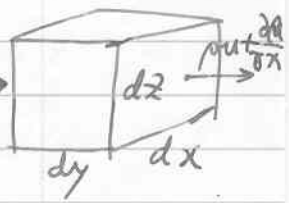
Chapter 6. Solution of Partial Differential Equations

§1. PDE's: ~~classific~~ introduction

(a) Arise when there're more than 1 independent variables (time + space; multiple space coord)

* Mass conservation in fluid flow:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$



or: $$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

* Heat transfer:

$$\rho C_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \quad : \quad 1\text{-D}$$

$$\rho C_p \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) : \quad 3\text{-D}$$

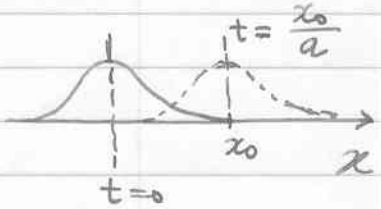
Steady state Heat-transfer: $\nabla^2 T = 0$

Laplace Eqn.

* Wave propagation:

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0$$

Solution: $u = u(x \pm at)$



(b) Classification:

* Order: 1st order, 2nd order
* Linearity: linear: coefficients depend on indep. variables only

quasi-linear: coeffi. contain lower-order derivatives

nonlinear: coeff. contain same (highest) order derivative.

* Characteristics for 2nd-order equations:

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + f(x, y, u) = 0$$

Why group them this way? Behavior different, B.C.

- $b^2 - ac < 0$: elliptic (Multi-D Heat transfer, diffusion process, Navier-Stokes equation.)
- $b^2 - ac = 0$: parabolic (1D unsteady diffusion, heat equation.)
- $b^2 - ac > 0$: hyperbolic (wave equation.)

First order equations: $\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = f(x, t, \rho)$ are known as transport equations, considered hyperbolic.

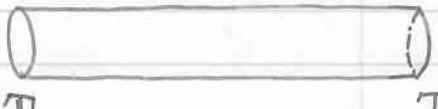
(c) Boundary conditions and initial conditions

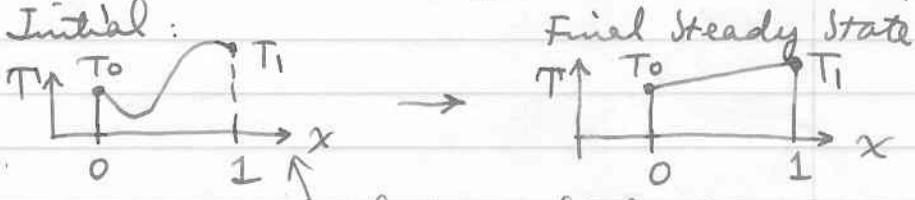
Take $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$ as an example.

* Dirichlet conditions: dependent variables are given as functions of independent variables:

$T(x,t)$: $T(x,0) = f(x)$: initial T profile

$\begin{cases} T(0,t) = f_1(t) \\ T(1,t) = f_2(t) \end{cases}$ 2-point B.V.P.





What's the final solution like?

* Neumann conditions : derivatives of unknown variable is given :

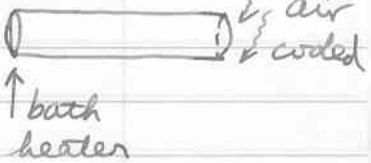
$$k\frac{\partial T}{\partial x}\Big|_{x=0} = \begin{cases} 0 & : \text{insulated} \\ q_0 & : \text{heat flux} \end{cases}$$

can be mixed with a Dirichlet condition on the other boundary (Cauchy Conditions :

Initial condition in Neumann form : velocity of projectile.

* *Robin* conditions : a relationship between unknown variable and its derivatives :

- at left end $x=0$: $T = T_0$ or $k\frac{\partial T}{\partial x} = q_0$



- at right end $x=1$:

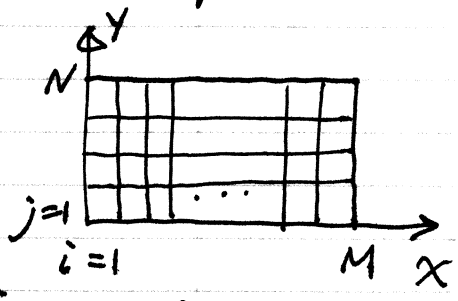
$$k\frac{\partial T}{\partial x} = h(T - T_{air})$$

- Initial Value problem : Domain "open" for at least 1 indep. variable
- Boundary Value Problem : Domain closed for all.

