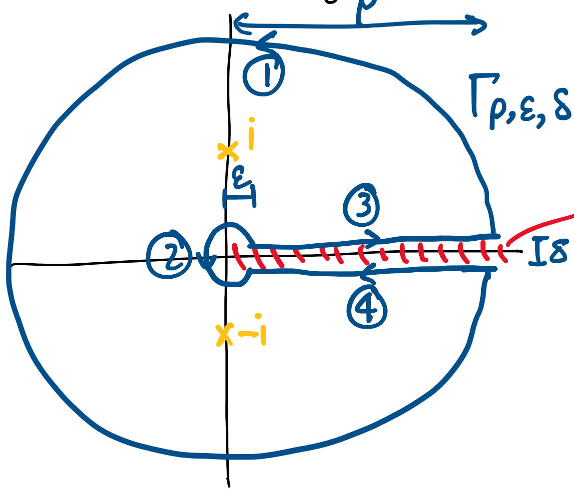
Homework 4 Solutions

Each question 4 pts
Mark out of 28

(All questions from 6.6)

Q1 Contour to integrate over:



By taking $\delta \to 0$, $\varepsilon \to 0$, $\rho \to \infty$:

$$\int_{\Gamma_{\rho,\varepsilon,\delta}} \frac{z^{1/2}}{z^2+1}\,dz = \overset{①}{\int_{\substack{\text{Large arc}\\ (\text{radius } \rho)}}} \frac{z^{1/2}}{z^2+1}\,dz + \overset{②}{\int_{\substack{\text{small arc}\\ (\text{radius } \varepsilon)}}} \frac{z^{1/2}}{z^2+1}\,dz$$

$$+ \overset{③}{\int_{\substack{\text{upper line}\\ \text{segment}}}} \frac{z^{1/2}}{z^2+1}\,dz + \overset{④}{\int_{\substack{\text{Lower line}\\ \text{segment}}}} \frac{z^{1/2}}{z^2+1}\,dz$$

① $\to 0$ since by M-L estimates, $2\pi\rho \cdot \frac{\rho^{1/2}}{\rho^2-1} \to 0$

② $\to 0$ since arc length $\to 0$ and $\frac{z^{1/2}}{z^2+1} \to 0$

③ $\to \int_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx$ ④ $\to -\int_0^\infty \frac{-\sqrt{x}}{x^2+1}\,dx$

(this branch of $z^{1/2}$ approaches $\begin{cases} +\sqrt{x} & \text{from above} \\ -\sqrt{x} & \text{from below} \end{cases}$)

$\Rightarrow$ Integral $= 2\int_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx$

By Residue Theorem,

both poles order 1

Integral $= 2\pi i \left(\mathrm{Res}(f; i) + \mathrm{Res}(f; -i) \right)$

$$\text{Integral} = 2\pi i\left(\operatorname{Res}(f;i) + \operatorname{Res}(f;-i)\right)$$
$$= 2\pi i\left(\frac{e^{i\pi/4}}{2i} + \frac{e^{i3\pi/4}}{-2i}\right)$$
$$= 2\pi i\left(\frac{i+1}{2\sqrt{2}i} - \frac{i-1}{2\sqrt{2}i}\right) = 2\pi i\,\frac{2}{2\sqrt{2}i} = \sqrt{2}\,\pi$$
$$\Rightarrow \int_0^\infty \frac{\sqrt{x}}{x^2+1}\,dx = \frac{\pi}{\sqrt{2}} \qquad \square$$

Q4 Same contour as before, but this time we have $f(z) = \dfrac{z^\alpha}{(z^2+1)^2}$

Use branch defined by $\dfrac{e^{\alpha \log z}}{(z^2+1)^2}$ where $\log z \in [0, 2\pi)$

Replace integrand with $f(z)$:

