Homework 3 Solutions Mark out of 25

3.3 Q4 $\text{Log}\, e^z = \ln|e^z| + i\,\text{Arg}\, e^z$ 2pts

Set $z = x + iy$:

$\ln|e^z| = \ln|e^{x+iy}| = \ln e^x = x$

By definition, $\text{Arg}\, e^z \in (-\pi, \pi]$ and

$y = \text{Arg}\, e^z + 2\pi k$ for some $k \in \mathbb{Z}$

$e^{x+iy} = e^{x+i\,\text{Arg}\, e^z} = e^x e^{i\,\text{Arg}\, e^z} = e^x e^{iy} = e^z$

(since $e^{i2\pi k} = 1$)

Thus, $\text{Log}\, e^z = z \iff y = \text{Im}\, z \in (-\pi, \pi]$ $\square$

3.3 Q5(b) $\text{Log}(z^2 - 1) = \frac{i\pi}{2} \Rightarrow z^2 - 1 = e^{i\pi/2} = i$ 2 pts

$z^2 = i + 1 = \sqrt{2}\, e^{i\pi/4} = \sqrt{2}\, e^{i9\pi/4}$

$\Rightarrow$ **$z = 2^{1/4} e^{i\pi/8}$ or $z = 2^{1/4} e^{i9\pi/8}$**

3.3 Q11 This branch of log should be analytic at 3pts

$z^2 + 2z + 3 \big|_{z=-1} = 1 - 2 + 3 = 2$

We can use the branch **$\text{Log}(z^2 + 2z + 3)$** since

$z^2 + 2z + 3 \big|_{z=-1} \notin (-\infty, 0]$ and Log is analytic at 2

$$\frac{d}{dz}(z^2 + 2z + 3)\Big|_{z=-1} = \frac{2z+2}{z^2+2z+3}\Big|_{z=-1} = 0$$

3.3 Q15 For all z in the upper half plane, 2 pts

$\ln|z| \in (0, \infty)$ and $\text{Arg}(z) \in (0, \pi)$. Then $z \mapsto \text{Log}\, z$

maps the upper half plane to the strip $\{u + iv : u \in \mathbb{R}, v \in (0, \pi)\}$

maps the upper half plane to the strip $\{u+iv : u\in\mathbb{R}, v\in(0,\pi)\}$

Then $z \to \frac{\text{Log } z}{\pi}$ maps the upper half plane to

$\{u+iv:\ u\in\mathbb{R},\ v\in(0,1)\} = \mathcal{H}$

3.5 Q1(d) $(1+i)^{1-i} = e^{(1-i)\log(1+i)}$ **2pts**

$$= e^{(1-i)(\ln\sqrt{2}+i(\pi/4+2\pi k))} \quad (k\in\mathbb{Z})$$

$$= e^{\ln\sqrt{2}+(\pi/4+2\pi k)}\, e^{i(\pi/4+2\pi k-\ln\sqrt{2})}$$

$$= e^{\ln\sqrt{2}+(\pi/4+2\pi k)}\, e^{i(\pi/4-\ln\sqrt{2})} = \mathbf{(1+i)e^{(\pi/4+2\pi k)}e^{-i\ln\sqrt{2}}}$$

3.5 Q3 (c) $(1+i)^{1+i} = e^{(1+i)\text{Log}(1+i)}$ **2pts**

$$= e^{(1+i)(\ln\sqrt{2}+i\pi/4)}$$

$$= e^{\ln\sqrt{2}-\pi/4}\, e^{i(\ln\sqrt{2}+\pi/4)} = \mathbf{(1+i)\,e^{-\pi/4}e^{i\ln\sqrt{2}}}$$

3.5 Q8 Using **Formula (5) [see p. 134]:** **3 pts**

$$z = \sin^{-1}(2) = -i\log\left(2i + (-3)^{1/2}\right)$$

$$= -i\log\left(i(2\pm\sqrt{3})\right)$$

gives two sets of solutions for $k\in\mathbb{Z}$:

$$-i\left(\ln(2+\sqrt{3})+i(\pi/2+2\pi k)\right) \qquad -i\left(\ln(2-\sqrt{3})+i(\pi/2+2\pi k)\right)$$

$$= (\pi/2+2\pi k) - i\ln(2+\sqrt{3}) \qquad = (\pi/2+2\pi k) - i\ln(2-\sqrt{3})$$

$$= (\pi/2+2\pi k) + i\ln(2+\sqrt{3})$$

[Note: $(2+\sqrt{3})(2-\sqrt{3}) = 1$ so $-\ln(2+\sqrt{3}) = \ln(2-\sqrt{3})$]

