The Strengthened Hanna Neumann Conjecture I: A Combinatorial Proof*

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Abstract

We prove the Strengthened Hanna Neumann Conjecture, using a common “fibre product in graphs” formulation of the conjecture. Our original approach used sheaf theory, although here we give a proof with no reference to sheaf theory. Rather, we translate the sheaf theory into combinatorial terms. This gives us an equivalent formulation of the conjecture in terms of “balancing functions.” The balancing function approach, like the sheaf approach, allows for a type of inductive attack on the conjecture. Our proof uses the balancing function approach, the fibre product approach, and previously settled cases of the conjecture.

1 Introduction

Hanna Neumann conjectured ([Neu56, Neu57]) that if $H, K$ are nontrivial finitely generated subgroups of a free group, $F$, then

$$\text{rank}(H \cap K) - 1 \leq \lceil \text{rank}(H) - 1 \rceil \lceil \text{rank}(K) - 1 \rceil.$$
The main goal of this paper is to prove a strengthened form of this conjecture, known as the Strengthened Hanna Neumann Conjecture (or SHNC) and described below; this conjecture first studied and proven to hold for numerous pairs \((H, K)\) by Walter Neumann ([Neu90]).

One main idea we use involves sheaf theory on graphs; however, in this paper we give a complete proof of the SHNC without sheaf theory. That is, we have converted the sheaf theory ideas into purely combinatorial ideas. The sheaf theoretic approach will be described in [Frib].

Our approach uses a beautiful reformulation of the SHNC and the original Hanna Neumann Conjecture (or HNC) in terms of fibre products in graph theory; according to Dicks ([Dic94]) the fibre product graph formulation was used by Howson ([How54]) and later expressed by Imrich ([Imr77a, Imr77b]) and others ([Sta83, Ger83, Ser83, Nic85]). This paper and its sequel ([Frib]) will give a number of different ways of viewing the HNC and SHNC. We will describe why these different ways may lead to alternate and possibly simpler proofs than the ones we currently have.

The main views of the SHNC we use in this paper and its sequel are graph theoretic, namely (1) the fibre product question, (2) the existence of balancing functions, and (3) the vanishing of twisted homology groups of \(\rho\)-kernels. All three have their strengths and weaknesses. For example, one of the weaknesses of (3) is that it requires developing a theory of twisted homology groups on graphs that, as we write this article, is not entirely complete; furthermore the reader may view this theory as somewhat burdensome. Formulation (1) shows that if the SHNC holds in certain cases, it is easily seen to hold when we take disjoint unions and coverings of the known cases; this is not clear in formulations (2) and (3). A weakness of (1) is that it is difficult to use when taking various “inductive approaches,” whereas (2) and (3) are amenable to such approaches; this is essential to our proof, so we will elaborate on this a bit.

Let us now give an example of what is meant by an inductive approach, one analogous to the approach we take, although simpler. Following the fibre product approach, the SHNC is shown to be equivalent to proving that

\[
\rho(H \times_{B_2} K) \leq \rho(H)\rho(K),
\]

where \(H, K\) are directed graphs of a certain type with a certain edge colouring, \(H \times_{B_2} K\) is a certain product based on this colouring, and \(\rho\) measures the “reduced cyclicity” of a graph; this concepts are defined precisely later.
in this section. An inductive approach might find subgraphs, $H_1$ and $H_2$, of $H$ such that $H_1 \cup H_2 = H$; then we can prove the conjecture for $H$ based on that for $H_1, H_2$, provided that $H_1 \cap H_2$ is sufficiently "benign" so that the disjoint union, $H_1 \sqcup H_2$, strongly reflects $H$ in the SHNC.

The problem here concerns the notion of "benign" for $H_1 \cap H_2$. There is a natural notion of benign for the SHNC, which is a graph all of whose connected components have Euler characteristic zero (i.e., have the same number of edges as vertices, i.e., homotopically are cycles). This notion seems to work quite well for approaches (2) and (3), balancing functions and twisted homology. It seems quite awkward for (1), in that if $H_1 \cap H_2$ is benign, then $\rho(H) = \rho(H_1) + \rho(H_2)$, yet there are generally many $K$ for which $(H_1 \cap H_2) \times_{B_2} K$ is not benign. This means that if we fix $K$, while in principle we can reconstruct $H \times_{B_2} K$ from the $H_i \times_{B_2} K$, $i = 1, 2$, and $(H_1 \cap H_2) \times_{B_2} K$, it is not clear how derive $\rho$ estimates without digging deeply into this reconstruction.

It is quite amusing to note that matters are reversed when it comes to $H_1 \sqcup H_2$. In approach (1), the fibre product formulation, it is immediate that if the $H_i$ satisfy the SHNC for one $K$ or for all $K$ then the same is true of $H_1 \sqcup H_2$. However, for approach (2), we have no direct way of seeing that if the $H_i$ have balancing functions then so does $H_1 \sqcup H_2$. For (3), while homology of direct sums of sheaves is the direct sum of the individual homology groups, this is not clearly true for the $\rho$-kernels of the sheaves\(^1\). So in this paper we use the strengths of both approaches.

It is helpful to see that the notion of benign we use makes sense in the fibre product approach, (1), at least philosophically. Namely, say that $H$ is *universal for the SHNC* if for any $K$ the SHNC holds for $H, K$. We will show equivalences between statements (1) and (2) below, and that (3) implies\(^2\) (1) and (2) for a given graph, $H$: (1) $H$ is universal for the SHNC, (2) $H$ has a balancing function, and (3) $H$ has a $\rho$-kernel with vanishing twisted homology groups. So induction on $H$ is roughly equivalent in any of these views; however, when we seek to show the SHNC holds for $H, K$ by inducting on $H$, we need to consider the information this holds over all $K$ while inducting on $H$, rather than fixing a given $K$.

Now let us briefly describe our approach, that combines previous results.

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\(^1\)An earlier version of this work was circulated ([Fri]) that contained just this oversight, and incorrectly claimed a proof of the SHNC. See [Frib] for a more complete account.

\(^2\)Actually we hope to show that (3) is equivalent to (1) and (2) in [Frib]
on the SHNC, the fibre product formulation, and balancing functions. We shall show that if $H$ is universal for the SNHC, then it has a balancing function. Now Tardos (see [Tar92]) has shown that $H$ is universal for the SHNC if $\rho(H) = 1$ (and Dicks and Fromanek, in [DF01] if $\rho(H) = 2$); hence any disjoint union of such graphs satisfies the SHNC. Hence any disjoint union of graphs with $\rho = 1$ has a balancing function. This is what we use from previous results and the fibre product formulation. Next we give the steps involving balancing function and our type of “induction.”

Our graphs, $H$, may be taken to consist only of vertices of degree two and three (using [JKM03]). This will mean that it has $2\rho(H)$ vertices of degree three; we shall divide the vertices in pairs and join each pair via a path in $H$ in a certain way; we shall call these paths “canonical.” Each path, in isolation, becomes a graph with $\rho = 1$ after we add self-loops to the endpoints. Taking the disjoint union of these paths plus self-loops gives us a graph, $M$. First, we know $M$ has a balancing function. Second, we shall show that $M$ is “close enough” to $H$ to induce a function that functions just like a balancing function on $H$. This will complete the proof of the SHNC. The canonical paths give us our form of “induction” in $H$, and can alternatively be viewed as reducing general $K$ to the case $\rho(K) = 1$.

Before giving a precise statement of our main theorems, we wish to make a few additional remarks on balancing functions.

It is instructive to note that the balancing function on $M$ pushes forward to a function on $H$ that is not literally a balancing function. This is analogous to the fact that if we decompose $H$ as $H_1 \cup H_2$, with $H_1 \cap H_2$ benign, $H_1 \cap H_2$ still enters into the calculation somewhere. In sheaf theory, $H_1 \cap H_2$ enters in a trivial manner into a long exact sequence. In balancing functions, we get a function that behaves like a balancing function, but has some “excess” part due to the overlap (analogous to $H_1 \cap H_2$) that is irrelevant in the end (the excess part is the excess weight in a “superweight;” see especially Theorem 3.2).

We wish to reiterate that while we have successfully pushed forward a balancing function from $M$ to $K$ as above, there are almost no other cases where we can build a balancing function out of other balancing functions. Given a covering map $K' \to K$, we don’t know how to directly construct a balancing function on $K'$ from $K$ and vice versa. We don’t know how to directly a balancing function on $K \amalg K'$ from ones on $K$ and $K'$ and vice versa. Even for the special case $\rho(K) = 1$, we do not know how to construct balancing functions except in certain special cases; this is why we need the
result of Tardos.

The task of finding a balancing function can be viewed as a linear program, and as such we have found balancing functions in some $\rho = 1$ and related examples that eluded us by hand. It is interesting that we apply the duality theory of linear programming to show that if a balancing function for a graph, $H$, does not exist, then the dual linear program provides us with data to construct a $K$ for which $H, K$ does not satisfy the SHNC.

