Homework 1 Solutions Grade out of 30

5.5 Q1(b) Laurent Series of $\frac{1}{z+z^2}$ in $1<|z|$: 2pts

Centering at $z=0$.

$$\frac{1}{z+z^2} = \frac{1}{z}\left(\frac{1}{1+z}\right)$$

Write $\frac{1}{1+z}$ as a geometric series:

$$\frac{1}{1+z} = \frac{1/z}{1/z+1} = \frac{1}{z}\left(\frac{1}{1-(-1/z)}\right) = \frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(-\frac{1}{z}\right)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n z^{-(n+1)} \qquad \left(\text{converges where } \left|\tfrac{1}{z}\right|<1 \Leftrightarrow |z|>1\right)$$

$$\Rightarrow \frac{1}{z+z^2} = \sum_{n=0}^{\infty}(-1)^n z^{-(n+2)}$$

5.5 Q1(c) Laurent Series of $\frac{1}{z^2+z}$ in $0<|z+1|<1$: 2pts

Centering at $z=-1$.

$$\frac{1}{z^2+z} = \frac{1}{z+1}\cdot\frac{1}{z} = \frac{1}{z+1}\cdot\left(\frac{-1}{1-(z+1)}\right)$$

$$= \frac{1}{z+1}\cdot(-1)\cdot\sum_{n=0}^{\infty}(z+1)^n \qquad (\text{converges where } |z+1|<1)$$

$$= -1\cdot\sum_{n=0}^{\infty}(z+1)^{n-1}$$

5.5 Q6 Laurent series for $z^2\cos\left(\frac{1}{3z}\right)$ in $|z|>0$: 3 pts

Centering at $z=0$.

Recall Taylor Series Expansion of $\cos z$ at $z=0$:

$$\cos z = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}$$

$$\cos\left(\frac{1}{3z}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{3z}\right)^{2n}$$

$$z^2\cos\left(\frac{1}{3z}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\,3^{2n}}z^{2-2n}$$

5.6 Q1 Isolated singularities of functions

(d) $\frac{1}{e^z-1}$: 2pts

Singularities where $e^z-1=0 \Rightarrow e^z=1$

Occur at $z=2\pi ni$ for all $n\in\mathbb{Z}$

Since e^z is $2\pi i$-periodic, we can write for each $n\in\mathbb{Z}$:

$$\frac{z-2\pi ni}{e^z-1} = \frac{z-2\pi ni}{e^{z-2\pi ni}-1}$$

$$\lim_{z\to 2\pi ni}\frac{z-2\pi ni}{e^{z-2\pi ni}-1} \overset{\text{(change of variables)}}{=} \lim_{z\to 0}\frac{z}{e^z-1} = \frac{1}{\frac{d}{dz}e^z\big|_{z=0}} = 1$$

Since $\frac{1}{}(z-2\pi ni)$ gives a nonzero removable

$$\lim_{z\to 2\pi n i} \frac{\quad}{e^{z-2\pi n i}-1} = \quad \lim_{z\to 0} \frac{\quad}{e^{z}-1} = \frac{\quad}{\frac{d}{dz}e^{z}\Big|_{z=0}} = 1$$

Since $\frac{1}{e^z-1}(z-2\pi n i)$ gives a nonzero removable singularity at $z=2\pi n i$, each point $\mathbf{z=2\pi n i}$ is a **pole of order 1**.

(e) $\tan z$: **2 pts**

By definition, $\tan z = \frac{\sin z}{\cos z}$

Singularities occur where $\cos z = 0$

$\Rightarrow z = \frac{(2n+1)\pi}{2}$ for all $n \in \mathbb{Z}$

$$\lim_{z\to\frac{(2n+1)\pi}{2}} \left(z - \frac{(2n+1)\pi}{2}\right) \tan z$$

$$= \lim_{z\to\frac{(2n+1)\pi}{2}} \frac{z-\frac{(2n+1)\pi}{2}}{\cos z} \cdot \sin z$$

$$= \frac{\sin\left(\frac{(2n+1)}{2}\pi\right)}{\frac{d}{dz}\cos z\Big|_{z=\frac{(2n+1)\pi}{2}}} = \frac{\sin\left(\frac{(2n+1)}{2}\pi\right)}{-\sin\left(\frac{(2n+1)}{2}\pi\right)} = -1$$

Thus $\mathbf{z=\frac{(2n+1)}{2}\pi}$ is a **pole of order 1**.