§2. Finite difference methods: generalities

Take 2D heat transfer for example:

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$



(a) Explicit scheme:

Discretization on rectangular domain, using, for example, Euler forward in time and central differencing in space.

$$\left(\frac{\partial T}{\partial t} \right)_{ij}^{n} = \frac{T_{ij}^{n+1} - T_{ij}^{n}}{\Delta t}$$

$$\left(\frac{\partial^2 T}{\partial x^2} \right)_{ij}^{n} = \frac{T_{i+1,j}^{n} - 2T_{ij}^{n} + T_{i-1,j}^{n}}{\Delta x^2}$$

$$\left(\frac{\partial^2 T}{\partial y^2} \right)_{ij}^{n} = \frac{T_{i,j+1}^{n} - 2T_{ij}^{n} + T_{i,j-1}^{n}}{\Delta y^2}$$

For simplicity, assume $\Delta x = \Delta y = \Delta$. Also $S = \frac{\alpha \Delta t}{\Delta x^2}$:

$$T_{ij}^{n+1} = T_{ij}^{n} + \frac{\alpha \Delta t}{\Delta^2} \left[T_{i+1,j}^{n} + T_{i-1,j}^{n} + T_{i,j+1}^{n} + T_{i,j-1}^{n} - 4T_{ij}^{n} \right]$$

$$= (1 - 4S) T_{ij}^{n} + S \left(T_{i+1,j}^{n} + T_{i-1,j}^{n} + T_{i,j+1}^{n} + T_{i,j-1}^{n} \right)$$

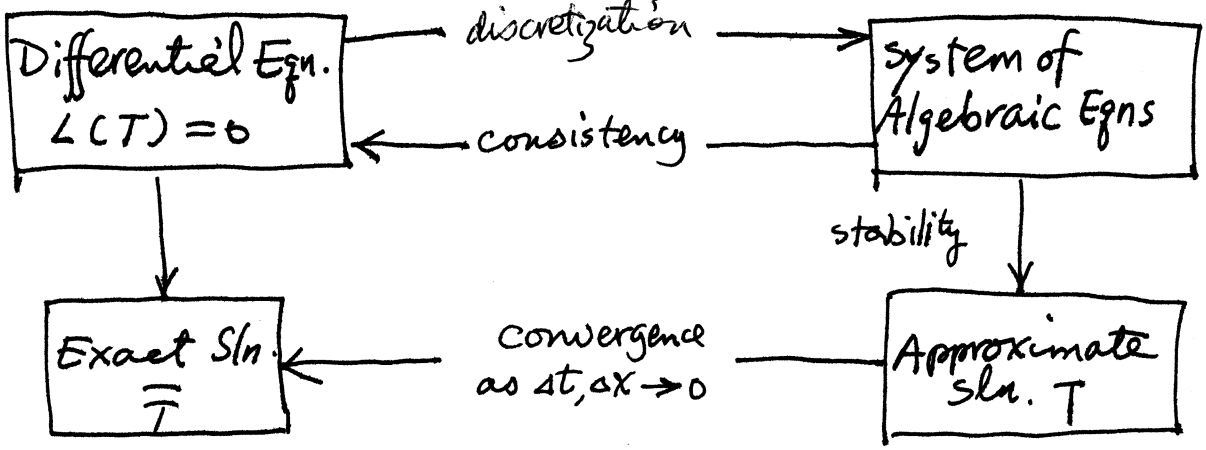
Explicit formula, amenable to time-marching.

(b) Consistency, stability and convergence.

Convergence: with $\Delta t, \Delta \to 0$, numerical solution $\to$ exact solution.

Ultimate goal, but hard to establish theoretically.

Indirect route to convergence:



"Consistency": discretized algebraic equations should be equivalent to PDE at each node in the limit $\Delta x \to 0$.