In case the balancing function can be taken to be zero for a graph, we say the graph is “balanced.” The fact that balanced graphs satisfy the SHNC follows from Walter Neumann’s work ([Neu90], also discussed in [JKM03]). So one could view this paper’s contribution as a consideration of non-zero balancing functions; this extends the SHNC already known to hold on balanced graphs.

We discovered the combinatorial notions like that of a “balancing function” and “balanced graph” via sheaf theory, where these notions are quite simple and intuitive: a “balancing function” relates to the vanishing of twisted homology groups of a sheaf we call a $\rho$-kernel, and a “balanced graph” is one in which the $\rho$-kernels are the discrete analogues of vector bundles. This will be explained in the sequel, [Frib]. From a combinatorial point of view these notions seemed at first a bit ad hoc, but it is not hard to give them a somewhat intuitive and purely combinatorial meaning.

1.1 The Strengthened Hanna Neumann Conjecture

Howson (in [How54]) showed that if $H, K$ are nontrivial finitely generated subgroups of a free group, $F$, then $H \cap K$ is finitely generated, and moreover that

$$\text{rank}(H \cap K) - 1 \leq 2 \text{rank}(H) \text{rank}(K) - \text{rank}(H) - \text{rank}(K). \quad (1)$$

Then Hanna Neumann, in [Neu56, Neu57], gave the estimate

$$\text{rank}(H \cap K) - 1 \leq 2[\text{rank}(H) - 1][\text{rank}(K) - 1]. \quad (2)$$

She conjectured that one could remove the factor of 2 in the above estimate, which has become known as the Hanna Neumann Conjecture (or HNC); this conjecture can be rewritten as

$$\text{rk}_{-1}(H \cap K) \leq \text{rk}_{-1}(H) \text{rk}_{-1}(K), \quad (3)$$
where $\text{rk}_n(G)$ denotes $\max(\text{rank}(G) + n, 0)$.

A stronger conjecture was proposed and studied in [Neu90], known as the Strengthened Hanna Neumann Conjecture (or SHNC), which conjectures that moreover

$$\sigma(H, K) \leq \text{rk}_{-1}(H) \text{rk}_{-1}(K),$$

(4)

where

$$\sigma(H, K) = \sum_{x \in H \setminus F/K} \text{rk}_{-1}(H \cap x^{-1}Kx),$$

the summation being over the double coset, $H \setminus F/K$, representatives, $x$; discarding all $x$’s but the identity recovers the original Hanna Neumann Conjecture. We say that the HNC or SHNC, respectively, holds for $(H, K)$ if equation (3) or (4), respectively, holds; we say that $H$ is universal for the HNC or SHNC, respectively, if for any $K$ the same conjecture holds for $(H, K)$.

Much progress has been made on these conjectures, see, for example, [Bur71, Ger83, Sta83, Neu90, Tar92, Dic94, Tar96, Iva99, Arz00, DF01, Iva01, Kha02, MW02, JKM03, Neu07, Eve08]; both conjectures have been proven in many cases, and equation (2) has been improved. To date, the best general bound for the HNC or SHNC is

$$\sigma(H, K) \leq \text{rk}_{-1}(H) \text{rk}_{-1}(K) + \text{rk}_{-3}(H) \text{rk}_{-3}(K),$$

(5)

due to Dicks and Formanek ([DF01]) based on the “amalgamated graph conjecture” formulation of Dicks ([Dic94]), representing a series of improved estimates on $\sigma(H, K)$ of starting with Walter Neumann’s original $2 \text{rk}_{-1}(H) \text{rk}_{-1}(K)$ in [Bur71, Neu90, Tar92, Tar96]. Results on the HNC or SHNC include, roughly speaking:

1. the SHNC holds if one of the groups is of rank at most 3, in view of equation (5), with rank 2 settled earlier by Tardos ([Tar92]);

2. the SHNC holds if at least one of $H, K$ is positively generated (see [Kha02, MW02, Neu07]);

3. the SHNC holds provided that either $K$ or $H$ do not have a “large” number of degree three vertices of one “type” (see [Neu90], also discussed in [JKM03]);

4. in particular, for “most” $H$ the SHNC holds for $(H, K)$ for all $K$;
5. the HNC holds either for $H$ and $K$ or for $H$ and $K'$, for any $H, K$ that are subgroups of $F_2$, the free group on two generators, where $K'$ is obtained from $K$ by interchanging the first and second generators of $F_2$ (see [JKM03]); (To prove either the HNC or SHNC, it suffices to prove it for subgroups of $F_2$.)

6. the SHNC holds if at least one of $H, K$ has at most three of the four “types” of degree two vertices incident upon both colours (see [MW02], Corollary 3.2); There are generalizations of the HNC, such as in [DI08]. There are equivalent formulations of the HNC and SHNC, such as the fibre product in graphs question, and the amalgamated graph conjecture of Dicks ([Dic94]). It is also known that the SHNC is related to the coherence problem in one-relator groups ([Wis05]).

1.2 The Fibre Product Graph Theoretic Formulation

From this point on we shall use the beautiful formulation of the HNC and SHNC (see [Sta83], but also [How54, Imr77a, Imr77b, Ger83, Neu90] and other references in [Dic94]) in terms of directed graphs, which we now describe. We must allow directed graphs to have multiple edges and self-loops; so in this paper a directed graph consists of tuple $G = (V_G, E_G, t_G, h_G)$ where $V_G$ and $E_G$ are sets—the vertex and edge sets—and $t_G: E_G \to V_G$ is the “tail” map and $h_G: E_G \to V_G$ the “head” map. Throughout this paper a graph is assumed to be finite, i.e., the vertex and edge sets are finite.

To state the fibre product formulation of the HNC and SHNC, recall that fibre products exist for directed graphs (see [Fri93], or [Sta83], where fibre products are called “pullbacks,” for example), and the fibre product, $K = G_1 \times_G G_2$, of morphisms $\pi_1: G_1 \to G$ and $\pi_2: G_2 \to G$ has

$$
V_K = \{(v_1, v_2) \mid v_i \in V_{G_i}, \pi_1 v_1 = \pi_1 v_2\},
$$

$$
E_K = \{(e_1, e_2) \mid e_i \in E_{G_i}, \pi_1 e_1 = \pi_1 e_2\},
$$

$$
t_K = (t_{G_1}, t_{G_2}), \quad \text{and} \quad h_K = (h_{G_1}, h_{G_2}).
$$

We say that $\pi: K \to G$ is a covering map (respectively, étale\(^3\)) if for each $v \in V_K$, $\pi$ gives a bijection (respectively, injection) of incoming edges of

\(^3\)Stallings, in [Sta83], uses the term “immersion;” see [Frib] for one explanation.
v (i.e., those edges whose head is v) with those of π(v), and a bijection (respectively, injection) of outgoing edges of v and π(v). If π: K → G is a covering map and G is connected, then the degree of π, denoted [K : G], is the number of preimages of a vertex or edge in G under π (which does not depend on the vertex or edge).

By a bicoloured digraph, or simply a bigraph, we mean a directed graph \( G = (V_G, E_G) \), such that each edge is labelled either “1” or “2.” The edge labelling is equivalent to partitioning \( E_G \) into two sets, \( E_{G,i} \), for \( i = 1, 2 \), with \( E_{G,i} \) being the edges of label \( i \). It is also equivalent to giving a directed graph homomorphism \( \pi: G \to B_2 \), where \( B_2 \) is the graph with one vertex and two self-loops, one labelled “1” and the other “2.”

Given a bidigraph, \( G \), we view \( G \) as an undirected graph (by forgetting the directions along the edges), and let \( h^i(G) \) denote the \( i \)-th Betti number of \( G \), and \( \chi(G) \) its Euler characteristic; hence

\[
h^0(G) - h^1(G) = \chi(G) = |V_G| - |E_G|.
\]

Let \( \text{conn}(G) \) denote the connected components of \( G \), and let

\[
\rho(G) = \sum_{X \in \text{conn}(G)} \max(0, h^1(X) - 1),
\]

and

\[
\rho'(G) = \max_{X \in \text{conn}(G)} \left( \max(0, h^1(X) - 1) \right).
\]

The HNC is shown in [Sta83] to be equivalent to

\[
\rho'(H \times_{B_2} K) \leq \rho(H)\rho(K)
\]

for all étale \( H \) and \( K \) over \( B_2 \); the SHNC is equivalent to

\[
\rho(H \times_{B_2} K) \leq \rho(H)\rho(K)
\]

for all étale \( H \) and \( K \) over \( B_2 \). We shall work with this form of the SHNC. Again, we say the HNC or SHNC holds for a pair of étale bigraphs, \((H, K)\), over \( B_2 \) if the corresponding equation holds for that particular \( H \) and \( K \); and again, we say that \( H \) is universal for the HNC or SHNC, respectively, if for any \( K \) the conjecture holds for \((H, K)\).

To help understand progress on the HNC and SHNC in graph theoretic terms, the groups \( H, K \) in original formulation correspond to graphs \( H', K' \);
viewing $H, K$ as subgroups of the free group $F_2$ (which can always be done) is equivalent to giving étale morphisms $H' \to B_2$ and $K' \to B_2$; the rank of the groups in the original formulation corresponds to the first Betti number of the corresponding graph in the fibre product formulation; finally $H \cap K$ represents one connected component of $H' \times_{B_2} K'$, and the sum of $H \cap x^{-1}Kx$ represents all of $H' \times_{B_2} K'$. Now we can view all the results on the HNC and SHNC stated earlier as graph theoretic results.

