MATH 301 HW 10

PROBLEM 1 CONSIDER $y' + y = f(t)$ WITH $f(t+1) = f(t)$ WITH INITIAL VALUE $y(0) = y_0$.

(i) SHOW HOW TO FIND y_0 SO THAT THE SOLUTION IS PERIODIC WITH $y(t+1) = y(t)$.

(ii) FOR THIS VALUE OF y_0 CALCULATE THE PERIODIC SOLUTION WHEN
$f(t) = 1$ FOR $0 \le t \le 1/2$ AND $f(t) = 0$ FOR $1/2 \le t \le 1$ WITH $f(t+1) = f(t)$.

PROBLEM 2 CONSIDER THE FOLLOWING INITIAL VALUE PROBLEM FOR $y(t)$:

$$y'''' + k y''' + y'' + y' = e^{-t} \quad t \ge 0, \quad k \text{ REAL WITH } k \ge 0$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1, \quad y'''(0) = 4$$

(i) calculate the Laplace transform of $y(t)$, denoted by $Y(s)$, in the form $Y(s) = P(s)/Q(s)$ where P, Q polynomials.

(ii) IF $k = 2$ prove that $y(t)$ IS BOUNDED AS $t \to \infty$ AND calculate $\lim_{t\to\infty} y(t)$.

(iii) FOR what range of k is $y(t)$ BOUNDED AS $t \to \infty$?

(iv) FIND THE BEHAVIOR OF $y(t)$ AS $t \to \infty$ when $k = 1$. what happens to $y(t)$ when $0 < k < 1$?

PROBLEM 3 USING LAPLACE TRANSFORMS FIND THE SOLUTION TO THE DIFFUSION EQUATION

$$u_t = u_{xx} \quad 0 < x < \infty, \ t > 0$$

$$u(0,t) = 1, \quad u(x,0) = e^{-x}, \quad u \to 0 \text{ AS } x \to \infty \text{ FOR } t \text{ FIXED}$$

PROBLEM 4 FIND AN INTEGRAL REPRESENTATION FOR THE SOLUTION TO

$$u_{tt} = -u_{xxxx} \quad -\infty < x < \infty, \quad t > 0$$

WITH $u(x,0) = g(x)$ AND $u_t(x,0) = 0$

USING FOURIER TRANSFORMS. WE ASSUME $u, u_x, u_{xx}, u_{xxx} \to 0$ AS $|x| \to \infty$.

SOLUTION 1

(i) WE TAKE LAPLACE TRANSFORMS TO OBTAIN $Y(s) = \mathcal{L}(y(t))$

$$sY - Y_0 + Y = \frac{F_0(s)}{1-e^{-s}} \qquad F_0(s) = \int_0^1 e^{-st} f(t)\,dt$$

THIS GIVES $$Y = \frac{H(s)F_0(s)}{1-e^{-s}} + \frac{Y_0}{s+1} \quad \text{WITH} \quad H(s) = \frac{1}{s+1}.$$

NOW $$Y = \left(\frac{H(s)F_0(s)}{1-e^{-s}} - A(s)\right) + \frac{Y_0}{s+1} + A(s) \qquad A(s) = \frac{F_0(-1)}{(1-e^{1})(s+1)}$$

THEN $$Y(s) = \frac{P_0(s)}{1-e^{-s}} + \left(\frac{Y_0}{s+1} + \frac{F_0(-1)}{(1-e^{1})(s+1)}\right)$$

WHERE $$P_0(s) = H(s)F_0(s) - A(s)(1-e^{-s})$$ IS ANALYTIC.

TO ENSURE THAT THERE IS NO TRANSIENT PART CHOOSE

$$Y_0 = -\frac{F_0(-1)}{(1-e^{1})} \qquad F_0(-1) = \int_0^1 e^{t} f(t)\,dt.$$

(ii) FOR THIS CASE WITH Y_0 CHOSEN THIS WAY WE HAVE

$$Y(s) = \frac{P_0(s)}{1-e^{-s}} \qquad P_0(s) = \frac{F_0(s)}{s+1} - \frac{F_0(-1)}{(1-e^{1})(s+1)}(1-e^{-s})$$

