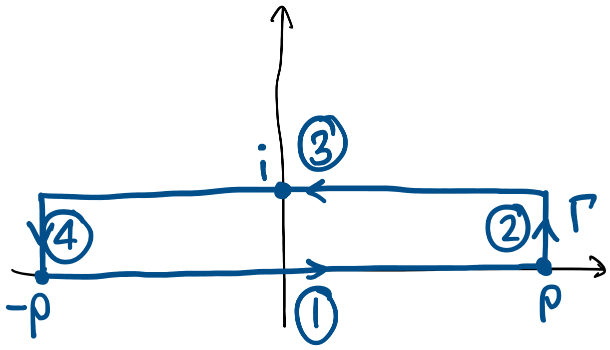
Homework 2 Solutions

Grade out of 30

6.3 Q9

4 pts

Contour to integrate over:



$$\int_\Gamma \frac{e^{2z}}{\cosh(\pi z)}\,dz = \underbrace{\int_{-\rho}^{\rho} \frac{e^{2x}}{\cosh(\pi x)}\,dx}_{①} + \underbrace{\int_0^1 \frac{e^{2(\rho+yi)}}{\cosh(\pi(\rho+yi))}\,dy}_{②}$$

$$- \underbrace{\int_{-\rho}^{\rho} \frac{e^{2(x+i)}}{\cosh(\pi(x+i))}\,dx}_{③} - \underbrace{\int_0^1 \frac{e^{2(-\rho+yi)}}{\cosh(\pi(-\rho+yi))}\,dy}_{④}$$

By residue theorem, $\frac{e^{2z}}{\cosh(\pi z)}$ has singularities where

$$\cosh(\pi z) = 0 \Leftrightarrow e^{\pi z} = -e^{-\pi z} \Leftrightarrow e^{2\pi z} = -1 \Leftrightarrow z = \frac{i(2n+1)}{2}$$

Only the singularity $z = \frac{i}{2}$ is inside the contour

$$\text{Residue} = \lim_{z\to i/2} \frac{e^{2z}}{\pi \sinh(\pi z)} = \frac{2e^{i}}{\pi\left(e^{i\pi/2} - e^{-i\pi/2}\right)} = \frac{2e^{i}}{\pi(2i)}$$

$\Rightarrow$ Total integral $= 2e^{i} = 2(\cos 1 + i \sin 1)$

Take $\rho \to \infty$.

① = Target integral $:= I$ as $\rho\to\infty$

②,④: Integrand $= \dfrac{2e^{2(\pm\rho+yi)}}{e^{\pi(\pm\rho+yi)} + e^{-\pi(\pm\rho+yi)}} = \dfrac{2}{e^{(\pi-2)(\pm\rho+yi)} + e^{(-\pi-2)(\pm\rho+yi)}}$

Since $\pi-2>0$, $-\pi-2<0$, ②,④ $\longrightarrow 0$ as $\rho\to\infty$.

Since $\pi - 2 > 0$, $-\pi - 2 < 0$, (2), (4) $\longrightarrow 0$ as $\rho \to \infty$ (denominator gets large)

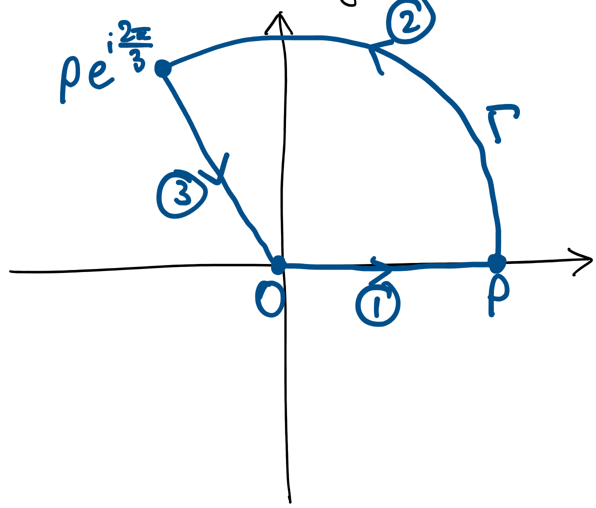
$$③ = \int_{-\rho}^{\rho} \frac{2e^{2x}e^{2i}}{e^{\pi x}\underbrace{e^{i\pi}}_{-1} + e^{-\pi x}\underbrace{e^{-i\pi}}_{-1}}\,dx = -e^{2i}\int_{-\rho}^{\rho}\frac{e^{2x}}{\cosh(\pi x)}\,dx \xrightarrow{\rho\to\infty} -e^{2i}\, I$$

$$\text{Total integral} \to \underbrace{(1+e^{2i})}_{=2e^{i}} I \quad \text{as } \rho \to \infty.$$

$$\Rightarrow I = \frac{2e^{i}}{1+e^{2i}} = \frac{2}{e^{i}+e^{-i}} = \frac{1}{\cos 1} = \sec 1 \quad \square$$