* Lax Equivalence Theorem: for linear PDE's, Consistency + stability $\Rightarrow$ convergence.

* Check consistency of explicit scheme above. Taylor-Expand exact solution $\bar{T}$ about n, i, j:

$$\bar{T}_{ij}^{n+1} = \bar{T}_{ij}^{n} + \frac{\partial \bar{T}^n}{\partial t}\Delta t + \frac{(\Delta t)^2}{2}\frac{\partial^2 \bar{T}^n}{\partial t^2} + O(\Delta t^3)$$

$$\frac{\bar{T}_{ij}^{n+1} - \bar{T}_{ij}^{n}}{\Delta t} = \frac{\partial \bar{T}^n}{\partial t} + \frac{\Delta t}{2}\frac{\partial^2 \bar{T}}{\partial t^2} + O(\Delta t^2)$$

Similarly: $$\frac{\bar{T}_{i+1,j}^{n} - 2\bar{T}_{ij}^{n} + \bar{T}_{i-1,j}^{n}}{\Delta^2} = \frac{\partial^2 \bar{T}^n}{\partial x^2} + \frac{\Delta^2}{12}\frac{\partial^4 \bar{T}}{\partial x^4} + O(\Delta^4)$$

similar for $\frac{\partial^2 T}{\partial y^2}$. $+ O(\Delta t^2, \Delta^4)$

$$\Rightarrow \left[\frac{\partial \bar{T}}{\partial t}\right]_{ij}^{n} - \alpha\left[\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial y^2}\right]_{ij}^{n} = -\frac{\Delta t}{2}\frac{\partial^2 \bar{T}}{\partial t^2} + \frac{\Delta^2}{12}\left(\frac{\partial^4 \bar{T}}{\partial x^4} + \frac{\partial^4 \bar{T}}{\partial y^4}\right)$$

Truncation error $E_{ij}^n = -\frac{\Delta t}{2}\frac{\partial^2 \bar{T}}{\partial t^2} + \frac{\Delta^2}{12}\left(\frac{\partial^4 \bar{T}}{\partial x^4} + \frac{\partial^4 \bar{T}}{\partial y^4}\right) + \alpha\left(\Delta t^2, \Delta^4\right)$

As $\Delta t \to 0, \Delta \to 0$, $E_{ij}^n \to 0$ for each i,j. $\Rightarrow$ Consistent.

*But consider Dufort-Frankel scheme (1D for simplicity)

$$\frac{T_i^{n+1} - T_i^{n-1}}{2\Delta t} - \alpha \frac{\left[T_{j-1}^n - \left(\overbrace{T_j^{n+1} + T_j^{n-1}}^{2T_j^n}\right) + T_{j+1}^n\right]}{\Delta^2} = 0$$

$$\Rightarrow T_j^{n+1} = \left(\frac{2s}{1+2s}\right)\left(T_{j-1}^n + T_{j+1}^n\right) + \frac{1-2s}{1+2s} T_j^{n-1}.$$

Taylor expansion:

$$\left[\frac{\partial \bar{T}}{\partial t} - \alpha \frac{\partial^2 \bar{T}}{\partial x^2}\right] = -\alpha\left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^2 \bar{T}}{\partial x^2} + O\left(\Delta t^2, \Delta^2\right)$$

Now consistency is compromised unless $\Delta t/\Delta \to 0$ as $\Delta \to 0$.

(c) Implicit scheme.

$$\left(\frac{\partial T}{\partial t}\right)_{ij}^{n+1} = \frac{T_{ij}^{n+1} - T_{ij}^n}{\Delta t}; \quad \nabla^2 T_{ij}^{n+1} = \frac{T_{i+1,j}^{n+1} + T_{i-1,j}^{n+1} + T_{i,j+1}^{n+1} + T_{i,j-1}^{n+1} - 4T_{ij}^{n+1}}{\Delta^2}.$$

$\Rightarrow$ Linear system.

$$T_{i+1,j}^{n+1} + T_{i-1,j}^{n+1} + T_{i,j+1}^{n+1} + T_{i,j-1}^{n+1} - \underbrace{\left(4 + \frac{\Delta^2}{\alpha \Delta t}\right)}_{\beta} T_{ij}^{n+1} = -\frac{\Delta^2}{\alpha \Delta t} T_{ij}^n$$

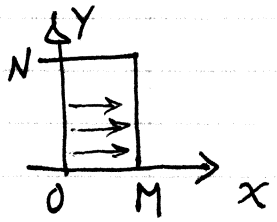
How to order unknowns?

