

- [20] 1. The parabola  $y = x^2 - 5x + 6$  has two tangent lines that intersect at the point (4, 1). Find the x and y coordinates of the two points at which the lines are tangent to the parabola.

2 marks ① Find  $y'$

$$y' = f'(x) = 2x - 5$$

10 marks ② Find general equation of tangent line for parabola, where  $x = a$ .

$$y = (a)^2 - 5(a) + 6 = a^2 - 5a + 6 \quad (x, y) = (a, a^2 - 5a + 6)$$

$$m = 2a - 5 \quad 2 \text{ marks}$$

$$y = mx + b \quad 1 \text{ mark}$$

$$b = y - mx = (a^2 - 5a + 6) - (2a - 5)a$$

$$b = 6 - a^2 \quad 2 \text{ marks}$$

$$\therefore \text{tangent line is } y = (2a - 5)x + (6 - a^2) \quad \text{at } (a, a^2 - 5a + 6) \quad 3 \text{ marks}$$

6 marks ③ Substitute a point on the tangent line into equation to solve for  $a$ , the  $x$ -coordinates for the 2 points we want to find.

$$(x, y) = (4, 1) \leftarrow \text{this point is on the tangent line.}$$

$$y = (2a - 5)x + (6 - a^2)$$

$$1 = (2a - 5)x + (6 - a^2)$$

$$1 = 8a - 20 + 6 - a^2$$

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$$\rightarrow a^2 - 8a + 15 = 0$$

3 marks

$$(a-3)(a-5) = 0. \quad 2 \text{ marks} \quad \begin{cases} a=3 \\ a=5 \end{cases}$$

(i.e.  $x=3$ ,  
 $x=5$ )

2 marks ④ Find the corresponding  $y$ -coordinates

$$y = x^2 - 5x + 6$$

$$x=3 : y = (3)^2 - 5(3) + 6 = 0$$

$$x=5 : y = (5)^2 - 5(5) + 6 = 6$$

The 2 points are (3, 0) and (5, 6).

2. Suppose a ball is thrown upwards with a velocity of 40 m/s by someone on a platform 200m above the ground. The acceleration due to gravity is  $g = -9.8 \text{ m/s}^2$ .

- [10] (a) When will the ball reach its maximum height?  
 [10] (b) When will the ball hit the ground?

a)  $g = -9.8 \frac{\text{m}}{\text{s}^2}$        $v_0 = 40 \frac{\text{m}}{\text{s}}$

$\begin{cases} v(t) = v_0 + gt & \text{Max. height when } v(t)=0 \\ 0 = 40 \frac{\text{m}}{\text{s}} - 9.8 \frac{\text{m}}{\text{s}^2} t \\ 9.8t = 40 & \boxed{t = \frac{40}{9.8} \text{ seconds}} \end{cases}$  ← 3 marks  
 $(\approx 4.08 \text{ s})$

b)  $h(t) = h_0 + v_0 t + \frac{1}{2} g t^2$  ← 3 marks       $h_0 = 200 \text{ m}$        $v_0 = 40 \frac{\text{m}}{\text{s}}$        $g = -9.8 \frac{\text{m}}{\text{s}^2}$   
 Ball hits ground when  $h(t) = 0$

1 mark  $\left\{ \begin{array}{l} D = 200 + 40t + \frac{1}{2}(-9.8)t^2 \\ -4.9t^2 + 40t + 200 = 0 \end{array} \right.$

using the quadratic equation

$t = \frac{-40 \pm \sqrt{(-40)^2 - 4(-4.9)(200)}}{2(-4.9)}$  ←  
 $t = \frac{40}{9.8} \pm \frac{\sqrt{1600 + 400(9.8)}}{-9.8}$  ←  $\sqrt{5520}$   
 $t = \frac{40}{9.8} \pm \frac{20\sqrt{13.8}}{-9.8}$  ←

$\therefore t \geq 0$  for problem

ans left like this ok if mention  
only positive values  
of  $t$  are used  
(-0.5 if not)

-0.5 mark if answer left in this form

$\therefore \boxed{t = \frac{40}{9.8} + \frac{20\sqrt{13.8}}{9.8}}$  or  $\frac{20}{9.8} (2 + \sqrt{13.8})$

↑  
3 marks

$\approx 11.66 \text{ seconds.}$

Sign issues: -0.5  
Value issues: -1

3. Consider the function  $y = f(x) = x^4 - 2x^3 + x^2$ .

[10] (a) Determine the locations of the critical points and the inflection points.

[10] (b) Sketch the curve.

a) ① Critical points (5 marks)

$$\rightarrow f'(x) = 4x^3 - 6x^2 + 2x$$

$$0 = 4x^3 - 6x^2 + 2x$$

$$0 = 2x(2x^2 - 3x + 1)$$

$$x = 0 \quad \text{or} \quad 2x^2 - 3x + 1 = 0$$

0.5 marks

critical points occur

when  $f'(x) = 0$ , change in sign

↑ should prove.