6.3 Q11 — 4 pts

Contour to integrate over:



$$\text{Integral } \int_\Gamma \frac{1}{z^3+1}\,dz = \underbrace{\int_0^\rho \frac{1}{r^3+1}\,dr}_{①} + \underbrace{\int_{arc} \frac{1}{z^3+1}\,dz}_{②} - \underbrace{\int_0^\rho \frac{e^{i\frac{2\pi}{3}}}{r^3+1}\,dr}_{③}$$

① = Target integral $:= I$ ③ $= e^{i\frac{2\pi}{3}} I$

$$② \le \frac{1}{\rho^3-1}\cdot\frac{2\pi}{3}\rho \xrightarrow{\rho\to\infty} 0 \quad \text{(M-L estimate)}$$

By taking $\rho \to \infty$, $\int_\Gamma \frac{1}{z^3+1}\,dz \to \left(1 - e^{i\frac{2\pi}{3}}\right) I$

By residue theorem, $\int_\Gamma \frac{1}{z^3+1} = 2\pi i \ \mathrm{Res}\left(\frac{1}{z^3+1}, e^{\frac{i\pi}{3}}\right)$

Since $e^{\frac{i\pi}{3}}$ is the ... in Γ

Since $e^{\frac{i\pi}{3}}$ is the only singularity in Γ,

$$= 2\pi i \lim_{z\to e^{i\pi/3}} \frac{1}{3z^2} = \frac{2\pi i}{3e^{i2\pi/3}}$$

$$\Rightarrow I = \frac{2\pi i}{3e^{i2\pi/3}(1-e^{i2\pi/3})} = \frac{2\pi i}{3(e^{i2\pi/3} - \underbrace{e^{i4\pi/3}}_{=e^{-i2\pi/3}})} = \frac{\pi}{3\sin(2\pi/3)} = \frac{\pi 2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}\pi \quad \square$$

6.3 Q17(a) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{(k+a)^2} = \pi^2 \csc^2 \pi a$ — 3 pts

Using the formula in **Q14**,

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k+a)^2} = -\sum_{\substack{w \text{ is a} \\ \text{pole of} \\ \frac{1}{(z+a)^2}}} \operatorname{Res}\left(\pi \cdot \frac{1}{(z+a)^2} \cdot \cot(\pi z),\ w\right)$$

Since $z=-a$ is the only pole, (of order 2) the sum is equal to

$$= -\operatorname{Res}\left(\pi \cdot \frac{1}{(z+a)^2} \cdot \cot(\pi z), -a\right)$$

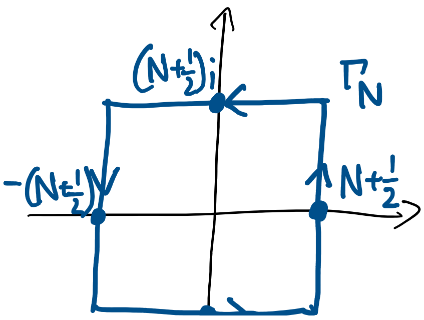
$$= -\frac{d}{dz}\left(\pi \cot(\pi z)\right)\Big|_{z=-a}$$

$$= \pi^2 \csc^2(-\pi a) = \pi^2 \csc^2(\pi a) \quad \square$$

6.3 Q18 $\displaystyle\sum_{k=-\infty}^{\infty} (-1)^k f(k) = -\sum_{\substack{w \text{ pole} \\ \text{of } f(z)}} \operatorname{Res}(\pi f(z)\csc(\pi z), w)$ — 4 pts

For $g(z) := \pi f(z) \csc(\pi z)$, $k \in \mathbb{Z}$,

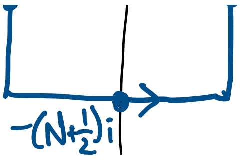
$$\operatorname{Res}(g; k) = \lim_{z\to k} \frac{\pi f(z)}{\pi \cos(\pi z)} = \frac{f(k)}{\cos(k\pi)} = (-1)^k f(k)$$