Therefore, the solutions are

$$\mathbf{(\pi/2+2\pi k) - i\ln(2\pm\sqrt{3})}$$

or

$$\mathbf{(\pi/2+2\pi k) \pm i\ln(2+\sqrt{3})} \qquad (k\in\mathbb{Z})$$

$(\pi/2 + 2\pi k) \pm i \ln(2+\sqrt{3})$

3.5 Q9 If $z = \cos w$, then $z = \dfrac{e^{iw} + e^{-iw}}{2}$ **3 pts**

Then $2z = e^{iw} + e^{-iw} \Rightarrow 2z\, e^{iw} = e^{2iw} + 1$

$\Rightarrow e^{2iw} - 2z\, e^{iw} + 1 = 0$

Using the quadratic formula,

$$e^{iw} = \frac{2z + (4z^2 - 4)^{1/2}}{2} \quad \text{(includes both values of } (\ldots)^{1/2}\text{)}$$

$$= z + (z^2 - 1)^{1/2}$$

Then $iw = \log\left(z + (z^2-1)^{1/2}\right)$

$w = -i \log\left(z + (z^2-1)^{1/2}\right) = \cos^{-1} z$

Hence formula (9).

Also, from (9),

$$\frac{d}{dz}\cos^{-1} z = -i \cdot \frac{1 + \frac{1}{2}\cdot 2z \cdot (z^2-1)^{-1/2}}{z + (z^2-1)^{1/2}}$$

$$= -i\, \frac{1 + z(z^2-1)^{-1/2}}{z + (z^2-1)^{1/2}}$$

$$= -i\, \frac{\cancel{(z^2-1)^{1/2} + z}}{(z^2-1)^{1/2}\cancel{\left(z + (z^2-1)^{1/2}\right)}} = \frac{-i}{\underbrace{(z^2-1)^{1/2}}_{i(1-z^2)^{1/2}}} = \frac{-1}{(1-z^2)^{1/2}}$$

Hence formula (11).

3.5 Q10 $\cos z = 2i$: Using formula (9), **3 pts**

$$z = -i \log\left(2i + (-5)^{-1/2}\right)$$

$$= -i \log\left(i(2 \pm \sqrt{5})\right)$$

has two sets of solutions for $k \in \mathbb{Z}$:

$$z = -i\left(\ln(2+\sqrt{5}) + i\left(\tfrac{\pi}{2} + 2\pi k\right)\right) \qquad z = -i\left(\ln(\sqrt{5}-2) + i\left(-\tfrac{\pi}{2} + 2\pi k\right)\right)$$

$$= \left(\frac{\pi}{2} + 2\pi k\right) - i\ln(\sqrt{5}+2) \qquad = \left(-\frac{\pi}{2} + 2\pi k\right) - i\ln(\sqrt{5}-2)$$

$$= \left(-\frac{\pi}{2} + 2\pi k\right) + i\ln(\sqrt{5}+2)$$

$$\left(\text{since } (\sqrt{5}+2)(\sqrt{5}-2) = 1\right)$$

Then the solutions are

$$\left(\frac{\pi}{2}+2\pi k\right) - i\ln(\sqrt{5}+2), \quad \left(-\frac{\pi}{2}+2\pi k\right) + i\ln(\sqrt{5}+2)$$

OR

$$\left(\frac{\pi}{2}+2\pi k\right) - i\ln(\sqrt{5}+2), \quad \left(-\frac{\pi}{2}+2\pi k\right) - i\ln(\sqrt{5}-2)$$

3.5 Q15b $(4+z^2)^{1/2} = e^{\frac{1}{2}\log(4+z^2)} = e^{\frac{1}{2}\left(\log z^2 + \log\left(\frac{4}{z^2}+1\right)\right)}$ **3 pts**

$$= z e^{\frac{1}{2}\log\left(\frac{4}{z^2}+1\right)}$$

On the slit from $-2i$ to $2i$, $z^2 \in [-4, 0]$

$$\frac{4}{z^2} \in (-\infty, -1]$$

$$\frac{4}{z^2}+1 \in (-\infty, 0]$$

So it suffices if the branch of log is analytic except on $(-\infty, 0]$, so use the principal Log

Thus, the branch $(4+z^2)^{1/2} = z e^{\frac{1}{2}\mathrm{Log}\left(\frac{4}{z^2}+1\right)}$ is analytic on $\mathbb{C} \setminus [-2i, 2i]$