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(1)}}{2(2)} = \frac{3 \pm \sqrt{9-8}}{4} \Rightarrow x = \frac{3 \pm 1}{4}$$

Critical points are at

$$(0, 0) \quad 0.5 \text{ mark}$$

$$x = 1 \text{ or } \frac{1}{2} \quad \begin{matrix} 0.5 \text{ mark} \\ \downarrow \\ x = \frac{1}{2} \end{matrix} \quad 0.5 \text{ mark}$$

$$\left(\frac{1}{2}, \left(\frac{1}{2}\right)^4 - 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2\right) = \left(\frac{1}{2}, \frac{1}{16} - 2\left(\frac{1}{8}\right) + \frac{1}{4}\right) = \left(\frac{1}{2}, \frac{1}{16}\right) \quad 0.5 \text{ mark}$$

(5 marks)

② Inflection points (P.I.)

$$f''(x) = 12x^2 - 12x + 2$$

$$0 = 12x^2 - 12x + 2$$

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(12)(2)}}{2(12)} = \frac{12 \pm \sqrt{144-96}}{24} = \frac{12 \pm \sqrt{48}}{24} = \frac{12 \pm 4\sqrt{3}}{24}$$

1 mark - work shown

P.I. : when  $f''(x) = 0$ , change in sign

↑ should prove.

$$\text{Inflection points: } \left( \frac{1}{2} + \frac{\sqrt{3}}{6}, \underbrace{\left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)^4 - 2\left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)^3 + \left(\frac{1}{2} + \frac{\sqrt{3}}{6}\right)^2}_{\approx 0.0278} \right) \quad 0.5 \text{ mark} = \frac{3 \pm \sqrt{3}}{6} = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$$

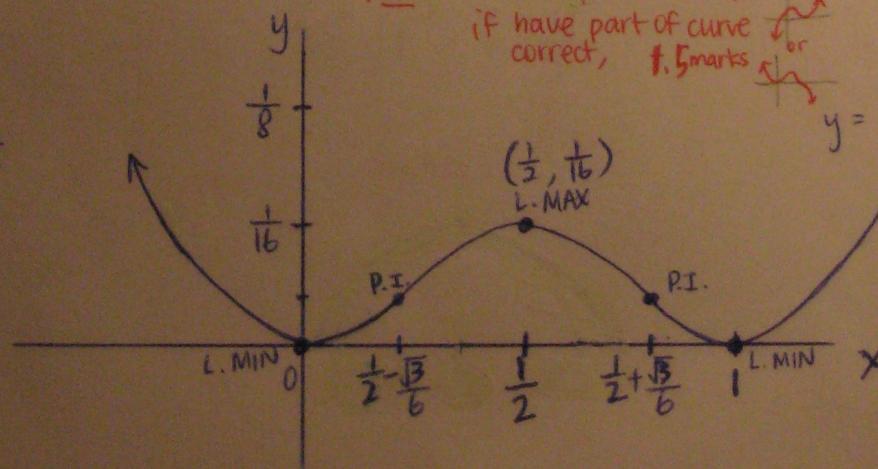
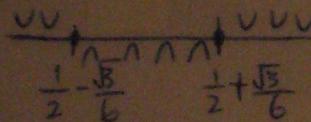
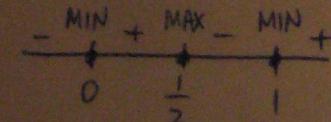
$$\left( \frac{1}{2} - \frac{\sqrt{3}}{6}, \underbrace{\left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)^4 - 2\left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)^3 + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)^2}_{\approx 0.0278} \right) \quad 0.5 \text{ mark}$$

should still produce correct curve shape

Note: for shape of curve,  
if have part of curve  
correct, 1.5 marks

approximately correct in case error  
in part a)

b) Sketch



(C.P)  
critical points and  
P.I.  
- 1 mark each, 5  
total

correct shape for  
curve - 2.5 marks

labeling for both  
axes - 2.5 mark

[20] 4. The function  $y = f(x) = \frac{5x^4}{16+x^4}$

is a Hill function with coefficient 4. Show that by defining new variables that are related to x and y this relationship can be re-expressed in linear form.

Let  $v = \frac{1}{y}, c = \frac{1}{x^4}$

Then  $y = \frac{1}{v}, x^4 = \frac{1}{c}$  2.5 marks 2.5 marks (substitute)

and the original equation is

$$\frac{1}{v} = \frac{5(\frac{1}{c})}{16 + \frac{1}{c}} \quad \leftarrow \text{6 marks} \quad (\text{insert new variables into equation})$$

$$\Rightarrow v = \frac{16 + \frac{1}{c}}{5(\frac{1}{c})}$$

$$v = \frac{16 + \frac{1}{c}}{5(\frac{1}{c})} \cdot \frac{c}{c} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \quad \begin{array}{l} 4 \text{ marks} \\ (\text{work is shown}) \end{array}$$

$$= \frac{16c + 1}{5}$$

$$\boxed{v = \frac{16}{5}c + \frac{1}{5}}$$

or  $\frac{1}{y} = \frac{16}{5} \left( \frac{1}{x^4} \right) + \frac{1}{5}$

↑ 0.5 ↑ 2 marks ↑ 0.5

2 marks 2 marks

[20] 5. Use the definition of the derivative to calculate the derivative of

$$y = f(x) = \frac{1}{1+x}$$

Note: No credit will be given unless the definition of derivative is used.

$$\begin{aligned}
 y' &= f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} && \leftarrow 5 \text{ marks} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{1}{1+x+h} - \frac{1}{1+x}}{h} && \leftarrow 5 \text{ marks} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{(1+x) - (1+x+h)}{(1+x)(1+x+h)}}{h} && \left. \begin{array}{l} \{ \text{correct variables are inserted; if have} \\ \text{this part but not def'n derivative,} \\ 10 \text{ marks} \end{array} \right\} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{-h}{(1+x)(1+x+h)} \right)
 \end{aligned}$$

$$\boxed{f'(x) = \frac{-1}{(1+x)^2}} \leftarrow 5 \text{ marks}$$

(-0.5 marks if negative sign missing)