For $z \in \Gamma_N$,

For $|\operatorname{Im} z| < 1/2$, we have $|\operatorname{Re} z| = N + \frac{1}{2}$

Then $|\pi \csc(\pi z)| = \left|\frac{\pi}{\cdots}\right| \le \pi$



(Note: $\sin(x+iy)$
$= \sin x \cosh y + i \cos x \sinh y$)

Then $\left|\pi \csc(\pi z)\right| = \left|\frac{\pi}{(-1)^N \cosh(\pi \operatorname{Im} z)}\right| \leq \pi$

For $|\operatorname{Im} z| \geq 1/2$,

$$\left|\pi \csc(\pi z)\right| = \left|\frac{2\pi i}{e^{i\pi z} - e^{-i\pi z}}\right| \leq \frac{2\pi}{e^{\pi/2} - e^{-\pi/2}}$$

$\therefore \left|\pi \csc(\pi z)\right|$ is bounded above by a constant $M = \pi$

For f a rational function $\frac{P(z)}{Q(z)}$ with $\deg Q - \deg P \geq 2$,

$$\left|\pi f(z) \csc(\pi z)\right| \leq M f(z)$$

$$\int_{\Gamma_N} \pi f(z) \csc(\pi z)\, dz \leq 8(N+\tfrac{1}{2})\, M \frac{k}{N^2} \text{ for } N \text{ sufficiently large}$$

$$\xrightarrow[N\to\infty]{} 0.$$

By the residue theorem,

$$0 = 2\pi i\left(\sum_{k=-\infty}^{\infty} \operatorname{Res}(\pi f(z) \csc(\pi z); k) + \sum_{\substack{w \text{ poles} \\ \text{of } f}}^{\infty} \operatorname{Res}(\pi f(z) \csc(\pi z); w)\right)$$

$$\Rightarrow -\sum_{\substack{w \text{ poles} \\ \text{of } f}}^{\infty} \operatorname{Res}(\pi f(z) \csc(\pi z); w) = \sum_{k=-\infty}^{\infty} (-1)^k f(k) \qquad \square$$

6.3 Q19

3 pts

Considering the square contour Γ_N as in **Q18**, the residue theorem says that for the function $g(z) = \frac{\pi}{z^2}\csc(\pi z)$, g has a pole of order 3 at $z=0$ and simple poles on all other $k \in \mathbb{Z}$ with residue $\frac{(-1)^k}{k^2}$. Then:

$$\int_{\Gamma_N} g(z)\, dz = 2\pi i\left(\sum_{\substack{|k|\leq N \\ k\neq 0}} \frac{(-1)^k}{k^2} + \operatorname{Res}(g(z); 0)\right)$$

By taking $N\to\infty$, LHS $\to 0$. Then:

$$\sum_{\substack{k\in\mathbb{Z} \\ k\neq 0}} \frac{(-1)^k}{k^2} = -\operatorname{Res}(g(z); 0)$$

Since $g(z)$ has a pole of order 3 at $z=0$, Since $g(z)$ has a pole of order 3 at $z=0$,

$$z^3 g(z) = \pi z \csc(\pi z)$$

$$= \frac{\pi z}{\pi z - \frac{\pi^3 z^3}{3!} + \cdots} = \frac{1}{1 - \frac{\pi^2 z^2}{3!} + \cdots} = 1 + \left(\frac{\pi^2 z^2}{3!} + \cdots\right) + \left(\frac{\pi^2 z^2}{3!} + \cdots\right)^2 + \cdots$$

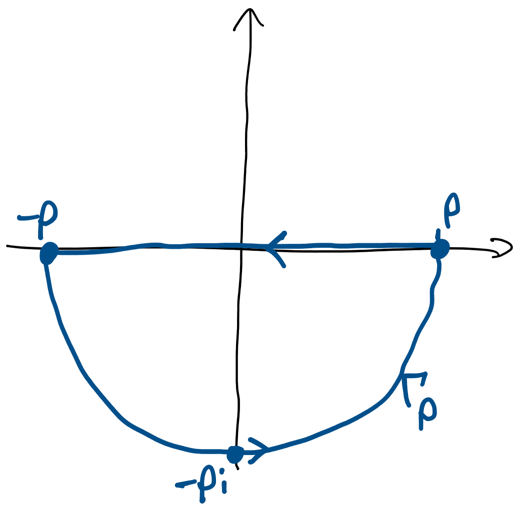
The z^2 term in the expansion ($= z^{-1}$ term in $g(z)$) has coefficient $\frac{\pi^2}{6}$, so this is $\mathrm{Res}(g(z);0)$.