1 $\to 0$ by M-L estimates: $\dfrac{\rho^\alpha}{(\rho^2-1)^2}\cdot 2\pi\rho = 2\pi\,\dfrac{\rho^{\alpha+1}}{(\rho^2-1)^2} \to 0$

since $\alpha+1<4$ by assumption and $(\rho^2-1)^2$ has degree 4

2 $\to 0$ by M-L estimates: $\dfrac{\varepsilon^\alpha}{(1-\varepsilon^2)^2}\cdot 2\pi\varepsilon = 2\pi\,\dfrac{\varepsilon^{\alpha+1}}{(1-\varepsilon^2)^2} \to 0$

since denominator $\to 1 \neq 0$ and numerator $\to 0$ given $\alpha+1>0$

3 $\to \displaystyle\int_0^\infty \frac{x^\alpha}{(x^2+1)^2}\,dx$ 4 $\to -\displaystyle\int_0^\infty \frac{x^\alpha e^{i2\pi\alpha}}{(x^2+1)^2}\,dx$

$\left(z^\alpha \text{ approaches } \begin{cases} x^\alpha & \text{from above} \\ e^{\alpha(\ln x + 2\pi i)} = x^\alpha e^{i2\pi\alpha} & \text{from below} \end{cases}\right)$

$$\Rightarrow \text{Integral} = (1 - e^{i2\pi\alpha})\int_0^\infty \frac{x^\alpha}{(x^2+1)^2}\,dx$$

By residue theorem,

poles order 2

$$\text{Integral} = 2\pi i\left(\operatorname{Res}(f;i) + \operatorname{Res}(f;-i)\right)$$

$$\operatorname{Res}(f;i) = \frac{d}{dz}\bigg|_{z=i} \frac{z^\alpha}{(z+i)^2} = \frac{\alpha z^{\alpha-1}}{(z+i)^2} - \frac{2z^\alpha}{(z+i)^3}\bigg|_{z=i}$$
$$= \frac{\alpha e^{i\pi(\alpha-1)/2}}{-4} - \frac{2e^{i\pi\alpha/2}}{-8i}$$
$$= i\frac{\alpha e^{i\pi\alpha/2}}{4} - i\frac{e^{i\pi\alpha/2}}{4} = \frac{ie^{i\pi\alpha/2}}{4}(\alpha-1)$$

$$\operatorname{Res}(f;-i) = \frac{d}{dz}\bigg| \quad \frac{z^\alpha}{} \quad \frac{\alpha z^{\alpha-1}}{} \quad \frac{2z^\alpha}{}\bigg|$$

$$\mathrm{Res}(f,-i) = \frac{d}{dz}\bigg|_{z=-i} \frac{z^\alpha}{(z-i)^2} = \frac{\alpha z^{\alpha-1}}{(z-i)^2} - \frac{2z^\alpha}{(z-i)^3}\bigg|_{z=-i}$$

$$= \frac{\alpha e^{3i\pi(\alpha-1)/2}}{-4} - \frac{2e^{3i\pi\alpha/2}}{8i}$$

$$= -i\,\frac{\alpha e^{3i\pi\alpha/2}}{4} + i\,\frac{e^{3i\pi\alpha/2}}{4} = \frac{ie^{3i\pi\alpha/2}}{4}(-\alpha+1)$$

$$= 2\pi i\left(\frac{ie^{i\pi\alpha/2}}{4}(\alpha-1) + \frac{ie^{3i\pi\alpha/2}}{4}(-\alpha+1)\right)$$

$$= -\pi\,\frac{e^{i\pi\alpha/2}}{2}\left((\alpha-1) + e^{i\pi\alpha}(-\alpha+1)\right)$$

$$= -\pi\,\frac{e^{i\pi\alpha/2}}{2}(\alpha-1)\left(1-e^{i\pi\alpha}\right)$$

$$\Rightarrow \int_0^\infty \frac{x^\alpha}{(x^2+1)^2}\,dx = -\frac{\pi e^{i\pi\alpha/2}}{2}(\alpha-1)\,\frac{1-e^{i\pi\alpha}}{1-e^{i2\pi\alpha}}$$