**Theorem 1.1** The Strengthened Hanna Neumann Conjecture holds. That is, if $G \to B_2$ and $K \to B_2$ are two étale digraphs over $B_2$, then

$$\rho(G \times_{B_2} K) \leq \rho(G) \rho(K).$$

The rest of this section is a brief outline of our proof of Theorem 1.1. We remind the reader that our results suggest that a number of other techniques may yield alternate proofs.

### 1.3 Our Proof and Balancing Functions

To prove Theorem 1.1 we will define a “balancing function.” It is easier and instructive to start by defining a degenerate case, where a graph is “balanced” (without a balancing function).

**Definition 1.2** Let $K$ be an étale bigraph. Let $A_{1h} = A_{1h}(K)$ be the vertices $v \in V_K$ that are the head of some edge of colour 1; similarly define $A_{1t}, A_{2h}, A_{2t}$. We say that $K$ is balanced if for all $U \subset V_K$ we have

$$\sum_{i=1}^{2} \sum_{j=h,t} |U \cap A_{ij}|_{\rho(K)} \leq 2|U|_{\rho(K)}, \quad (6)$$

where for a set, $S$, and non-negative integer, $m$ we set $|S|_m = \max(0, |S| - m)$.

**Theorem 1.3** If $K$ is a balanced bigraph, then $K$ is universal for the SHNC.

The “balanced” condition corresponds to a certain sheaf, a $\rho$-kernel, being a vector bundle (more or less); see [Frib]. We can give a fairly intuitive combinatorial description, although our description may seem more intuitive after reading the proof of Theorem 1.3; roughly speaking, $\rho(K \times_{B_2} K_1)$ is given by the maximum of $-\chi(H)$ over all subgraphs, $H$, of $K \times_{B_2} K_1$ (as
such $H$ will be the union of all components of $K \times_{B_2} K_1$ with negative or non-positive Euler characteristic); over each vertex of $K_1$ we have its fibre in $H$, which can be viewed as a subset of $V_K$; similarly for edge fibres; it turns out that if $K$ were not universal, the $K_1$ with a minimal number of edges and vertices would have all edge and vertex fibres of size at least $\rho$. Furthermore, if we subtract $\rho$ from each edge and vertex fibre in $-\chi(H)$, we get exactly $\rho(K)\rho(K_1)$. So if the balanced condition holds, then what’s left in $-\chi(H)$ is at most zero, i.e.,

$$\rho(K \times_{B_2} K_1) = -\chi(H) \leq \rho(K)\rho(K_1).$$

Theorem 1.3 proves universality for the SHNC only in the case of a balanced bigraph. For example, this proves universality of $K$ provided that each $A_{ij}$ contains at most $\rho$ vertices of degree three or four; this was established by Walter Neumann in [Neu90]. A stronger form of the above theorem is required for our proof of the SHNC; it involved “balancing functions,” which are a bit cumbersome to describe. The balanced bigraph case is the degenerate case when the balancing function can be taken to be zero. So the “balancing function” is probably the new combinatorial device in our proof of the SHNC, and we now describe them.

With notation as in the above theorem and definition, there is a natural bijection $\iota_{1,th}: A_{1t} \to A_{1h}$ which takes the tail of an edge of colour 1 to its head; we also have the inverse bijection $\iota_{1,ht}$, and similarly defined $\iota_{2,th}, \iota_{2,ht}$. Let $\mathcal{P}(A_{1t})$ denote the power set (i.e., the set of all subsets) of $A_{1t}$, and consider a function, $f: \mathcal{P}(A_{1t}) \to \mathbb{R}$. For any $U \subset V_K$, we can view $U \cap A_{1t}$ as a subset of $A_{1t}$, and it makes sense to write $f(U \cap A_{1t})$; also $U \cap A_{1h}$ can be viewed as a subset of $A_{1h}$, and therefore $\iota_{1,ht}(U \cap A_{1h})$ as a subset of $A_{1t}$. So any $f: \mathcal{P}(A_{1t}) \to \mathbb{R}$ gives rise to a function $g: \mathcal{P}(V_K) \to \mathbb{R}$

$$g(U) = f(U \cap A_{1t}) - f(\iota_{1,ht}(U \cap A_{1h}));$$

such a function, $g = g_f$, is called a borrowing arrangement through colour 1. Since $A_{1t}, A_{1h}$ are in bijection, we would get the same class of functions by taking $f: \mathcal{P}(A_{1h}) \to \mathbb{R}$ and considering

$$f(U \cap A_{1h}) - f(\iota_{1,th}(U \cap A_{1t})).$$

When equation (7) holds we shall say that $g$ arises from $f$; we shall say that $g$ is Lipschitz if it arises from an $f$ that satisfies

$$|f(S) - f(T)| \leq |S \setminus T|$$
for all $T \subset S \subset A_{1t}$, and pseudo-Lipschitz if the same is true when we limit ourselves to $T$ with $|T| \geq \rho(K)$. Borrowing arrangements through colour 1 are closed under addition. We similarly define borrowing arrangements through colour 2.

**Definition 1.4** Let notation be as in the previous paragraph. A borrowing function on $K$ is any real valued function defined on subsets of $V_K$ that is the sum of a borrowing arrangement through colour 1 and one through colour 2. It is pseudo-Lipschitz or Lipschitz, respectively, provided that the two borrowing arrangements can be chosen to be so.

**Definition 1.5** Let $K, A_{ij} = A_{ij}(K)$ be as in Definition 1.2. We say that $K$ is balanceable if there exists a pseudo-Lipschitz borrowing function, $f$, on $K$ such that for all $U \subset V_K$ we have

$$f(U) + \sum_{i,j} |U \cap A_{ij}|_{\rho} \leq 2|U|_{\rho},$$

where $\rho = \rho(K_1)$. If so, then such a function, $f$, is called a balancing function.

Using the fact that $|W|_m = |W| - m + \max(0, m - |W|)$, we may rewrite equation (8) as

$$f(U) + \sum_{u \in U} (\deg(u) - 2) + \sum_{i,j} \max(0, \rho - |U \cap A_{ij}(K)|)$$

$$\leq 2\rho + 2\max(0, \rho - |U|).$$

This is sometimes more convenient.

**Theorem 1.6** If $K$ is a balanceable étale bigraph, then $K$ is universal for the SHNC.

In sheaf theory this says when a $\rho$-kernel fails to be a vector bundle, it can still have vanishing twisted homology groups provided that the vertices that fail the vector bundle condition can “borrow” from others that have this condition “in excess.” In combinatorial terms of $H \subset K \times_{B_2} K_1$ as before, this means that the vertex fibres of $K_1$ in $H$ are allowed to borrow from each other.
At this point we are ready to describe the “inductive” step. Namely, we shall prove the existence of a borrowing function for every étale bigraph, $K$, by establishing their existence in the case $\rho(K) = 1$, and then combining them by a type of “pairing” of degree three vertices when $\rho(K) > 1$. Even for $\rho(K) = 1$, we do not have a general description of a borrowing function for $K$. However, finding a borrowing function amounts to a certain linear program. The linear program has allowed us to find a borrowing function with a computer where we have been unable to find it by hand (see Section 2). Moreover, the duality theory of this linear program enables us to prove the following converse to Theorem 1.6.

**Theorem 1.7** If $K$ is an étale bigraph that is universal for the SHNC, then $K$ is balanceable with a balancing function that is Lipschitz.

There is a one final crucial fact about balanceability that we need.

**Definition 1.8** For a set, $S$, an $f : \mathcal{P}(S) \to \mathbb{C}$, and a subset $S' \subset S$, we say that $f$ is definable on $S'$ if $f$ depends only on $S'$; i.e., for all $U \subset S$, $f(U) = f(U \cap S')$. In equation (7), we say that $g$ (which is a borrowing arrangement through colour 1) is definable on $E' \subset E_K$ if $f$ is definable on $t(E'_1)$ where $E'_1 = E' \cap E_{K,1}$ are the edges of $E'$ of colour 1. Similarly for borrowing arrangements through colour 2. Similarly we say that a borrowing function, $f$, is definable on $E'$ if it is the sum of a borrowing arrangement through colour 1 and one through colour 2 that are each definable on $E'$; this is equivalent to saying that $f$ arises as a borrowing function on the sub-bigraph $K' = (V_K, E', t_{K|E'}, h_{K|E'})$.

**Theorem 1.9** Let $K$ be a balanceable bigraph. Then $K$ is balanceable by a balancing function definable on $E_K$ with the self-loops discarded. Moreover, we may additionally discard all cycles consisting of edges of a single colour.

The result of Tardos (or, later, Dicks-Formanek) shows that if $\rho(K) = 1$, then $K$ is universal for the SHNC. It follows that the disjoint union of graphs with $\rho = 1$ is universal for the SHNC. Hence any disjoint union of graphs with $\rho = 1$ is balanceable.