NOW $$F_0(s) = \int_0^{1/2} t e^{-st}\,dt = -\frac{1}{2s}e^{-s/2} - \frac{1}{s^2}e^{-st}\Big|_0^{1/2} = \frac{1}{s^2} - \left(\frac{1}{2s} + \frac{1}{s^2}\right)e^{-s/2}$$

HENCE $$P_0(s) = \frac{1}{s^2(s+1)} - e^{-s/2}\frac{1}{s+1}\left(\frac{1}{s^2} + \frac{1}{2s}\right) - \frac{F_0(-1)}{(1-e^{1})(s+1)} + A(s)e^{-s}$$

inverse transform $= 0$ FOR $t \geq 1$

NOW
$$\left.\begin{aligned}\frac{1}{s^2(s+1)} &= \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \\ \frac{1}{2s(s+1)} &= \frac{1}{2s} - \frac{1}{2(s+1)}\end{aligned}\right\}$$ partial FRACTIONS.

HENCE, $$p_0(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2(s+1)} - \frac{F_0(-1)}{(1-e^1)(s+1)} - \frac{1}{2s(s+1)}e^{-s/2} - \frac{e^{-s/2}}{s^2(s+1)}\right]$$

SO $$p_0(t) = (t-1+e^{-t}) - \frac{F_0(-1)}{(1-e^1)}e^{-t} - U_{1/2}(t)\left((t-\tfrac{1}{2})-1+e^{-(t-1/2)}\right)$$
$$- U_{1/2}(t)\left(\frac{1}{2} - \frac{1}{2}e^{-(t-1/2)}\right).$$ ON $0<t<1$

THIS MEANS $$y(t) = \sum_{n=0}^{\infty} p_0(t-n)$$

PROBLEM 2 WE CALCULATE

$$\mathcal{L}(y') = s\,\mathcal{L}(y) - y(0)$$
$$\mathcal{L}(y'') = s\,\mathcal{L}(y') - y'(0) = s^2\mathcal{L}(y) - s\,y(0) - y'(0)$$
$$\mathcal{L}(y''') = s^3\mathcal{L}(y) - s^2 y(0) - s\,y'(0) - y''(0)$$
$$\mathcal{L}(y'''') = s^4\mathcal{L}(y) - s^3 y(0) - s^2 y'(0) - s\,y''(0) - y'''(0).$$

WE TAKE $\mathcal{L}(y'''') + K\,\mathcal{L}(y''') + \mathcal{L}(y'') + \mathcal{L}(y') = \dfrac{1}{s+1}$.

SO $(s^4\bar{Y} - s - 4) + K(s^3\bar{Y} - 1) + s^2\bar{Y} + s\bar{Y} = \dfrac{1}{s+1}$

HENCE $(s^4 + Ks^3 + s^2 + s)\bar{Y} = s + 4 + K + \dfrac{1}{s+1} = \dfrac{(s+4+K)(s+1)+1}{s+1}$

(i) THEREFORE WE OBTAIN $\bar{Y}(s) = \dfrac{P(s)}{Q(s)} = \dfrac{(s+4+K)(s+1)+1}{s(s+1)(s^3+Ks^2+s+1)}$

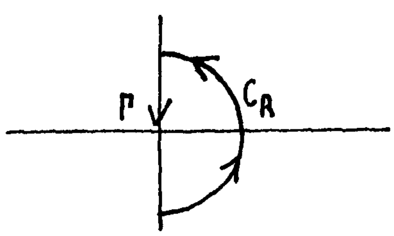
(ii) NOTICE we have a pole at $s=0$, $s=-1$ AND AT ROOTS OF $s^3+Ks^2+s+1=0$

THEREFORE IF ROOTS OF $s^3+Ks^2+s+1=0$ SATISFY $\mathrm{Re}\,s\le 0$

WE CAN GUARANTEE that all poles of $\bar{Y}(s)$ satisfy $\mathrm{Re}(s)\le 0 \to y(t)$

IS BOUNDED AS $t\to\infty$.