$$= -\frac{\pi e^{i\pi\alpha/2}}{2}(\alpha-1)\,\frac{1}{1+e^{i\pi\alpha}}$$

$$= -\frac{\pi}{2}(\alpha-1)\frac{1}{e^{i\pi\alpha/2}+e^{-i\pi\alpha/2}} = -\frac{\pi}{2}(\alpha-1)\frac{1}{2\cos(\pi\alpha/2)}$$

$$= \frac{\pi(1-\alpha)}{4\cos(\pi\alpha/2)} \quad \square$$

Q5 Same contour and branch as before. $f(z) = \dfrac{z^{\alpha-1}}{z^2+z+1}$

① $\to 0$ by M-L estimates: $\dfrac{\rho^{\alpha-1}}{\rho^2-\rho-1}\cdot 2\pi\rho = 2\pi\dfrac{\rho^\alpha}{\rho^2-\rho-1} \to 0$ since $\alpha<2$

② $\to 0$ by M-L estimates: $\dfrac{\varepsilon^{\alpha-1}}{1-\varepsilon-\varepsilon^2}\cdot 2\pi\varepsilon = 2\pi\dfrac{\varepsilon^\alpha}{1-\varepsilon-\varepsilon^2} \to 0$ since $\alpha>0$, denominator $\to 1\neq 0$

③ $\to \displaystyle\int_0^\infty \frac{x^{\alpha-1}}{x^2+x+1}\,dx$ ④ $\to -\displaystyle\int_0^\infty \frac{x^{\alpha-1}e^{i2\pi(\alpha-1)}}{x^2+x+1}\,dx$

$\Rightarrow$ Integral $= \left(1-e^{i2\pi(\alpha-1)}\right)\displaystyle\int_0^\infty \frac{x^{\alpha-1}}{x^2+x+1}\,dx$

Residues: Integral $= 2\pi i\left(\mathrm{Res}\left(f;\underbrace{\tfrac{-1+\sqrt3 i}{2}}_{w_1}\right) + \mathrm{Res}\left(f;\underbrace{\tfrac{-1-\sqrt3 i}{2}}_{w_2}\right)\right)$ poles order 1

$$= 2\pi i\left(\frac{w_1^{\alpha-1}}{\ \cdots\ } + \frac{w_2^{\alpha-1}}{\ \cdots\ }\right)$$

(Note: $w_1 = e^{i2\pi/3}$, $w_2 = e^{i4\pi/3}$

$$= 2\pi i \left(\frac{w_1^{\alpha-1}}{w_1 - w_2} + \frac{w_2^{\alpha-1}}{w_2 - w_1} \right) \qquad \left(\text{Note: } w_1 = e^{i2\pi/3},\ w_2 = e^{i4\pi/3}\right)$$

$$= 2\pi i \left(\frac{e^{i2\pi(\alpha-1)/3} - e^{i4\pi(\alpha-1)/3}}{\sqrt{3}\, i} \right) = \frac{2\pi}{\sqrt{3}} \left(e^{i2\pi(\alpha-1)/3} - e^{i4\pi(\alpha-1)/3} \right)$$

$$\Rightarrow \int_0^\infty \frac{x^{\alpha-1}}{x^2+x+1}\,dx = \frac{2\pi}{\sqrt{3}} \frac{e^{i2\pi(\alpha-1)/3} - e^{i4\pi(\alpha-1)/3}}{1 - e^{i2\pi(\alpha-1)}} = \frac{2\pi}{\sqrt{3}} \frac{e^{i2\pi\alpha/3}e^{-i2\pi/3} + e^{i4\pi\alpha/3}e^{-i\pi/3}}{1 - e^{i2\pi\alpha}}$$