At this point we assume (d’après Jitsukawa-Khan-Myasnikov) that $K$ has all vertices of degree two or three, and we connect the degree three vertices in pairs with paths. Based on these paths we form a disjoint union of graphs of $\rho = 1$, and show that the balancing function of the disjoint union gives rise to a function on $K$ that is balancing modulo a harmless term (arising from a “superweight” as in Definition 3.1; see Section 4).
1.4 Structure of This Paper and Acknowledgments

We now summarize the organization of this paper. In Section 2 we give examples of balancing functions for various graphs, especially paths with self-loops at the endpoints; then we explain the idea of “canonical” paths and a few simplifications one can make to the SHNC. Finally we explain the “uninverse” principle, useful both in the construction balancing functions and as a possible approach to a much simpler proof of the SHNC. In Section 3 we prove Theorems 1.3, 1.6, 1.7, and 1.9; we also discuss covering maps in relation to balanced graphs and balancing functions. In Section 4 we finish the proof of the SHNC. In Section 5 we make some concluding remarks. As mentioned before, in a sequel to this paper, [Frib], we will develop a sheaf homology theory for graphs and discuss how to view the SHNC in terms of homology and a sheaf we call a ρ-kernel; we will also give some theorems in linear algebra and vanishing of twisted homology on graphs that may be of interest independent of the SHNC.

We wish to thank Laurent Bartholdi, Pierre Pansu, Luc Illusie, Sadok Kallel, and Aaron Friedman for discussions. In particular, Laurent Bartholdi described the Hanna Neumann Conjecture to us and discussed with us a number of early efforts. Luc Illusie discussed with us our earlier ideas on the conjecture using the sheaf theory, duality, and cohomology; he asked us about the case where $H \to B$ or $K \to B$ is a covering map or Galois, leading to our applying Galois theory here. Discussions with Aaron Friedman particularly helped with the exposition and a number of details. Our understanding of sheaves on finite spaces grew out of an attempt (unsuccessful to date) to tackle circuit complexity with cohomology over certain Grothendieck topologies; we wish to thank the many people who have discussed these matters with us (acknowledged in in [Fri05, Fri06]).

We also wish to thank Goulnara Arjantseva, Bernt Everitt, Daniel Wise, Igor Mineyev, Warren Dicks, and Richard Kent for comments on an earlier paper on this line of research ([Fria]); we have incorporated these comments into this paper. We extend a special thanks for Warren Dicks and Richard Kent, correspondences with whom lead to discovering an error in the earlier paper.

We thank many people over the last year for encouragement to work to overcome the mistake in [Fria], including family, mathematicians, people from the Beanery Café, the Birdcoop, and Latin Funk Dance, and other friends.

4 Or “un-inverse.”
(not that these groups are mutually exclusive).

Finally, we thank Alain Valette and the Centre Bernoulli at the EPFL for hosting us during a programme on limits of graphs, where we met Bartholdi and Pansu and began this work.

2 Examples and Starting Considerations

In this section we give some examples of balancing functions on some graphs, $K$, with $\rho(K) = 1$. We introduce the “uninverse principle,” which can be used to construct balancing functions for a large family of graphs, $K$, with $\rho(K) = 1$. Furthermore we describe an idea called “pairing” or “canonical paths” which (1) will be crucial to our proof of the SHNC, and (2) has some hope of giving a much simpler proof of the SHNC when combined with the uninverse principle.

2.1 Paths and Types

We wish to pause for some very useful definitions that are a bit formal.

If $G$ is a bigraph and $v \in V_G$, then the “type” of $v$ refers to which of the four $A_{ij}(G)$ $v$ belongs. For example, if $v$ is of degree four, then $v$ belongs to all the $A_{ij}$, and we say that $v$ is of type $\{A_{1h}, A_{1t}, A_{2h}, A_{2t}\}$. Historically most authors use $a$ for $A_{1h}$, $a^{-1}$ for $A_{1t}$, $b$ for $A_{2h}$ and $b^{-1}$ for $A_{2t}$, and we shall also do so at times. There are four possible types for a vertex of degree three, corresponding to the four subsets of order three of $\{a, a^{-1}, b, b^{-1}\}$.

Roughly speaking, an undirected walk in a directed graph is a walk where we are allowed to traverse edges in either direction. The only subtlety is that when we walk along a self-loop we must specify the orientation we are taking; we will not use walks that traverse self-loops, but for completeness we give a precise discussion below. An undirected walk in a bigraph determines a colour pattern, one letter from $\{a, a^{-1}, b, b^{-1}\}$, for each step of the walk. The rest of this subsection just makes the above precise.

If $G$ is a directed graph, then an oriented edge is the formal expression consisting of an edge with an exponent of a sign, either $+$ or $-$; for $e \in E$ we refer to $e^+$ as “$e$ traversed forward (or positively)” and $e^-$ as “$e$ traversed backwards (or negatively).” By an undirected path in a directed graph we mean an alternating sequence of vertices and oriented edges,

$$v_0, e_1^{\sigma_1}, v_1, \ldots, e_k^{\sigma_k}, v_k$$
such that for $i = 1, \ldots, k$ we have $te_i = v_{i-1}$ and $he_i = v_i$ provided that $\sigma_i = +$, and otherwise $he_i = v_{i-1}$ and $te_i = v_i$. If $G$ is a bigraph, then each undirected path gives its \textit{colour pattern}, which a word $w = w_1 \ldots w_k$ of $k$ letters in $\{a, a^{-1}, b, b^{-1}\}$ given by letting $w_i$ be $a$ if $e_1$ is of colour 1 and $\sigma = +$, $a^{-1}$ if the same but $\sigma = -$, and similarly for colour 2.

If the definition of undirected path given above seems somewhat ad hoc, here is a more natural but more involved explanation. An \textit{undirected graph} (or \textit{graph}) is a directed graph, $G$, with an involution, $\iota$, of its edges (i.e., $\iota : E_G \to E_G$ such that $\iota^2 = \text{id}$) such that $t \iota = h$. To each directed graph, $G$, we associate an undirected graph, $G'$, by doubling its edges in reverse orientation, i.e., such that $V_{G'} = V_G$, $E_{G'} = E_G \parallel E_{G'}$, with $t_{G'} = (t_G, h_G)$, $h_{G'} = (h_G, t_G)$, and $\iota$ takes an edge in the first component of $E_G \parallel E_{G'}$ to its copy in the second, and vice versa. An \textit{undirected morphism} from $G$ to $K$ of directed graphs is a morphism of $G'$ to $K'$ of their associated undirected graphs. The \textit{path of length $k$} is the graph with vertex set $\{0, 1, 2, \ldots, k\}$ and one edge from $i$ to $i + 1$ for $i = 0, \ldots, k - 1$. An \textit{undirected walk of length $k$} in a directed graph, $G$, is an undirected morphism from the path of length $k$ to $G$; hence this is the same as giving alternating sequence of vertices and edges,

$$v_0, e_1, v_1, \ldots, e_k, v_k$$

such that for $i = 1, \ldots, k$ we have $\{te_i, he_i\} = \{v_{i-1}, v_i\}$, and if $e_i$ is a self-loop, then we specify whether it is traversed “forward” or “backward;” this is equivalent to our first definition of undirected walk.

\subsection*{2.2 Fundamental Examples of Balancing Functions}

Let $w$ be a non-backtracking or irreducible word over $\Pi = \{a, a^{-1}, b, b^{-1}\}$; i.e., $w = w_1 w_2 \ldots w_k$ with $w_i \in \Pi$ and $w_i \neq w_{i+1}^{-1}$ for $i = 1, \ldots, k - 1$. A letter $a$ or $a^{-1}$ or a $w_i$ of either of those values is said to be “of colour 1;” similarly with $b$ replacing $a$ and “colour 2” replacing “colour 1.” We associate to $w$ its \textit{minimal model}, which is an étale bigraph, $M = M(w)$, as follows. In brief, we turn $w$ into a path of length $k$ and add self-loops to both endpoints; $M$ will have all vertices of degree two except at the two “endpoint vertices” that have degree three because of the self-loops. More precisely, let $V_M = \{v_0, \ldots, v_k\}$; $E_M$ consists of a self-loop at $v_0$, a self-loop at $v_k$, and one edge joining $v_{i-1}$ to $v_i$ for $i = 1, \ldots, k$; the colours and head/tails of $E_M$ are determined in the natural way: $v_0$ has a self-loop of colour 1 if $w_1$ is of colour 2 and vice versa;
similarly for $v_k$ according to the colour of $w_k$; the edge from $v_{i-1}$ to $v_i$ has its head in $v_i$ if $w_i = a$ or $w_i = b$, and otherwise its tail in $v_i$; the colour of the edge from $v_{i-1}$ to $v_i$ is the colour of $w_i$.

For example, if $w = b$, then $M(w)$ is a graph with two vertices, $x_0$ and $x_1$, each of degree three, and

$$A_{1h} = A_{1t} = \{v_0, v_1\}, \quad A_{2h} = \{v_1\}, \quad A_{2t} = \{v_0\}.$$  

We easily verify that $M(b)$ is balanced. Similarly $M(w)$ is balanced in the other three case cases a one word of length 1, i.e., $w = b^{-1}, a, a^{-1}$.