Then $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = \frac{1}{2} \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} \frac{(-1)^k}{k^2} = \frac{1}{2}\left(-\frac{\pi^2}{6}\right) = -\frac{\pi^2}{12}$ □

(Note: The formula in Q18 cannot be directly applied with $f(z) = \frac{1}{z^2}$ because f has a pole at the integer $z=0$.)

6.4 Q6 Choose $f(z) = \dfrac{e^{-2iz}}{z^2+4}$ **4 pts**

Contour to integrate over:



$$\int_{\Gamma_\rho} f(z)\,dz = \underbrace{-\int_{-\rho}^{\rho} \frac{e^{-2ix}}{x^2+4}\,dx}_{1} + \underbrace{\int_{\text{arc}} \frac{e^{-2iz}}{z^2+4}\,dz}_{2}$$

Taking $\rho\to\infty$, (2) $\to 0$ by **Jordan's Lemma**.

(1) $\to$ p.v. $-\displaystyle\int_{-\infty}^{\infty} \frac{e^{-2ix}}{x^2+4}\,dx$

By residue theorem, we have a simple pole of f at $z=\pm 2i$

Since $z=-2i$ lies inside Γ_ρ, integral $= -2\pi i\, \mathrm{Res}(f;-2i)$

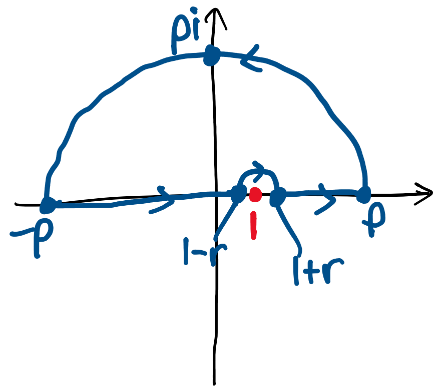
$$= -2\pi i \lim_{z\to -2i} \frac{e^{-2iz}}{2z} = -2\pi i \frac{e^{-4}}{-4i} = \frac{\pi e^{-4}}{2}$$

$$= -2\pi i \lim_{z \to -2i} \frac{e^{-2iz}}{2z} = -2\pi i \frac{e^{-4}}{-4i} = \frac{\pi e^{-4}}{2}$$

6.5 Q6

4 pts

Consider $\int_{\Gamma_{\rho,r}} \frac{e^{iz}}{(z^2+4)(z-1)} dz$ where $\Gamma_{\rho,r}$ is the contour



This function has one simple pole at $z = 2i$ inside $\Gamma_{\rho,r}$. By Residue Theorem, $\int_{\Gamma_{\rho,r}} \frac{e^{iz}}{(z^2+4)(z-1)} dz = 2\pi i \operatorname{Res}\left(\frac{e^{iz}}{(z^2+4)(z-1)} ; 2i\right)$

$$= 2\pi i \frac{e^{-2}}{4i \cdot (2i-1)} = \frac{\pi e^{-2}}{4i-2}$$

By separating the integral into parts, this is

$$\underbrace{\int_{-\rho}^{1-r} \frac{e^{ix}}{(x^2+4)(x-1)} dx}_{1} + \underbrace{\int_{\text{Small arc}} \frac{e^{iz}}{(z^2+4)(z-1)} dz}_{2} + \underbrace{\int_{1+r}^{\rho} \frac{e^{ix}}{(x^2+4)(x-1)} dx}_{3} + \underbrace{\int_{\text{large arc}} \frac{e^{iz}}{(z^2+4)(z-1)} dz}_{4}$$

$$= \frac{\pi e^{-2}}{4i-2}$$

Taking $\rho \to \infty$, $r \to 0$,

Target integral $= 1 + 3 = \text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+4)(x-1)} dx$

$2 \to -i\pi \operatorname{Res}\left(\frac{e^{iz}}{(z^2+4)(z-1)} ; 1\right) = -i\pi \frac{e^{i}}{5}$ as $r \to 0$ by **Lemma 4**