(g) $\frac{\sin(3z)}{z^2} - \frac{3}{z}$: **2 pts**

$z=0$ is the only singularity

Recall the Taylor Series expansion of sin:

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

Then $\frac{\sin(3z)}{z^2} = \frac{3z}{z^2} - \frac{(3z)^3}{3!\,z^2} + \frac{(3z)^5}{5!\,z^2} - \cdots$

and $\frac{\sin(3z)}{z^2} - \frac{3}{z} = -\frac{3^3}{3!}z + \frac{3^5}{5!}z^3 - \cdots$

is the Laurent series centered at $z=0$

Since it has no negative part, $\mathbf{z=0}$ is a **removable singularity**.

5.6 Q7 $f(z)$ being analytic on $0<|z|\le 1$ makes it bounded on the outer boundary $|z|=1$. **2pts**

$z=0$ cannot be a removable singularity as this would make $f(z)$ bounded near $z=0$ (since $\lim_{z\to 0} f(z)$ exists), contradicting the assumption.

If $z=0$ is a pole of order m, this means $z^m \cdot f(z)$ has a limit at $z=0$, contradicting the assumption that $z^{\ell}\cdot f(z)$ is unbounded for every integer ℓ.

Therefore, $z=0$ must be an **essential singularity**.

6.1 Q1

(b) $\frac{z+1}{z^2-3z+2}$: 2 pts

Factor $z^2-3z+2=(z-2)(z-1)$. This gives the singularities $z=1$ and $z=2$.

$z=1$ is clearly a pole of order 1, and has residue $\lim_{z\to 1}\frac{(z+1)(z-1)}{z^2-3z+2}=\lim_{z\to 1}\frac{z+1}{z-2}=-2$

$z=2$ is also a pole of order 1, and has residue $\lim_{z\to 2}\frac{(z+1)(z-2)}{z^2-3z+2}=\lim_{z\to 2}\frac{z+1}{z-1}=3$

(g) $\tan z$: 2 pts

$\tan z=\frac{\sin z}{\cos z}$. Singularities are where $\cos z=0$

$\Rightarrow z=\frac{(2n+1)\pi}{2}$ $\forall n\in\mathbb{Z}$.

(See 5.6 Q1(e)) Residue at each singularity is

$\lim_{z\to(2n+1)\pi i}(\tan z)\cdot(z-(2n+1)\pi i)=-1$

6.1 Q2 Cauchy's Integral Formula says that if Γ is a closed contour and z_0 is a point inside Γ; 2pts

$\frac{1}{2\pi i}\oint_\Gamma\frac{f(z)}{z-z_0}dz=f(z_0)$ for all analytic functions f inside Γ.

Since $\frac{f(z)}{z-z_0}$ has a singularity at $z=z_0$, with residue $f(z_0)$, the residue theorem gives the result

$\oint_\Gamma\frac{f(z)}{z-z_0}dz=2\pi i\,\mathrm{Res}\left(\frac{f}{z-z_0},z_0\right)=2\pi i\,f(z_0)$, which is the same as Cauchy's integral formula

6.1 Q3

(a) $\frac{\sin z}{z^2-4}$ has two singularities $z=2$ and $z=-2$ 3 pts

since $z^2-4=(z+2)(z-2)$

Both singularities lie inside $|z|=5$

$\mathrm{Res}(f;z_0=2)=\lim_{z\to 2}\frac{(\sin z)(z-2)}{z^2-4}=\lim_{z\to 2}\frac{\sin z}{z+2}=\frac{\sin 2}{4}$

$\mathrm{Res}(f;z_0=-2)=\lim_{z\to -2}\frac{(\sin z)(z+2)}{z^2-4}=\lim_{z\to -2}\frac{\sin z}{z-2}=\frac{-\sin 2}{-4}$