Similarly, $M(w)$ will be balanced whenever $v_0$ and $v_k$ are of different type, meaning that the edge colour and orientation missing from $v_0$ is not the same as that missing from $v_k$; this is the same as saying that $w_1 \neq w_k^{-1}$. In particular, this is true for all $w$ with $|w| = 2$, for $w$ is non-backtracking, and hence $w_1 \neq w_2^{-1}$.

Conversely, if $w_1 = w_k^{-1}$, i.e., $v_0$ and $v_k$ are of the same type, then $M(w)$ will not be balanced (take $U = \{v_0, v_k\}$ in equation (6)). For example, $M(bab^{-1})$ is not balanced, but has the balancing function

$$f(U) = -u_0u_3 + u_1u_2,$$  

where

$$u_i = \begin{cases} 1 & \text{if } v_i \in U, \\ 0 & \text{otherwise}. \end{cases}$$

This example can be generalized as follows.

**Definition 2.1** Consider a non-backtracking walk in a bigraph, $p = (v_0, e_1, v_1, \ldots, e_r, v_r)$. Let $s$ be the smallest non-negative integer such that $e_s, e_{r+1-s}$ are not of the same colour and opposite orientation. (Such an $s$ must exist: for $s = [(r + 1)/2]$, $e_s, e_{r+1-s}$ are either the same edge or adjacent edges, and in either case they cannot be of the same colour and opposite orientation.) The uninverse point(s) of $p$ refers to the two vertices $v_{s-1}, v_{r-s}$.

This discussion implies what we call the “uninverse principle,” that the word in $\{a, a^{-1}, b, b^{-1}\}$ associated to the sequence $e_1, \ldots, e_r$ cannot be its own inverse; when we speak of the “uninverse principle” we shall also refer to the existence of $s$ and/or the uninverse points.

Consider a word, $w = w_1 \ldots w_k$, for which $M(w)$ is unbalanced, i.e., $w_1 = w_k^{-1}$, but such that the missing colour and orientation of the edge incident upon $v_0$ (and, therefore, upon $v_k$) occurs in one of the uninverse points, $v_s, v_t$. Then

$$f(U) = -u_0u_k + u_su_t,$$
is a balancing function. For example, for $M(abab^{-1}a^{-1})$, $f(U) = -u_0u_6 + u_2u_4$.

We wish to point out a number of mysterious properties (at least to us, at present) of balancing functions. First, do not know how to combine balancing functions for two graphs into one for the disjoint union. For example, while $M(aba^{-1})$ and $M(a^{-1}ba)$ are unbalanced, their disjoint union is balanced (since each $A_{ij}$ contains at most $\rho = 2$ vertices of degree three or four). For another example, while $M(aba)$ is balanced, any balancing function for the disjoint union of $M(aba)$ and $M(a^{-1}ba)$ must borrow across (i.e., involve) $M(aba)$ edges; indeed, the disjoint union has $\rho = 2$, but $M(a^{-1}ba)$ will not satisfy equation (8) with $\rho = 2$ for any borrowing function, $f$, on $M(a^{-1}ba)$.

A second mysterious property is that it seems tricky to find balancing functions for $M(w)$ even for certain small $w$. For example, consider $M(abba^{-1}b^{-1}a)$. We know of no extremely simple or intuitive balancing function for this case. In principle balancing functions are determined by a set of inequalities of equation (8) that are linear in the values of the functions $\mathcal{P}(A_{it}) \rightarrow \mathbb{R}$, $i = 1, 2$, that determine the borrowing arrangements (as in $f: \mathcal{P}(A_{it}) \rightarrow \mathbb{R}$ given in equation (7)). Using this we have found balancing functions for $M(abba^{-1}b^{-1}a)$, such as

$$f_1(U) = [-u_5u_1(1 - u_2)(1 - u_6)(1 - u_0) + u_4u_2(1 - u_3)(1 - u_6)(1 - u_0)]$$

$$+ [-u_6u_0 + u_5u_1]$$

and

$$f_2(U) = [-u_5u_1(1 - u_2) + u_4u_2(1 - u_3)]$$

$$+ [-\max(0, u_0 + u_6 + u_4 - 1) + \max(0, u_1 + u_5 + u_3 - 1)]$$

In both cases the first term in brackets is a borrowing arrangement through colour 1, the second term through colour 2; $f_1$ was the first balancing function that the computer gave us, while we obtained $f_2$ by adding the constraint that the balancing function not “borrow through the self-loops,” i.e., the borrowing function through colour 1 should not depend on $u_0$ or $u_6$—according to Theorem 1.9, such a balancing function exists. But at present we have little understanding of the balancing functions for $M(abba^{-1}b^{-1}a)$. The reader may be amused to perform some such computations on small graphs\(^5\).

\(^5\)The author will post his Maple programs for doing this on his website, although the programs are currently a bit light on documentation.
A third mysterious property of balancing functions is that we don’t know much about how large the set of balancing functions is for a given bigraph. For example, as a convex subset of the real vector space of functions \( P(A_1) \to \mathbb{R} \), we could ask to know its dimension, especially to know if this dimension is non-zero, i.e., if there is a unique balancing function. For example, in \( M(abba^{-1}b^{-1}a) \), even if we give the “no borrowing through self-loops” constraint, the balancing function is not unique; for example,

\[
f_3(U) = [-\max(0, u_1 + u_2 + u_5 - 1) + \max(0, u_2 + u_3 + u_4 - 1)]
+[-\max(0, u_0 + u_6 + u_4 - 1) + \max(0, u_1 + u_5 + u_3 - 1)]
\]

is another.

2.3 Simplifications

By modifying our \( \tilde{\text{etale}} \) bigraphs we can make several assumption on them. First, following a sequence of ideas culminating in a clever construction of Jitsukawa-Khan-Myasnikov ([JKM03]), it suffices to prove the SHNC (or HNC) for bigraphs with all vertices of degree two or three, and all degree three vertices not being the head of an edge of colour 2 (or all degree three vertices being of one specified “type”). We describe this below.

If \( G \) is a bigraph and \( e \in E_G \) and \( w \) is a non-empty word over the letters \( \{a, a^{-1}, b, b^{-1}\} \), we define a \( w \)-graft at \( e \) to be a bigraph, \( G' \), obtained by replacing \( e \) with a path of colour pattern \( w \); i.e., we remove \( e \) and introduce \( |w| - 1 \) new vertices of degree two, and \( |w| \) edges that form a path of length \( |w| \) from \( te \) to \( he \) of colour pattern \( w \) (\( G' \) is unique up to unique isomorphism). Given a bigraph, \( G \), and words \( w_a, w_b \in \{a, a^{-1}, b, b^{-1}\} \), we define the \( (w_a, w_b) \)-transform of \( G \), denoted \( T(G; w_a, w_b) \), to be the graph obtained by a \( w_a \) graft at each edge of colour 1 and a \( w_b \) graft at each edge of colour 2.

Consider a map \( T = T(G) \) taking a bigraph, \( G \), and returning another bigraph with the following properties: (1) \( \rho(T(G)) = \rho(G) \), (2) if \( G \) is \( \tilde{\text{etale}} \), then so is \( T(G) \), and (3) for any bigraphs \( G, K \) we have

\[
T(G \times_{B_2} K) \cong T(G) \times_{B_2} T(K).
\]

We call such a map a \( \text{SHNC reduction} \). To prove the SHNC, it suffices to prove it for all graphs in the image of \( T \).
For example $T(G; a^2, b^2)$ is the graph obtained by replacing each edge of colour 1 or 2 by a path of length two of the same colour and orientation. In the sheaf theory we will use in [Frib], the Grothendieck topology associated to a graph, $G$, is actually a topological space, provided $G$ has no self-loops. Clearly for any bigraph, $G$, $T(G; a^2, b^2)$ has no self-loops. We can easily verify that the $(a^2, b^2)$-transformation is an SHNC reduction, and so this transformation allows us to work exclusively with topological spaces for the SHNC.

Consider the $(a^2, ab^{-1}a^{-1}b)$-transformation on an étale bigraph, $G$. This takes an étale bigraph vertex, $v \in V_G$, of degree 4 to a vertex of degree 4 incident upon two edge heads of colour 1 (and an edge tail of colour 1 and a tail of colour 2). By identifying the two incoming colour 1 edges, we get two vertices of degree 3.

**Definition 2.2** By the Jitsukawa-Khan-Myasnikov (or JKM) transformation we mean the $(a^2, ab^{-1}a^{-1}b)$-transformation followed by the identification of any two incoming edges of colour 1 incident upon the same vertex.

We can verify (see [JKM03]) that the JKM transformation is a SHNC reduction on étale bigraphs. Furthermore, a bigraph arising from this transformation has no vertices of degree 4 and all vertices of degree 3 are of the same type (namely type $\{a, a^{-1}, b^{-1}\}$).

