THE UNIVERSITY OF BRITISH COLUMBIA

Math 102 Section 101

No calculators or notes allowed
Midterm begins at 10:00am and ends at 10:50am

MIDTERM 2 November 9, 2009

Solutions

NAME

STUDENT NUMBER

1. A certain cell culture grows exponentially. It takes 7 hours to double in size. Initially the culture weighs 5 grams and as time goes by the weight is w(t).

[10] (a) What is the weight of the culture after one day?

[10] (b) How long will it take until the weight is 500 grams?

$w(t) = w_0 e^{kt}$ $\quad w_0 = 5$ grams

a) $w(7) = 2w_0 = 10$ $\quad w(7) = 2w_0 = w_0 e^{k(7)}$

$2 = e^{7k}$ $\quad \ln 2 = \ln e^{7k} = 7k$ $\quad k = \frac{\ln 2}{7}$

$w(24) = 5e^{\left(\frac{\ln 2}{7}\right)24} = 5e^{(\ln 2)\frac{24}{7}}$ (can also write this as $w(24) = 5(2^{24/7})$)

b) $500 = 5e^{\frac{\ln 2}{7}t}$

$100 = e^{\frac{\ln 2}{7}t}$ $\quad \ln 100 = \ln e^{\left(\frac{\ln 2}{7}\right)t}$

$\ln 100 = \left(\frac{\ln 2}{7}\right)t$ $\quad t = \frac{7\ln 100}{\ln 2}$

[Alternatively can write $100 = (e^{\ln 2})^{t/7}$ and $100 = 2^{t/7}$ and $\log_2 100 = \log_2 2^{t/7} = (\log_2 2)\,t/7 = t/7$ and $t = 7\log_2 100$]

[20] 2. Suppose that when two fish are at distance x > 0 from one another, they are attracted with force Fa and repelled with force Fr given by

$$F_a = 3e^{-x/4} \qquad F_r = 5e^{-x/2}$$

Find the distance at which the forces exactly balance.

$$3e^{-x/4} = 5e^{-x/2}$$

$$e^{-x/4} = \tfrac{5}{3}e^{-x/2}$$

$$e^{-x/4+x/2} = \tfrac{5}{3}e^{-x/2+x/2} = \tfrac{5}{3}$$

$$e^{x/4} = \tfrac{5}{3} \qquad x/4 = \ln e^{x/4} = \ln \tfrac{5}{3}$$

$$x = 4\ln \tfrac{5}{3}$$

Alternative solution

$$\ln 3e^{-x/4} = \ln 5e^{-x/2}$$

$$\ln 3 + \ln e^{-x/4} = \ln 5 + \ln e^{-x/2}$$

$$(\ln 3) - x/4 = \ln 5 + (-x/2)$$

$$-x/4 + x/2 = \ln 5 - \ln 3$$

$$x/4 = \ln \tfrac{5}{3}$$

$$x = 4\ln \tfrac{5}{3}$$

[20] 3. Find an equation for the tangent line to the curve

$xe^{y} = -2y$ at the point $(x, y) = (-4e^{-2}, 2)$.

$$xe^{y}\frac{dy}{dx} + e^{y} = -2\frac{dy}{dx}$$

$$\frac{dy}{dx}(xe^{y}+2) = -e^{y}$$

$$\frac{dy}{dx} = \frac{-e^{y}}{xe^{y}+2}$$

at $(-4e^{-2}, 2)$

$$\frac{dy}{dx} = \frac{-e^{2}}{(-4e^{-2})e^{2}+2} = \frac{-e^{2}}{-4+2} = \frac{1}{2}e^{2}$$

Tangent line $y = mx+b$

$m = \frac{1}{2}e^{2}$ so $y = \frac{1}{2}e^{2}x + b$

at point $(-4e^{-2}, 2)$ we have

$2 = \frac{1}{2}e^{2}(-4e^{-2}) + b \qquad 2 = (-2e^{2-2}) + b$

$b = 2-(-2) = 4 \qquad \boxed{y = \frac{1}{2}e^{2}x + 4}$

4. The **Reaction** R(x) of a patient to a drug dose of size x depends on the type of drug. For a certain drug, it was determined that a good description of the relationship is:

$$R(x) = Ax^2(B - x)$$

Where A and B are positive constants. The **Sensitivity** S(x) of the patient's body to the drug is defined to be S(x) = dR/dx.