$4 \to 0$ as $\rho \to \infty$ by **Jordan's Lemma**

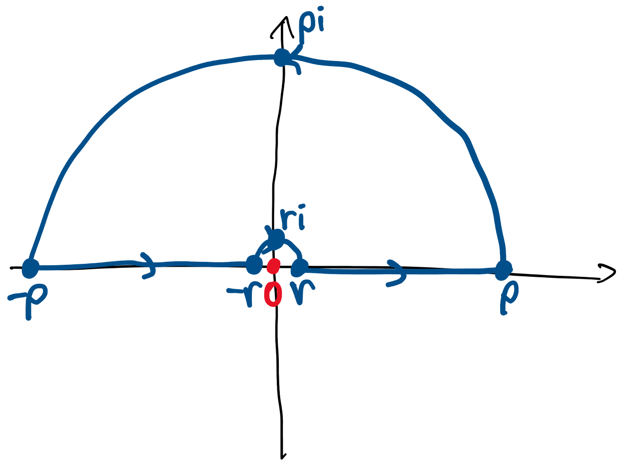
$$\Rightarrow \text{Integral} = \frac{\pi e^{-2}}{4i-2} + i\pi\frac{e^{i}}{5} = \frac{\pi e^{-2}(-2i-1)}{10} + \frac{\pi(-\sin(1)+i\cos(1))}{5}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{(x^2+4)(x-1)}\,dx = \operatorname{Im}\left[\int_{-\infty}^{\infty}\frac{e^{ix}}{(x^2+4)(x-1)}\,dx\right] = -\frac{\pi e^{-2}}{5} + \frac{\pi\cos(1)}{5}$$

$$= \frac{\pi}{5}\left(\cos(1) - e^{-2}\right) \quad \square$$

6.5 Q12 Consider the integral $\int_{\Gamma_{\rho,r}} \frac{e^{iaz}}{z(z^2+b^2)}\,dz$ over the contour $\Gamma_{\rho,r}$:

4 pts



Function has singularity $z=bi$ inside of contour, simple pole.

$$\text{Residue} = \operatorname{Res}\left(\frac{e^{iaz}}{z(z^2+b^2)}; bi\right) = \frac{e^{-ab}}{bi(2bi)} = -\frac{e^{-ab}}{2b^2}$$

$$\Rightarrow \int_{\Gamma_{\rho,r}} \frac{e^{iaz}}{z(z^2+b^2)}\,dz = -\frac{\pi e^{-ab} i}{b^2}$$

Split into four parts:

$$= \underbrace{\int_{-\rho}^{-r}\frac{e^{iax}}{x(x^2+b^2)}\,dx}_{1} + \underbrace{\int_{\text{small arc}}\frac{e^{iaz}}{z(z^2+b^2)}\,dz}_{2} + \underbrace{\int_{r}^{\rho}\frac{e^{iax}}{x(x^2+b^2)}\,dx}_{3} + \underbrace{\int_{\text{large arc}}\frac{e^{iaz}}{z(z^2+b^2)}\,dz}_{4}$$

2 $\xrightarrow{r\to 0} -\pi i \operatorname{Res}\left(\frac{e^{iaz}}{z(z^2+b^2)}; 0\right) = -\frac{\pi i}{b^2}$ by **Lemma 4**

4 $\xrightarrow{\rho\to\infty} 0$ by **Jordan's Lemma**

1 $\xrightarrow[r\to 0]{\rho\to\infty,} \int_{-\infty}^{0}\frac{e^{iax}}{x(x^2+b^2)}\,dx = \int_{0}^{\infty}\frac{e^{iax}}{x(x^2+b^2)}\,dx$

$$\int_{-\infty} \frac{\quad}{x(x^2+b^2)} \, dx \qquad \int_0 \frac{\quad}{x(x^2+b^2)} \, dx$$

③ $$\xrightarrow[r\to 0]{\rho\to\infty,} \int_0^\infty \frac{e^{iax}}{x(x^2+b^2)} \, dx$$

$$\Rightarrow 2\int_0^\infty \frac{e^{iax}}{x(x^2+b^2)} \, dx - \frac{\pi i}{b^2} = -\frac{\pi e^{-ab} i}{b^2}$$

$$\Rightarrow \int_0^\infty \frac{e^{iax}}{x(x^2+b^2)} \, dx = \frac{\pi i}{2b^2}\left(1 - e^{-ab}\right)$$

$$\Rightarrow \int_0^\infty \frac{\sin(ax)}{x(x^2+b^2)} \, dx = \operatorname{Im}(\cdots) = \frac{\pi}{2b^2}\left(1 - e^{-ab}\right) \quad \square$$