Finally, to prove the SHNC, it suffices to work with graphs without vertices of degree 0 (that can be discarded) or degree 1 (that can be pruned).

**Definition 2.3** By a JKM bigraph we mean an étale bigraph with all vertices of degree 2 or 3, and all degree 3 vertices of one type.

The JKM transformation allows us to restrict a proof of the SHNC to JKM bigraphs.

2.4 The Canonical Pairing in General

We say that a walk in a graph is non-backtracking (sometimes called reduced) when no edge is immediately followed by its inverse. Let $K$ be any bigraph,
and let \( v \) be a vertex in \( K \) of degree three. Then \( v \) is incident upon two edges of one colour, and we call this colour the **majority colour**; it has one edge of the other colour, which we call the **terminus** of \( v \). For each vertex of degree 4, choose either colour 1 or colour 2 to be declared the **majority colour** and the two other edges to be **termini**. For each vertex of degree 1 we declare its incident edge to be its **terminus**. Consider the non-backtracking walk which begins in a vertex, \( v \), of degree 4, 3, or 1 and takes a (or the) terminus edge, continuing this walk under the following rules:

1. we stop the walk at a vertex once we have traversed one of its terminus edges;
2. we continue the walk in the non-backtracking direction when we reach a vertex of degree two;
3. we continue the walk as non-backtracking along the edge of the same colour when we reach a vertex of degree four or three along a majority colour edge.

We easily see by induction that such a walk enters a new edge on every step. Hence such a walk is finite and ends after traversing a terminus. We also see by induction that starting in the final (terminal) vertex and edge and following the same rules will result in the reverse walk.

In a JKM bigraph, this gives us a pairing of all vertices of degree three. We call this the **canonical pairing**. More generally, this gives us a pairing of all terminus edges, and can be said to pair the vertices of degree 1, 3, and 4, with the understanding that vertices of degree 4 appear twice in the pairing.

Let \( K \) be a connected JKM bigraph. If \( n_3 \) is the number of vertices of \( K \) of degree three, we easily see that \(|E_K| - |V_K| = n_3/2\). But we also have \( \rho(K) = h^1(K) - 1 = \chi(K) \), and hence \( \rho(K) = n_3/2 \). Hence the canonical pairing on \( K \) yields \( \rho(K) \) edge disjoint (non-backtracking) paths whose endpoints comprise the \( 2\rho(K) \) vertices of degree 3. We call these paths the **canonical paths**. (Canonical paths are, of course, defined for any étale bigraph, although the endpoints may be of degree one or four, and the paths depend on a majority colour chosen for each degree four vertex; the number of paths will be \( 2\rho(K) \) provided there are no vertices of degree 1.)

We remark that much of our discussion could allow vertices of degree four two choose any two edges to be declared as “majority edges” and the other two as termini.
2.5 Canonical Paths and The Uninverse Principle

It seems possible that there could be a much shorter proof of the SHNC based on canonical paths and the uninverse principle directly. Here is one possible approach.

We start with an example. Fix a positive integer, \( k \), and let \( M_1 \) and \( M_2 \) be the disjoint unions of graphs of the form \( M(w) \) where \( w = w_1 \ldots w_k \) is a non-backtracking word of length \( k \) over \( \{a, a^{-1}, b, b^{-1}\} \) with \( w_1 = a \) and \( w_k = a^{-1} \). We claim that \((M_1, M_2)\) satisfy the SHNC. First note that for any graph, \( G \), we have

\[
\chi(G) = (1/2) \sum_{v \in V_G} (2 - \deg(v)).
\]

Also, if each component of \( G \) is acyclic, we have

\[
\rho(G) = h^1(G) - h^0(G) = -\chi(G).
\]

In this case we have \( \rho(G) = n_4 + (n_3 - n_1)/2 \), where \( n_i \) is the number of vertices of \( G \) of degree \( i \). Both \( M_1 \) and \( M_2 \) are JKM graphs, and so \( \rho(M_i) = 2n_3(M_i) \).

Any vertex of degree 3 in \( M_1 \times_{B_2} M_2 \) arises from vertices of degree 3 in each \( M_i \), and conversely, and hence

\[
n_3(M_1 \times_{B_2} M_2) = n_3(M_1)n_3(M_2) = 4\rho(M_1)\rho(M_2).
\]

Let \( S \) be \( M_1 \times_{B_2} M_2 \) with all its acyclic components discarded. Since \( M_1 \times_{B_2} M_2 \) has no vertices of degree 4, we have that

\[
n_4(S) = 0, \quad n_3(S) \leq 4\rho(M_1)\rho(M_2),
\]

and hence \( \rho(S) \leq 2\rho(M_1)\rho(M_2) \). The SHNC must take the \( 4\rho(M_1)\rho(M_2) \) degree three vertices of \( M_1 \times_{B_2} M_2 \) and must show that \( S \) can only “benefit” from half of them. Indeed, the uninverse principle says that if \( M(w) \) is a component of \( M_1 \), and \( M(w') \) a component of \( M_2 \), then either \( w \neq w' \) or \( w^{-1} \neq w' \). It follows that of the four degree three vertices in \( M(w) \times_{B_2} M(w') \), at most two are connected by following the canonical paths in \( M(w) \) and \( M(w') \), and at least two must encounter a vertex of degree one. So all four degree 3 vertices may be in \( S \), but at least two of them are offset by degree 1 vertices. Summing over all components yields

\[
\rho(S) \leq \rho(M_1)\rho(M_2).
\]
There are a number of variants of this argument that we can try to carry out. For example, for any JKM graphs, \(K_1, K_2\), consider a degree three vertex \(v = (v_1, v_2)\) of \(K_1 \times_{B_2} K_2\), and follow it along its canonical path to its terminus \((v'_1, v'_2)\). By Galois theory we can find coverings of \(K_1, K_2\) with no cycles as short as any canonical path in \(K_1 \times_{B_2} K_2\); the canonical paths have the same length in any covering, so we may assume \(v_1 \neq v'_1\) and \(v_2 \neq v'_2\). Now we’d like to argue that of the four vertices,

\[(v_1, v_2), (v'_1, v_2), (v_1, v'_2), (v'_1, v'_2),\]

only two can contribute to \(\rho(S)\) as before without being offset by vertices of degree 1.

We remark that it is possible that \((v_1, v_2)\) be connected to \((v'_1, v'_2)\) and \((v_1, v'_3)\) be connected to \((v'_1, v_2)\). As an example, take \(L\) to be a star, with one vertex, \(v_0\), of degree 3 connected to three leaves, \(v_1, v_2, v_3\) by paths starting at \(v_0\) of type \(ba^{-1}b, aba^{-1}b, a^{-1}ba^{-1}b\); set \(v'_1 = v_3\) and \(v'_2 = v_1\); then let \(K_1 = K_2\) be \(L\) with a self-loop of colour 1 added to each leaf. However, in this case the paths in \(M_1, M_2\) connecting \(v_i\) to \(v'_i\) have a degree three vertex in the middle (namely \(v_0\)); in the example given, there are nine degree three vertices arising from the pairs of degree three vertices along these paths (namely \(v_1, v_2, v_0\) paired with \(v'_1 = v_3, v'_2 = v_1, v_0\)) and there are two sets of four vertices such that at most two vertices in each set could contribute to \(\rho(S)\). So this particular example has enough “cancellation,” but we have to go beyond the simplest cancellation. Another source of trouble is that while the canonical path from \((v_1, v_2)\) may look to \((v'_1, v_2)\) for cancellation, another vertex, \((v_3, v_2)\), could have a canonical path connecting to \((v'_1, v'_2)\), and hence also look to \((v'_1, v_2)\) for cancellation. The examples we have studied so far all have some sort of “larger cancellation” involving all the degree three vertices needed to construct such examples. However, we have no systematic way, at present, to carry out this cancellation in a general situation.

3 Balancing and the SHNC

The goal of this section is to prove Theorems 1.3, 1.6, 1.7, and 1.9. Although the first proof we found of the first two theorems involved sheaf theory, we prove all theorems here combinatorially, without reference to sheaf theory. The proofs without sheaf theory seem less intuitive to us; also some of the proofs with sheaf theory are much shorter, but that is provided that one has
a sheaf and cohomology theory in place (which takes some work and which some people may consider burdensome).

3.1 Proof of Theorems 1.3 and 1.6

We begin with Theorem 1.3.

Assume there is an étale bigraph $K_1$ such that

$$\rho(K \times_{B_2} K_1) > \rho(K)\rho(K_1), \quad (10)$$

and consider such a $K_1$ with $|V_{K_1}| + |E_{K_1}|$ minimal. First note that $K_1$ contains no isolated vertices, for such vertices may be discarded without affecting $\rho(K_1)$ and $\rho(K \times_{B_2} K_1)$. Next note that $K_1$ has no vertices of degree one, for any such vertex and its incident can be discarded (which is often called "pruning a leaf"). Also, note that $K_1$ is connected, for otherwise it could be replaced by one of its connected components. Also $\rho(K_1) \geq 1$, for otherwise $K_1$ contains no vertices of degree three or four (without vertices of degree zero and one, we have $\rho(K_1) = -\chi(K_1) = n_4 + (n_3/2)$, where $n_i = n_i(K_1)$ is the number of vertices of degree $i$); in this case $K \times_{B_2} K_1$ has no vertices of degree greater than two, and hence $\rho(K \times_{B_2} K_1) = 0$, contradicting equation (10).