Go along shorter edge (say $M < N$).

$T_{11}, T_{21}, T_{31}, \cdots, T_{M1}, T_{12}, T_{22}, \cdots, T_{M2},$

$\cdots$

$T_{1N}, T_{2N}, T_{3N}, \cdots, T_{MN}$

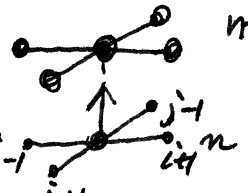


Implicit case: we end up with coupled linear system.

$$T_{i+1,j}^{n+1} + T_{i-1,j}^{n+1} + T_{i,j+1}^{n+1} + T_{i,j-1}^{n+1} - \beta T_{ij}^{n+1} = -\frac{\Delta^2}{\alpha \Delta t} T_{ij}^{n}$$

$$\beta = 4 + \frac{\Delta^2}{\alpha \Delta t}$$

(i) How large is the system? $MN \times MN$ unknowns if all Neumann conditions.



How many FD equations? Only on interior points $\Leftarrow$ must be supplemented by B.C.'s.

Dirichlet $\Rightarrow$ Gives boundary T values, reduces # of unknowns;

Neumann or Robbins $\Rightarrow$ provides additional relationship for boundary points.

(ii) For all Dirichlet BC: unknowns have to be ordered. Which direction? $\Rightarrow$ has to do with bandwidth, choose shorter dimension.

Say: $T_{2,2}, T_{3,2}, \cdots, T_{M-1,2}$

$T_{2,3}, T_{3,3}, \cdots, T_{M-1,3}$

$\vdots$

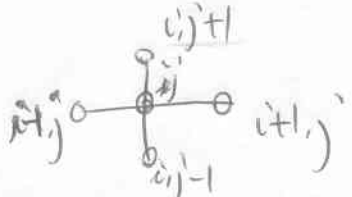
$T_{2,N-1}, T_{3,N-1}, \cdots, T_{M-1,N-1}$

$\Big\}$ $(M-2) \times (N-2)$ unknowns

Let's now write out the linear system:

$$\begin{bmatrix} -\beta & 1 & \cdots & 0 & 1 & 0 & & \\ 1 & -\beta & \ddots & & 0 & 1 & 0 & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & 1 & & \ddots & \ddots & \ddots & \ddots & 0 \\ & 0 & \ddots & & \ddots & \ddots & \ddots & 1 \\ 0 & & \ddots & 0 & 1 & 0 & 1 & -\beta \end{bmatrix} \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} \underline{T} = \begin{bmatrix} \text{B.C.} \\ \\ \\ -\frac{\Delta^2}{\alpha \Delta t} T_{ij}^{n} \\ \\ \\ \\ \text{B.C.} \end{bmatrix}$$

Will be different for Neumann or other conditions, or combinations thereof.



$$\begin{pmatrix} -\beta & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & & 0 \\ 1 & -\beta & 1 & \cdots & 0 & 1 & 0 & & & \\ & \ddots & \ddots & \ddots & & \ddots & \ddots & \ddots & & \\ 1 & 0 & \cdots & 1 & -\beta & 1 & & 1 & & \\ 0 & 1 & \cdots & & 1 & -\beta & 1 & \cdots & 1 & \\ & \ddots & \ddots & & & \ddots & \ddots & \ddots & \ddots & 1 \\ & & & & & & & & & 1 \\ 1 & 0 & \cdots & 0 & 1 & -\beta & 1 & & & \\ & 1 & 0 & \cdots & 0 & 1 & -\beta & & & \end{pmatrix}$$

minimize bandwidth

Band Width M

Next line

neighboring

$T_{i,j}$ $T_{i+1,j}$ $T_{i,j+1}$ $T_{i,j-1}$ $T_{i+1,j}$

(c) Solving linear or nonlinear systems.

(Newton's method again!)