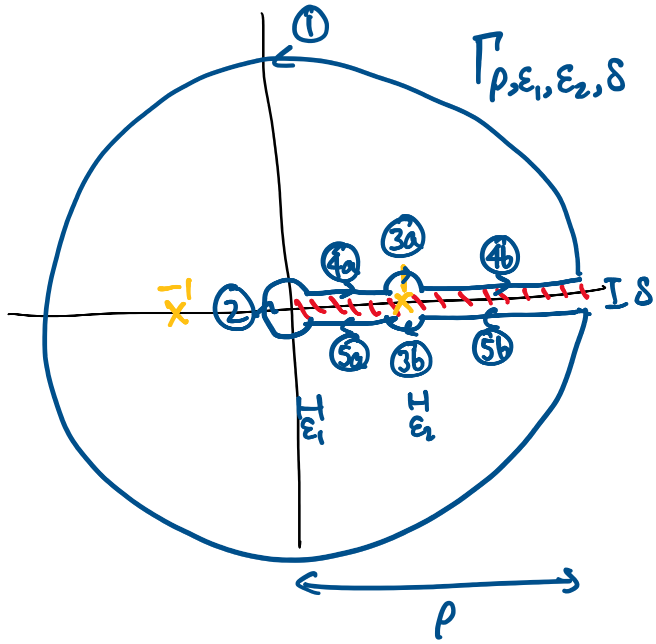
$$= \frac{2\pi}{\sqrt{3}} \frac{e^{i\pi\alpha}e^{-i\pi/2}\left(e^{-i\pi\alpha/3}e^{-i\pi/6} + e^{i\pi\alpha/3}e^{i\pi/6}\right)}{e^{i\pi\alpha}\left(e^{-i\pi\alpha} - e^{i\pi\alpha}\right)}$$

$$= \frac{2\pi}{\sqrt{3}} (\cancel{-i}) \frac{\cos\left(\frac{2\pi\alpha+\pi}{6}\right)}{\cancel{-i}\, \sin(\pi\alpha)}$$

$$= \frac{2\pi}{\sqrt{3}} \cos\left(\frac{2\pi\alpha+\pi}{6}\right) \csc(\pi\alpha) \qquad \square$$

Q6 New contour:

$$f(z) = \frac{z^\alpha}{z^2-1}$$



[Contour now has detours around singularity $z=1$ given by curves (3a) and (3b), splits original (3) and (4) into (4a), (4b) and (5a), (5b) respectively]

Taking $\delta \to 0$, $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, $\rho \to \infty$:

(1) $\to 0$ by M-L estimates: $\frac{\rho^\alpha}{\rho^2-1} \cdot 2\pi\rho \to 0$ since $\alpha+1 < 2$, denom. has deg 2

(2) $\to 0$ by M-L estimates: $\frac{\varepsilon^\alpha}{1-\varepsilon^2} \cdot 2\pi\varepsilon \to 0$ since $\alpha+1>0$, denom. $\to 1 \neq 0$

(3a) $\to -\pi i \underbrace{\lim_{w \to 1 (\text{Im}+)}}_{\text{approaching 1 from}} \text{Res}(f; w) = \left.\frac{z^\alpha}{z+1}\right|_{z \to 1 (\text{Im}+)} = -\frac{\pi i}{2}$

$$(3a) \rightarrow -\pi i \underbrace{\lim_{w\to 1(\mathrm{Im}+)}}_{\text{approaching 1 from the positive imaginary}} \mathrm{Res}(f; w) = \frac{\quad}{z+1}\Big|_{z\to 1(\mathrm{Im}+)} = \frac{\quad}{2}$$

$$(3b) \rightarrow -\pi i \underbrace{\lim_{w\to 1(\mathrm{Im}-)}}_{\text{approaching 1 from the negative imaginary}} \mathrm{Res}(f; w) = \frac{z^\alpha}{z+1}\Big|_{z\to 1(\mathrm{Im}-)} = -\frac{\pi i e^{i2\pi\alpha}}{2}$$