Take $H$ to consist of all cyclic components of $K \times_{B_2} K_1$, i.e., all connected components with $\chi \leq 0$, i.e., at least as many edges as vertices; we have

$$\rho(K \times_{B_2} K_1) = \rho(H) = -\chi(H).$$

Furthermore, notice that $-\chi(L)$ with $L$ ranging over all subgraphs of any graph (in particular, the graph $K \times_{B_2} K_1$) is maximized when $L$ consists of all cyclic components of the graph.

Say that an edge of $K_1$ is *alive* if it has at least $\rho(K)$ edges of $H$ lying over it (we could require *alive* to mean having more than $\rho(K)$ edges of $H$ lying over it and nothing would change). In fact, each edge of $K_1$ must be alive, for if we remove an edge, $e$, that isn’t alive, we get a graph $K'_1$ with $\rho(K'_1) = \rho(K_1) - 1$ (since $K_1$ is connected with $\rho(K_1) \geq 1$), while $H'$ obtained by removing the (fewer than $\rho(K)$) edges over $e$ has

$$-\chi(H') \geq -\chi(H) - \rho(K) > \rho(K)\rho(K_1) - \rho(K) = \rho(K)\rho(K'_1).$$

Hence

$$\rho(K \times_{B_2} K'_1) > \rho(K)\rho(K'_1),$$
contradicting the minimality of \( K_1 \).

Since \( K_1 \) has no isolated vertices, and every edge of \( K_1 \) is alive, then similarly every vertex of \( K_1 \) is alive (meaning its fibre in \( H \) consists of at least \( \rho(K) \) vertices). If \( \phi : H \to K_1 \) is the projection, then

\[
-\chi(H) = \sum_{e \in E_{K_1}} |\phi^{-1}(e)| - \sum_{v \in V_{K_1}} |\phi^{-1}(v)|
\]

\[
= -\chi(K_1)\rho(K) + \sum_{e \in E_{K_1}} |\phi^{-1}(e)|_{\rho(K)} - \sum_{v \in V_{K_1}} |\phi^{-1}(v)|_{\rho(K)}.
\]

Since \( \rho(K_1) = -\chi(K_1) \), we would contradict the existence of \( K_1 \) by showing that

\[
\sum_{e \in E_{K_1}} |\phi^{-1}(e)|_{\rho(K)} \leq \sum_{v \in V_{K_1}} |\phi^{-1}(v)|_{\rho(K)}.
\]

Let \( U_v = \pi \phi^{-1}(v) \) for each \( v \in V_{K_1} \), where \( \pi \) is the projection \( H \to K \). For each \( e \in E_{K_1} \) of colour 1 we have

\[
|\phi^{-1}(e)|_{\rho(K)} \leq |U_{he} \cap A_{1h}|_{\rho(K)}
\]

and similarly

\[
|\phi^{-1}(e)|_{\rho(K)} \leq |U_{te} \cap A_{1t}|_{\rho(K)}.
\]

Similarly for edges of colour 2. Hence

\[
2 \sum_{e \in E_{K_1}} |\phi^{-1}(e)|_{\rho(K)} \leq \sum_v \sum_{ij} |U_v \cap A_{ij}|_{\rho(K)}.
\]

Given that \( K \) is balanced we have

\[
\sum_v \sum_{ij} |U_v \cap A_{ij}|_{\rho(K)} \leq 2 \sum_v |U_v|_{\rho(K)} = 2 \sum_v |\phi^{-1}(v)|_{\rho(K)}
\]

and combining the last two inequalities (and dividing by 2) yields equation (11).

Next we explain the modifications needed above to prove Theorem 1.6. Actually, we will need a slightly stronger form of Theorem 1.6, so we first give a precise statement.

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Definition 3.1 Let $K$ be an étale bigraph. Let $w = \{w_v\}_{v \in V_K}$ be a collection of reals. For $U \subset V_K$ we define the weighted size of $U$ with respect to $w$ to be
\[ \|U\| = \|U\|_w = \sum_{v \in U} w_v. \]
(We often write $\|U\|$ for $\|U\|_w$ if $w$ is clear from the context.) If all $w_v \geq 1$, or equivalently $\|U\|_w \geq |U|$ for all $U$, we say that $w$ or $\| \cdot \|_w$ is a superweight. We say that a pseudo-Lipschitz borrowing function, $f$, on $K$ is a balancing function for $w$ or $\| \cdot \|_w$ if
\[ f(U) + \sum_j \|U \cap A_{1j}\|_\rho + \sum_j \|U \cap A_{2j}\|_\rho \leq 2\|U\|_\rho, \tag{13} \]
where $\rho = \rho(K_1)$ and we use the notation $\|W\|_m = \max(0, \|W\| - m)$. (We wish to emphasize that “pseudo-Lipschitz” in the last sentence means with respect to $\| \cdot \|$ as defined before, not with respect to $\| \cdot \|_w$.) In this case we say that $w$ or $\| \cdot \|_w$ is balanceable. We shall sometimes use the notation
\[ \delta(U) = \delta_w(U) = \|U\|_w - |U| = \sum_{v \in U} (w_v - 1). \]

Theorem 3.2 Let $K$ is an étale bigraph with a balanceable superweight, $\| \cdot \|$, and assume that $\|\{v\}\| > 1$ implies that $v \in A_{1h} \cap A_{1t}$. Then $K$ is universal for the SHNC.

The reader will notice an obvious asymmetry in our definition and theorem. Actually one could allow $V_K$ and each $A_{ij}$ to have their own superweight. The reader will see that theorem then holds provided that for all $v$ with $\|\{v\}\| > 1$ we have that $2\delta(v)$ in the $V_K$ weight equals the sum of the $\delta(v)$ in the four $A_{ij}$ weights. For the purpose of this paper, Theorem 3.2 suffices.

Proof Assume there is a an étale bigraph, $K_1$, for which
\[ \rho(K \times_{B_2} K_1) > \rho(K)\rho(K_1). \]
Again, assume a $K_1$ with sum of number of edges and of vertices minimized. As before, we argue that if $H \subset K \times_{B_2} K_1$ has $-\chi(H)$ maximized, and hence equal to $\rho(K \times_{B_2} K_1)$, that $H$ has at least $\rho(K)$ edges (respectively, vertices) above each edge (respectively, vertex) of $K_1$. Let notation be as before, so, in particular, $\pi, \phi, \text{ respectively,}$ are the projections of $H$ or $K \times_{B_2} K_1$ to
their two components, $K, K_1$, respectively. As before, it suffices to establish equation (11).

For an $e \in E_{K_1}$ let $V_{e,t} = \pi t \phi^{-1}(e)$ be the tails of the edges over $e$ projected to their $K$ component, and similarly for $V_{e,h}$. We have that $t_{1,ht}$ as in equation (7) gives an isomorphism from $V_{e,h}$ to $V_{e,t}$. For any function $f_1 : \mathcal{P}(A_{1t}) \rightarrow \mathbb{C}$ we have

$$f_1(V_{e,t}) = f_1 t_{1,ht}(V_{e,h}).$$

Hence for any such $f_1$ we have

$$\sum_{e \in E_{K_1,1}} f_1(V_{e,t}) - \sum_{e \in E_{K_1,1}} f_1 t_{1,ht}(V_{e,h}) = 0, \quad (14)$$

Similarly for $f_2 : \mathcal{P}(A_{2t}) \rightarrow \mathbb{C}$. Now

$$\sum_{e \in E_{K_1}} 2|\phi^{-1}(e)|_{\rho(K)} = \sum_{e \in E_{K_1}} (|V_{e,h}|_{\rho(K)} + |V_{e,t}|_{\rho(K)})$$

$$= \sum_{i=1,2} \sum_{\substack{j=h,t \atop j'=v}} \left( |U_v \cap A_{ij}|_{\rho(K)} - |(U_v \cap A_{ij}) \setminus V_{e,j}| \right)$$

(since $V_{e,j} \subset U_v \cap A_{ij}$ for all $e, i, j, v$ with $j = v$ and since $|V_{e,h}|$, $|V_{e,t}|$, and $|U_v \cap A_{ij}|$ are all at least $\rho(K)$)

$$= \sum_{j=h,t} \sum_{\substack{i=1,2 \atop j'=v}} \left( |U_v \cap A_{ij}|_{\rho(K)} - |(U_v \cap A_{ij}) \setminus V_{e,j}| - \delta(U_v \cap A_{ij}) \right)$$

$$+ \sum_{j=h,t} \sum_{\substack{i=1,2 \atop j'=v}} \left( |U_v \cap A_{ij}|_{\rho(K)} - |(U_v \cap A_{ij}) \setminus V_{e,j}| \right)$$