NOW LET $K=2$ THEN $s^3+2s^2+s+1 \equiv p(s)$. FIND ROOTS OF $p(s)=0$. WE USE ARGUMENT PRINCIPLE: $N(p)$: # ZEROES of p in $\mathrm{Re}\,s>0$



$$(*)\quad \frac{1}{2\pi}\Delta_\Gamma \arg(p(iy)) + \frac{1}{2\pi}\Delta_{\substack{C_R\\ R\to\infty}} \arg(p) = N(p)$$

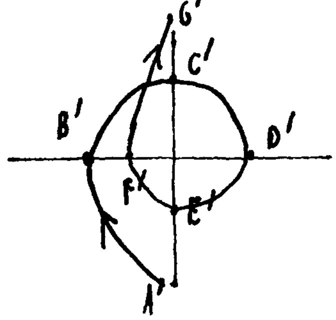
BUT $\Delta_{\substack{C_R\\ R\to\infty}} \arg(p) = 3\pi$.

$\mathrm{RE}\,p = 0 \to y = y_{R_\pm} = \pm\dfrac{1}{\sqrt{2}}$

$\mathrm{IM}\,p = 0 \to y = 0,\ y_{I_\pm} = \pm 1$

NOW WE CALCULATE $p(iy) = iy(1-y^2) + (1-2y^2)$

	y	RE	IM
A	∞	<0	<0
B	y_{I_+}	<0	$=0$
C	y_{R_+}	$=0$	>0
D	0	>0	$=0$
E	y_{R_-}	$=0$	<0
F	y_{I_-}	<0	$=0$
G	$-\infty$	<0	>0



$\Rightarrow \Delta_\Gamma \arg(p(iy)) = -3\pi$

SO $N(p) = 0$ FROM $(*)$

NOW FOR $K=2$ THE POLE WITH largest Real PART IS AT $S=0$.

NEAR $S=0$ WE CALCULATE $Y(S) \sim \frac{5+K}{S} = \frac{7}{S}$ SINCE $K=2$.

HENCE $\lim_{t\to\infty} y(t) = 7$, WHEN $K=2$.

(iii) NOW LET K BE ARBITRARY THEN $p(S) = S^3 + KS^2 + S + 1$ AND SO

$$p(iy) = iy(1-y^2) + (1-Ky^2)$$

$\mathrm{RE}\, p = 0$ when $y_{R_\pm} = \pm 1/\sqrt{K}$

$\mathrm{IM}\, p = 0$ WHEN $y_{I_\pm} = 1$, $y = 0$.

IT IS CLEAR THAT $\Delta_\Pi\, p(iy)) = -3\pi$ when $y_{R_+} < y_{I_+}$

THIS MEANS $\frac{1}{\sqrt{K}} < 1$ OR $K > 1$ FOR THAT

ALL THE ROOTS OF $p(S) = 0$ ARE IN $\mathrm{RE}(S) < 0$.

HENCE WE CLAIM THAT FOR $K > 1$, $y(t)$ IS BOUNDED

AS $t \to \infty$; SINCE $Y(S) \sim \frac{5+K}{S}$ NEAR $S=0 \Rightarrow y(t) \to 5+K$ AS $t \to \infty$

WHEN $K > 1$.

(iv) WHEN $K=1$ WE GET $p(S) = S^3 + S^2 + S + 1 = 0 \rightarrow S^2(S+1) + (S+1) = 0$

HENCE $(S^2+1)(S+1) = 0$ SO $S = -1$ AND $S = \pm i$ HENCE

$$Y(S) = \frac{(S+5)(S+1)+1}{S(S+1)^2(S+i)(S-i)}$$ WE CLAIM $y(t) = \mathcal{L}^{-1}[Y(S)]$ IS

OSCILLATORY AS $t \to \infty$. IN FACT SINCE DOMINANT POLES ARE AT $\pm i$

$$y(t) \sim \frac{(5+i)(1+i)+1}{i(1+i)^2\, 2i} e^{it} + \frac{(5-i)(1-i)+1}{-i(1-i)^2(-2i)} e^{-it} \qquad t \gg 1$$

THIS YIELDS $$y(t) \sim \left(\frac{6-5i}{4}\right) e^{it} + \left(\frac{6+5i}{4}\right) e^{-it} \qquad t \gg 1$$

THUS $y(t) \sim \frac{6}{2}\left(\frac{e^{it} + e^{-it}}{2} \right) + \frac{\boxed{5}\,2}{4}\left(\frac{e^{-it+i\pi/2} + e^{it-i\pi/2}}{2} \right)$

HENCE $y(t) \sim 3\cos t + \frac{5}{2}\cos(t - \pi/2)$ AS $t \to \infty$, when $K = 1$.

