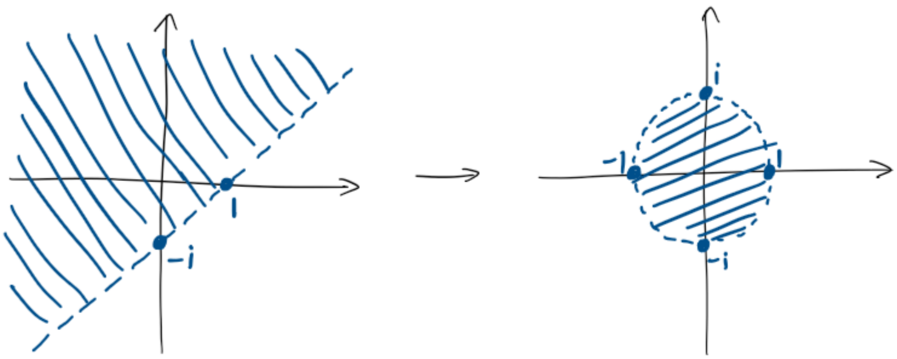
$\{ 0 < \operatorname{Arg} w_3 < \pi \} = \{ \operatorname{Im} w_3 > 0 \}$

$\rightarrow$ 

$\rightarrow$ 

Thus, $\mathbf{w = w_3 = e^{-\pi i \left(\frac{z+2}{z-2}\right)}}$

7.3 Q12 3pts



A suitable Möbius transform $w=f(z)$ maps boundary to boundary

By conformality, following the boundary $-i \to 1 \to \infty$ (original)

$-i \to 1 \to i$ (mapped)

with interior on left-hand side (relative to direction of movement), a possible Möbius transform maps

$-i \mapsto -i, \quad 1 \mapsto 1, \quad \infty \mapsto i$

Let $f(z) = \dfrac{az+b}{cz+d}$

Then $\dfrac{a}{c} = i$, $\quad \dfrac{-ai+b}{-ci+d} = \dfrac{-ai+b}{-a+d} = -i$, $\quad \dfrac{a+b}{c+d} = \dfrac{a+b}{-ai+d} = 1$

$c = -ia$

$-ai+b = ai - di$

$2ai = b + di$

$a+b = -ai+d$

$a(1+i) = d-b$

$$\frac{b+di}{2i} = \frac{d-b}{1+i}$$

$$-\frac{i}{2}b + \frac{1}{2}d = \frac{-1+i}{2}b + \frac{1-i}{2}d$$

$$\frac{1-2i}{2}b = \frac{-i}{2}d$$

$$d = (2+i)b$$

Set $a=1$, then $c=-i$, $\quad d-b = 1+i = (1+i)b$

$b=1, \quad d = 2+i$

Then $f(z) = \dfrac{z+1}{-iz+(2+i)}$ (other solutions possible)