Then $\oint_{|z|=5}\frac{\sin z}{z^2-4}dz=2\pi i\left[\mathrm{Res}(f;2)+\mathrm{Res}(f;-2)\right]$

$=2\pi i\cdot\frac{\sin 2}{2}=(\sin 2)\pi i$

(b) $\frac{e^z}{z(z-2)^3}$ has two singularities $z=0$ and $z=2$, both inside $|z|=3$ 3 pts

$\mathrm{Res}(f;z_0=0)=\lim_{z\to 0}\frac{e^z\cdot z}{z(z-2)^3}=\lim_{z\to 0}\frac{e^z}{(z-2)^3}=\frac{1}{-8}$

Since $z=2$ is a pole of order 3,

$\mathrm{Res}(f;z_0=2)=\lim_{z\to 2}\frac{1}{2}\frac{d^2}{dz^2}\frac{e^z\cdot(z-2)^3}{\quad}$

Since $z=2$ is a pole of order 3,

$$\text{Res}(f; z_0=2) = \lim_{z\to 2} \frac{1}{2}\frac{d^2}{dz^2} \frac{e^z \cdot (z-2)^3}{z(z-2)^3}$$

$$= \lim_{z\to 2} \frac{1}{2}\frac{d^2}{dz^2}\frac{e^z}{z} = \lim_{z\to 2}\frac{1}{2}\left(\frac{e^z}{z} - \frac{2e^z}{z^2} + \frac{2e^z}{z^3}\right)$$

$$= \frac{1}{2}\left(\frac{e^2}{2} - \frac{2e^2}{4} + \frac{2e^2}{8}\right) = \frac{e^2}{8}$$

$$\oint_{|z|=3} \frac{e^z}{z(z-2)^3}\,dz = 2\pi i\left(-\frac{1}{8} + \frac{e^2}{8}\right) = \mathbf{\frac{e^2-1}{4}\pi i}$$

(f) $\frac{3z+2}{z^4+1}$ has four singularities: 3 pts

$$z = \frac{1}{\sqrt{2}}(1+i),\quad z = \frac{1}{\sqrt{2}}(-1+i),\quad z = \frac{1}{\sqrt{2}}(-1-i),\quad z = \frac{1}{\sqrt{2}}(1-i)$$

$$\underset{z_1}{\shortparallel}\ (=e^{i\pi/4}) \qquad \underset{z_2}{\shortparallel}\ (=e^{i3\pi/4}) \qquad \underset{z_3}{\shortparallel}\ (=e^{i5\pi/4}) \qquad \underset{z_4}{\shortparallel}\ (=e^{i7\pi/4})$$

all simple poles

$$\text{Res}(f; z_1) = \frac{3z_1+2}{\frac{d}{dz}(z^4+1)\big|_{z=z_1}} = \frac{3z_1+2}{4z_1^3} = \frac{3e^{i\pi/4}+2}{4e^{i3\pi/4}} = \frac{3}{4}e^{-i\pi/2} + \frac{1}{2}e^{-i3\pi/4}$$

$$\text{Res}(f; z_2) = \quad \text{''} \quad \text{''} \quad = \frac{3e^{i3\pi/4}+2}{4e^{i9\pi/4}} = \frac{3}{4}e^{-i3\pi/2} + \frac{1}{2}e^{-i\pi/4}$$

$$\text{Res}(f; z_3) = \quad \text{''} \quad \text{''} \quad = \frac{3e^{i5\pi/4}+2}{4e^{i15\pi/4}} = \frac{3}{4}e^{-i5\pi/2} + \frac{1}{2}e^{i\pi/4}$$

$$\text{Res}(f; z_4) = \quad \text{''} \quad \text{''} \quad = \frac{3e^{i7\pi/4}+2}{4e^{i21\pi/4}} = \frac{3}{4}e^{-i7\pi/2} + \frac{1}{2}e^{i3\pi/4}$$

(replacing z_1 with z_2, z_3, z_4)

Summing the residues, we get $\oint_{|z|=3} \frac{3z+2}{z^4+1}\,dz = \mathbf{0}$