$\downarrow$

$\sin(t)$

WHEN $0 < K < 1$, $\longrightarrow$ $y_{R_+} > y_{I_+} \Rightarrow N_0(p) = 2$ SINCE $\Delta_\Gamma \arg p(iy) = +\pi$

IN THIS CASE $y(t)$ IS UNBOUNDED AS $t \to \infty$ WITH

EXPONENTIAL GROWTH.

PROBLEM 3

$$U_t = U_{xx}$$
$$U(0,t) = 1, \quad U(x,0) = e^{-x}.$$

NOW LET $\bar{U}(x,s) = \mathcal{L}(U(x,t))$. HENCE

$$s\bar{U} - U(x,0) = \bar{U}_{xx}$$
$$\bar{U}(0,s) = \frac{1}{s}$$

THIS LEADS TO
$$\bar{U}_{xx} - s\bar{U} = -e^{-x}$$
$$\bar{U}(0,s) = \frac{1}{s}, \quad \bar{U} \to 0 \text{ AS } x \to \infty.$$

THE SOLUTION IS
$$\bar{U} = Ae^{-\sqrt{s}x} + Be^{\sqrt{s}x} + \frac{1}{s-1}e^{-x}.$$

HOWEVER $B = 0$ SO THAT $\bar{U} \to 0$ AS $x \to +\infty$ AND $A = \left(\frac{1}{s} - \frac{1}{s-1}\right)$

THIS YIELDS
$$\bar{U}(x,s) = \left(\frac{1}{s} - \frac{1}{s-1}\right)e^{-\sqrt{s}x} + \frac{1}{s-1}e^{-x}$$

WHICH SATISFIES $\bar{U}(0,s) = 1/s$. NOTICE THERE IS <u>NO POLE</u> AT $s=1$.

RECALL
$$\mathcal{L}^{-1}\left[\frac{1}{s}e^{-\sqrt{s}x}\right] = \mathrm{ERFC}\left(x/2\sqrt{t}\right)$$
$$\mathcal{L}^{-1}\left[e^{-\sqrt{s}x}\right] = \frac{x}{2\sqrt{\pi}t^{3/2}}e^{-x^2/4t}$$

HENCE
$$\mathcal{L}^{-1}(\bar{U}(x,s)) = \mathcal{L}^{-1}\left(\frac{1}{s}e^{-\sqrt{s}x}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-1}e^{-\sqrt{s}x}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-1}e^{-x}\right)$$

$$U(x,t) = \mathrm{ERFC}\left(x/2\sqrt{t}\right) - \int_0^\infty e^{t-\tau}\frac{x}{2\sqrt{\pi}\tau^{3/2}}e^{-x^2/4\tau}\,d\tau + e^{-x+t}$$

OR
$$U(x,t) = \mathrm{ERFC}\left(x/2\sqrt{t}\right) + e^t\left(e^{-x} - \frac{x}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-x^2/4\tau}\,e^{-\tau}}{\tau^{3/2}}\,d\tau\right)$$

PROBLEM 4 DEFINE $\hat{U}(K,t) = \mathcal{F}(u(x,t))$.

WE CALCULATE $\mathcal{F}(U_{tt}) = -\mathcal{F}(U_{xxxx}) = -(iK)^4 \mathcal{F}(U)$

$$\longrightarrow \quad \hat{U}_{tt} = -K^4 \hat{U}$$

$$\hat{U}(K,0) = G(K) = \mathcal{F}(g(x)), \quad \hat{U}_t(K,0) = 0.$$

THEN WE CALCULATE $\hat{U}(K,t) = A\cos(K^2 t) + B\sin(K^2 t)$.