$$(4a) + (4b) \rightarrow \int_0^1 \frac{x^\alpha}{x^2-1}dx + \int_1^\infty \frac{x^\alpha}{x^2-1}dx = \text{p.v.} \int_0^\infty \frac{x^\alpha}{x^2-1}dx$$

$$(5a) + (5b) \rightarrow -\int_0^1 \frac{x^\alpha e^{i2\pi\alpha}}{x^2-1}dx - \int_1^\infty \frac{x^\alpha e^{i2\pi\alpha}}{x^2-1} = -e^{i2\pi\alpha}\, \text{p.v.} \int_0^\infty \frac{x^\alpha}{x^2-1}dx$$

$$\Rightarrow \text{Integral} = (1-e^{i2\pi\alpha})\, \text{p.v.} \int_0^\infty \frac{x^\alpha}{x^2-1}dx - \frac{\pi i}{2}(1+e^{i2\pi\alpha})$$

Residues:

$$\text{Integral} = 2\pi i\, \mathrm{Res}(f, -1) = 2\pi i \frac{e^{i\pi\alpha}}{-1-1} = -\pi i\, e^{i\pi\alpha}$$

$$\Rightarrow \text{p.v.} \int_0^\infty \frac{x^\alpha}{x^2-1}dx = \frac{-\pi i\, e^{i\pi\alpha} + \frac{\pi i}{2}(1+e^{i2\pi\alpha})}{1-e^{i2\pi\alpha}}$$

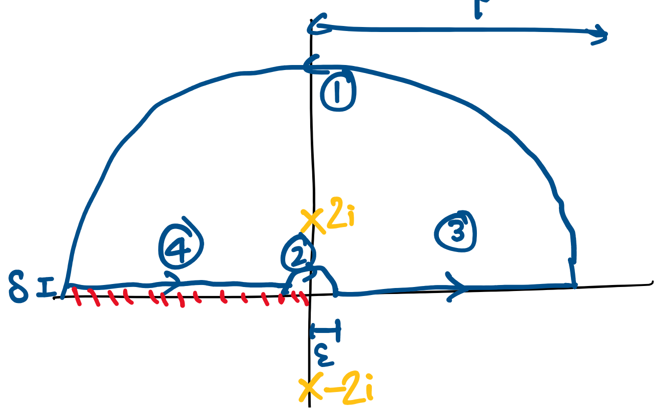
$$= \frac{-\pi i + \frac{\pi i}{2}(e^{-i\pi\alpha} + e^{i\pi\alpha})}{e^{-i\pi\alpha} - e^{i\pi\alpha}}$$

$$= \pi i \frac{-1 + \cos(\pi\alpha)}{-2i \sin(\pi\alpha)}$$

$$= \frac{\pi(1-\cos(\pi\alpha))}{2\sin(\pi\alpha)} \qquad \square$$

Q8

Contour:



$$f(z) = \frac{\mathrm{Log}\, z}{z^2+4}$$

Taking $\delta \to 0$, $\varepsilon \to 0$, $\rho \to \infty$:

① $\to 0$ by M-L estimates: $\dfrac{\ln \rho}{\rho^2 - 4} \cdot \pi\rho \to 0$

② $\to 0$ by M-L estimates: $\dfrac{\ln \varepsilon}{4 - \varepsilon^2} \cdot \pi\varepsilon \to 0$

③ $\to \displaystyle\int_0^\infty \frac{\operatorname{Log} x}{x^2+4}\,dx$ ④ $\to \displaystyle\int_{-\infty}^0 \frac{\operatorname{Log} x}{x^2+4}\,dx$

$$= \int_{-\infty}^0 \frac{\operatorname{Log}|x| + \pi i}{x^2+4}$$

$$\Rightarrow \text{Integral} = \underbrace{\text{p.v.} \int_{-\infty}^\infty \frac{\operatorname{Log}|x|}{x^2+4}\,dx}_{\text{Real}} + \underbrace{\int_0^\infty \frac{\pi i}{x^2+4}\,dx}_{\text{Imaginary}}$$