[10] (a) For what value of x is the reaction a maximum, and what is that maximum reaction value?

[10] (b) For what value of x is the sensitivity a maximum? What is the maximum sensitivity?

For both (a) and (b) be sure to prove that the critical points you find yield a local maximum.

$$R(X) = AX^2(B-X) = ABX^2 - AX^3$$

(a) $R'(X) = AX^2(-1) + (B-X)(2AX) = -AX^2 + 2ABX - 2AX^2$

and $R'(x) = -3AX^2 + 2ABX$

OR can write $R'(X) = (ABX^2 - AX^3)' = 2ABX - 3AX^2$

If $R' = 0 = A(2BX - 3X^2)$ then $2BX = 3X^2$ and

$X = 0$ OR $2B = 3X$ and $X = 2B/3$ and, using 2nd derivative test $R''(X) = -6AX + 2AB$ at $X = 2B/3$ we have

$$R''\left(\frac{2B}{3}\right) = -6A\left(\frac{2B}{3}\right) + 2AB = -2A(2B) + 2AB = -2AB$$

hence R is a maximum at $\boxed{X = \frac{2B}{3}}$ by 2nd derivative test for R(x).

ALTERNATIVELY one can use 1st derivative test for R(x)

$R' = AX(2B - 3X)$

R	INC	$2B/3$	DEC
R'	$+$	0	$-$
		LOC MAX for R	

$R' < 0$ for $X > \frac{2B}{3}$

$R' > 0$ for $x < \frac{2B}{3}$

MAX REACTION VALUE

$R\left(\frac{2B}{3}\right) = A\left(\frac{2B}{3}\right)^2\left(B - \frac{2B}{3}\right) = \frac{4AB^3}{27}$

(b) $S(X) = R'(X) = AX(2B - 3X) = 2ABX - 3AX^2$

$S'(X) = 2AB - 6AX = 0$ for $2A(B - 3X) = 0$ if $X = \frac{1}{3}B$

2nd derivative test for S(x) $S''(X) = -6A < 0$ Thus S(x) has a maximum at $\boxed{X = \frac{B}{3}}$

ALTERNATIVELY one can use 1st derivative test for S(X)

$S' = 2A(B - 3X)$

S	INC	$B/3$	DEC
S'	$+$	0 LOC MAX for S	

$S' < 0$ for $X > B/3$

$S' > 0$ for $X < B/3$

MAX S(X)

$S(B/3) = 2AB\frac{B}{3} - 3A\frac{B^2}{9} = \frac{AB^2}{3}$

5. A ball is thrown from a tower of height h_0. The height of the ball at time t is

$$h(t) = h_0 + v_0 t - (1/2)gt^2$$

where h_0, v_0, g are positive constants.

[10] When does the ball reach its highest point?
[10]How fast is it going just before it hits the ground?

$v(t) = v_0 - gt$ = velocity

(a) Ball reaches highest point when $v(t) = 0$

$0 = v_0 - gt$ yields $t = \frac{v_0}{g}$

(b) Need to find v when $h(t) = 0$

$h(t) = 0 = h_0 + v_0 t - \frac{1}{2}gt^2$. Need to solve

$-\frac{1}{2}gt^2 + v_0 t + h_0 = 0$ OR $t^2 - \frac{2v_0}{g}t - \frac{2h_0}{g} = 0$

$$t = -\left(-\frac{2v_0}{g}\right) \pm \sqrt{\left(-\frac{2v_0}{g}\right)^2 - 4(1)\left(-\frac{2h_0}{g}\right)}$$

$$t = \frac{\frac{2v_0}{g} \pm \sqrt{\frac{4v_0^2}{g^2} + \frac{8h_0}{g}}}{2} = \frac{v_0}{g} \pm \sqrt{\frac{v_0^2}{g^2} + \frac{2h_0}{g}}$$

Now $v = v_0 - gt = v_0 - g\left(\frac{v_0}{g} \pm \sqrt{\frac{v_0^2}{g^2} + 2\frac{h_0}{g}}\right)$ must choose + square root since ball hits after max. height

Thus $v = v_0 - v_0 - g\sqrt{\frac{v_0^2}{g^2} + \frac{2h_0}{g}}$ OR $v = \sqrt{v_0^2 + 2gh_0}$

Alternatively can write this as $v^2 - v_0^2 = 2gh_0$