NOW $\hat{U}(K,0) = G(K) \rightarrow A = G(K)$

$\hat{U}_t(K,0) = 0 \rightarrow B = 0$

SO $\hat{U} = G(K)\cos(K^2 t)$

THIS YIELDS $u(x,t) = \mathcal{F}^{-1}[\hat{U}(K,t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iKx} G(K)\cos(K^2 t)\,dK$

NOW WITH $G(K) = \int_{-\infty}^{\infty} e^{iKx'} g(x')\,dx'$ WE OBTAIN,

$$(*) \qquad u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x')\left(\int_{-\infty}^{\infty} e^{iK(x'-x)}\cos(K^2 t)\,dK\right)dx'$$

DEFINE
$$I = \int_{-\infty}^{\infty} e^{iK(x'-x)}\cos(K^2 t)\,dK = \int_{-\infty}^{\infty}\Big(\cos\big(K(x'-x)\big) + i\sin\big(K(x'-x)\big)\Big)\cos K^2 t$$

THE SECOND INTEGRAL IS ODD AND INTEGRATES TO ZERO.

HENCE $I = \int_{-\infty}^{\infty} \cos\big(K(x'-x)\big)\cos(K^2 t)\,dK$.

RECALL $\cos A \cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$

HENCE $I = \frac{1}{2}\left(RE\left[\int_{-\infty}^{\infty} e^{iK(x'-x) - iK^2 t}\,dK\right] + RE\left[\int_{-\infty}^{\infty} e^{iK(x'-x)+iK^2 t}\,dK\right]\right.$

NOW WE COMPLETE THE SQUARE TO GET

$$I = \frac{1}{2}\, \mathrm{RE}\left[e^{i(x'-x)^2/4t} \int_{-\infty}^{\infty} e^{-it\left(K - (x'-x)/2t\right)^2} dK + e^{-i(x'-x)^2/4t} \int_{-\infty}^{\infty} e^{it\left(K + \frac{(x'-x)}{2t}\right)^2} dK \right]$$

THIS IS SIMPLY BY SHIFTING VARIABLES

$$I = \frac{1}{2}\, \mathrm{RE}\left[e^{i(x'-x)^2/4t} \int_{-\infty}^{\infty} e^{-its^2} ds + e^{-i(x'-x)^2/4t} \int_{-\infty}^{\infty} e^{its^2} ds \right]$$

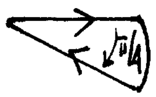
let $\varsigma = \sqrt{t}\, s$ in integrals to obtain $d\varsigma = \sqrt{t}\, ds$.

so $$I = \frac{1}{2\sqrt{t}}\, \mathrm{RE}\left[e^{i(x'-x)^2/4t} J_- + e^{-i(x'-x)^2/4t} J_+ \right]$$

WHERE $J_- = \int_{-\infty}^{\infty} e^{-i\varsigma^2} d\varsigma$ $\quad J_+ = \int_{-\infty}^{\infty} e^{i\varsigma^2} d\varsigma$.

RECALL THAT INTEGRALS LIKE $J_\pm$ WHERE CALCULATED IN PROBLEM 2 OF HOMEWORK 8. NOTICE $J_\pm = 2\int_0^{\infty} e^{\pm i\varsigma^2} d\varsigma$

WE TAKE A CONTOUR LIKE



$$\int_0^{\infty} e^{-i\varsigma^2} d\varsigma + e^{-i\pi/4} \int_{\infty}^{0} e^{-r^2} dr = 0$$

HOWEVER $\int_0^{\infty} e^{-r^2} dr = \sqrt{\pi}/2$. HENCE $\int_0^{\infty} e^{-i\varsigma^2} d\varsigma = e^{-i\pi/4} \frac{\sqrt{\pi}}{2}$.

THIS GIVES $J_- = 2\int_0^{\infty} e^{-i\varsigma^2} d\varsigma = \sqrt{\pi}\, e^{-i\pi/4}$

TAKING THE CONJUGATE $J_+ = \sqrt{\pi}\, e^{i\pi/4}$.

THUS $$I = \frac{1}{2\sqrt{t}} \sqrt{\pi}\, \mathrm{RE}\left[e^{i(x-x')^2/4t} e^{-i\pi/4} + e^{-i(x'-x)^2/4t} e^{i\pi/4} \right]$$

HENCE, WE OBTAIN $$I = \sqrt{\frac{\pi}{t}} \cos\left(\frac{(x-x')^2}{4t} - \frac{\pi}{4} \right)$$

FINALLY, WE RETURN TO $(*)$ ON PREVIOUS PAGE TO OBTAIN

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} g(x') \cos\left(\frac{(x-x')^2}{4t} - \frac{\pi}{4} \right) dx'.$$