Residues: $\text{Integral} = 2\pi i \operatorname{Res}(f; \underset{\substack{\uparrow \\ \text{simple pole}}}{2i}) = 2\pi i \dfrac{\operatorname{Log}(2i)}{4i} = \dfrac{\pi(\ln 2 + \frac{\pi}{2} i)}{2}$

$$= \underbrace{\frac{\pi \ln 2}{2}}_{\text{Real}} + \underbrace{\frac{\pi^2}{4}}_{\text{Imaginary}} i$$

$\Rightarrow$ Equating the real parts,

$$\text{p.v.} \int_{-\infty}^\infty \frac{\operatorname{Log}|x|}{x^2+4}\,dx = \frac{\pi \ln 2}{2} \qquad \square$$

Remark: One can also observe the elementary integral

$$\int_0^\infty \frac{\pi}{x^2+4}\,dx = \frac{\pi}{2} \arctan(x/2)\Big|_0^\infty = \frac{\pi}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{4}$$

Q11 Use the contour as in Q 8.

$$f(z) = \frac{\operatorname{Log} z}{(z^2+1)^2}$$

Take $\delta \to 0$, $\varepsilon \to 0$, $\rho \to \infty$:

① $\to 0$ by M-L estimates: $\dfrac{\operatorname{Log} \rho}{(\rho^2 - 1)^2} \cdot \pi\rho \to 0$

① $\to 0$ by M-L estimates: $\dfrac{\operatorname{Log} \rho}{(\rho^2-1)^2} \cdot \pi\rho \to 0$

②' $\to 0$ by M-L estimates: $\dfrac{\operatorname{Log} \varepsilon}{(1-\varepsilon^2)^2} \cdot \pi\varepsilon \to 0$

③ $\to \displaystyle\int_0^\infty \frac{\operatorname{Log} x}{(x^2+1)^2}\,dx$ ④ $\to \displaystyle\int_0^\infty \frac{\operatorname{Log} x + \pi i}{(x^2+1)^2}\,dx$

$$= \int_0^\infty \frac{\operatorname{Log} x}{(x^2+1)^2}\,dx + i\int_0^\infty \frac{\pi}{(x^2+1)^2}\,dx$$

$$\Rightarrow \text{Integral} = \underbrace{2\int_0^\infty \frac{\operatorname{Log} x}{(x^2+1)^2}\,dx}_{\text{Real}} + \underbrace{i\int_0^\infty \frac{\pi}{(x^2+1)^2}\,dx}_{\text{Imaginary}}$$

Residues: Integral $= 2\pi i \operatorname{Res}(f; i)$

Pole order 2

$$\operatorname{Res}(f; i) = \frac{d}{dz}\bigg|_{z=i} \frac{\log z}{(z+i)^2} = \frac{1}{z(z+i)^2} - \frac{2\log z}{(z+i)^3}\bigg|_{z=i}$$

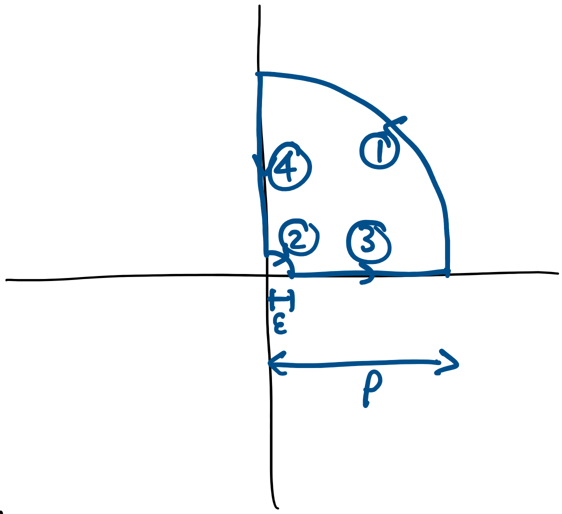
$$= \frac{1}{i(-4)} - \frac{\pi i}{-8i}$$

$$= 2\pi i\left(\frac{1}{-4i} + \frac{\pi}{8}\right) = \underbrace{-\frac{\pi}{2}}_{\text{Real}} + \underbrace{\frac{\pi^2}{4}\,i}_{\text{Imaginary}}$$

Equating real parts,

$$\int_0^\infty \frac{\operatorname{Log} x}{(x^2+1)^2}\,dx = -\frac{\pi}{4} \qquad \square$$

Q12 Contour:



$f(z) = e^{-z} z^{\alpha-1}$

Take $\varepsilon \to 0$, $\rho \to \infty$:

$$① = \int_0^{\pi/2} e^{-\rho e^{it}} \cdot \rho^{\alpha-1} e^{i(\alpha-1)t} \cdot \rho i e^{it}\, dt = \rho^{\alpha} i \int_0^{\pi/2} e^{-\rho\cos t} e^{-i\rho \sin t} e^{i\alpha t}\, dt$$

$$|\cdot| \le \rho^{\alpha} \int_0^{\pi/2} e^{-\rho\cos t}\, dt \le \rho^{\alpha} \int_0^{\pi/2} e^{-\rho\left(1-\frac{2t}{\pi}\right)}\, dt$$

$$= \frac{\rho^{\alpha}}{2\rho/\pi} e^{-\rho\left(1-\frac{2t}{\pi}\right)} \Big|_0^{\pi/2} = \rho^{\alpha-1} \frac{\pi}{2}\left(1-e^{-\rho}\right) \longrightarrow 0$$

since $\alpha - 1 < 0$

(This proof similar to proof of Jordan's Lemma, see pp. 332–334)

② $\to 0$ by M-L estimates: $\varepsilon^{\alpha-1} \cdot \frac{\pi}{2}\varepsilon = \frac{\pi}{2}\varepsilon^{\alpha} \longrightarrow 0$ since $\alpha > 0$

③ $\to \int_0^{\infty} e^{-x} x^{\alpha-1}\, dx$

$= \Gamma(\alpha)$

④ $\to -i\int_0^{\infty} e^{-ix} x^{\alpha-1} e^{i\pi(\alpha-1)/2}\, dx$

$= -i e^{i\pi(\alpha-1)/2} \int_0^{\infty} (\cos x - i \sin x)\, x^{\alpha-1}\, dx$

$= -e^{i\pi\alpha/2} \int_0^{\infty} (\cos x - i\sin x)\, x^{\alpha-1}\, dx$

Since there are no poles inside the contour,

$$\Gamma(\alpha) = e^{i\pi\alpha/2} \int_0^{\infty} (\cos x - i \sin x)\, x^{\alpha-1}\, dx$$

$$\Rightarrow \underbrace{\int_0^{\infty} x^{\alpha-1}\cos x\, dx}_{\text{Real}} - i \underbrace{\int_0^{\infty} x^{\alpha-1} \sin x\, dx}_{\text{Imaginary}} = \Gamma(\alpha) e^{-i\pi\alpha/2}$$

$$= \underbrace{\Gamma(\alpha)\cos\left(\frac{\pi\alpha}{2}\right)}_{\text{Real}} - \underbrace{\Gamma(\alpha)\sin\left(\frac{\pi\alpha}{2}\right) i}_{\text{Imaginary}}$$

Equate imaginary parts:

$$\int_0^{\infty} x^{\alpha-1} \sin x\, dx = \Gamma(\alpha)\sin\left(\frac{\pi\alpha}{2}\right) \quad \square$$