(with $\delta$ as in Definition 3.1)

$$\leq \sum_{v \in V_{K_1}} \left( 2|U_v|_{\rho(K)} - f(U) - \sum_{i=1,2} \sum_{\substack{j=h,t \atop j'=v}} |(U_v \cap A_{ij}) \setminus V_{e,j}| - \sum_{j=h,t} \delta(U_v \cap A_{ij}) \right)$$

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\[
\sum_{v \in V_{K_1}} \left( 2\|U_v\|_{\rho(K)} - \sum_{j = h, t} \delta(U_v \cap A_{ij}) \right) - \sum_{v \in V_{K_1}} \left( f(U) + \sum_{i = 1, 2} \left| (U_v \cap A_{ij}) \setminus V_{e,j} \right| \right)
\]

(15)

For the left quantity of the above we have

\[
\sum_{v \in V_{K_1}} \left( 2\|U_v\|_{\rho(K)} - \sum_{j = h, t} \delta(U_v \cap A_{ij}) \right) = \sum_{v \in V_{K_1}} 2\|U_v\|_{\rho(K)} - \sum_{v \in V_{K_1}} |\phi^{-1}(v)|_{\rho(K)}
\]

(where we have used \(2\| \cdot \| - 2\delta(\cdot) = 2|\cdot|\) and that \(\|\{v\}\| \neq 1\) implies that \(v \in A_{1t} \cap A_{1h}\)). So it suffices to bound the right quantity of equation (15), i.e., to show that

\[
\sum_{v \in V_{K_1}} \left( f(U) + \sum_{i = 1, 2} \left| (U_v \cap A_{ij}) \setminus V_{e,j} \right| \right) \geq 0.
\]

where \(V_{e,j}\) denotes \(V_{e,j}\) for the \(e\) of colour \(i\) such that \(je = v\). By equation (14) it suffices to show that for each \(v\) we have

\[
f(U_v) - f_1(V_{v1t}) + f_1 t_{1,ht}(V_{v1h}) - f_2(V_{v2t}) + f_2 t_{2,ht}(V_{v2h})
\]

\[
+ \sum_{i = 1, 2} \left| (U_v \cap A_{ij}) \setminus V_{e,j} \right| \geq 0
\]

for some choice of \(f_i : \mathcal{P}(A_{it}) \to \mathbb{R}\). Choose any \(f_1, f_2\) that give rise to \(f\); i.e., \(f = g_1 + g_2\) with \(g_1\) given as in equation (7) (where \(g\) is used instead of \(g_1\)) and \(g_2\) similarly. It then suffices to show that for any \(v\) we have

\[
f_1(U_v \cap A_{1t}) - f_1 t_{1,ht}(U_v \cap A_{1h}) - f_1(V_{v1t}) + f_1 t_{1,ht}(V_{v1h})
\]

\[
+ \left| (U_v \cap A_{1h}) \setminus V_{v1h} \right| + \left| (U_v \cap A_{1t}) \setminus V_{v1t} \right| \geq 0
\]

and similarly with the index 2 replacing 1 appropriately. But this follows from the fact that the \(f_1\) are pseudo-Lipschitz.
3.2 Linear Programming in Theorem 1.7

In this subsection we will prove Theorem 1.7. We assume the reader is familiar with linear programming and duality theory on the level of [Chv83].

The existence of a balancing function for a bigraph, \( K \), is equivalent to whether one can find arbitrary real numbers \( f_{\mathcal{S}\mathcal{T}} \) satisfying the following constraints:

\[
\forall \mathcal{U}, \quad a_{\mathcal{U}\cap \mathcal{A}_1} - a_{\mathcal{A}_1, h_t(\mathcal{U}\cap \mathcal{A}_1 h_t)} + b_{\mathcal{U}\cap \mathcal{A}_2} - b_{\mathcal{A}_2, h_t(\mathcal{U}\cap \mathcal{A}_2 h_t)} + \sum_{i=1}^2 \sum_{j=h,t} |\mathcal{U}\cap \mathcal{A}_{ij}|_{\rho(K)} \leq 2|\mathcal{U}|_{\rho(K)},
\]

(16)

plus the Lipschitz constraints

\[
\forall \mathcal{S} \subset \mathcal{T} \subset \mathcal{A}_1, \quad a_\mathcal{S} - a_\mathcal{T} \leq |\mathcal{T} \setminus \mathcal{S}|,
\]

(17)

\[
\forall \mathcal{S} \subset \mathcal{T} \subset \mathcal{A}_1, \quad a_\mathcal{T} - a_\mathcal{S} \leq |\mathcal{T} \setminus \mathcal{S}|,
\]

(18)

\[
\forall \mathcal{S} \subset \mathcal{T} \subset \mathcal{A}_2, \quad b_\mathcal{S} - b_\mathcal{T} \leq |\mathcal{T} \setminus \mathcal{S}|,
\]

(19)

\[
\forall \mathcal{S} \subset \mathcal{T} \subset \mathcal{A}_2, \quad b_\mathcal{T} - b_\mathcal{S} \leq |\mathcal{T} \setminus \mathcal{S}|.
\]

(20)

Of course, in the above we can restrict ourselves to the \( \mathcal{S}, \mathcal{T} \) pairs with \( |\mathcal{T} \setminus \mathcal{S}| = 1 \). Furthermore, if we required only that the balancing function be pseudo-Lipschitz, we could restrict ourselves to \( |\mathcal{S}| \geq \rho = \rho(K) \).

Duality theory of linear programming implies that a system of linear inequalities is unsolvable (i.e., has no solutions) iff it is inconsistent (there is a linear combination of them that give a contradiction, where all variables are eliminated and constants remain in a false inequality); see, for example, Theorem 9.2, page 144, of [Chv83]. In particular, the above inequalities have a solution iff there are non-negative real numbers—“multipliers” for the respective inequalities—\( \gamma_\mathcal{U}, \alpha_{\mathcal{S}, \mathcal{T}, 1}, \alpha_{\mathcal{S}, \mathcal{T}, 2}, \beta_{\mathcal{S}, \mathcal{T}, 1}, \beta_{\mathcal{S}, \mathcal{T}, 2} \) that produces an inconsistency, i.e., such that

\[
\sum_\mathcal{U} \gamma_\mathcal{U} \left( -2|\mathcal{U}|_{\rho(K)} + \sum_{i=1}^2 \sum_{j=h,t} |\mathcal{U}\cap \mathcal{A}_{ij}|_{\rho(K)} \right) - \sum_{\mathcal{S}, \mathcal{T}, i} |\mathcal{T} \setminus \mathcal{S}| \alpha_{\mathcal{S}, \mathcal{T}, i} - \sum_{\mathcal{S}, \mathcal{T}, i} |\mathcal{T} \setminus \mathcal{S}| \beta_{\mathcal{S}, \mathcal{T}, i} > 0,
\]

(21)
and

\[ \forall S \subset A_{1t}, \quad \sum_{U \cap A_{1t} = S} \gamma_U - \sum_{U \cap A_{1h} = \ell_{1,th}} \gamma_U + \sum_{T, i} \alpha_{S,T,i}(-1)^{i+1} - \sum_{R, j} \alpha_{R,S,i}(-1)^{j+1} = 0, \quad (22) \]

\[ \forall S \subset A_{2t}, \quad \sum_{U \cap A_{2t} = S} \gamma_U - \sum_{U \cap A_{2h} = \ell_{2,th}} \gamma_U + \sum_{T, i} \beta_{S,T,i}(-1)^{i+1} - \sum_{R, j} \beta_{R,S,i}(-1)^{j+1} = 0, \quad (23) \]

So assume the inconsistency exists, i.e., that there are non-negative reals \( \gamma_U \), \( \alpha_{S,T,i} \), \( \alpha_{R,S,i} \), \( \beta_{S,T,i} \), \( \beta_{R,S,i} \) satisfying equations (21)–(23). We shall produce a bigraph, \( K_1 \), for which \( \rho(K \times B_2, K_1) > \rho(K) \rho(K_1) \).

To illustrate our ideas, we first consider the case where all \( \alpha_{S,T,i} \) and \( \beta_{S,T,i} \) vanish. We claim that we may assume the \( \gamma_U \) are rationals and hence, after scaling, non-negative integers; this essentially follows from the simplex method applied to the following problem: maximize the left-hand-side of equation (21) subject to equation (22) and (23) and \( 0 \leq \gamma_U \leq 1 \) for all \( U \), with vanishing \( \alpha_{S,T,i} \) and \( \beta_{S,T,i} \); this is clearly a bounded linear program, and its optimum value occurs at some final dictionary (otherwise called a tableau); at such a dictionary, the values of the \( \gamma_U \) are rational functions of the original constants and coefficients (see equation (7.6) on page 99 of [Chv83]).

So we have non-negative, integral \( \gamma_U \) with

\[ \sum_{U \cap A_{1t} = S} \gamma_U = \sum_{U \cap A_{1h} = \ell_{1,th}} \gamma_U \]

and

\[ \sum_{U \cap A_{2t} = S} \gamma_U = \sum_{U \cap A_{2h} = \ell_{2,th}} \gamma_U \]

for appropriate \( S \). Form a graph \( K_1 \) as follows: \( V_{K_1} \) will be a set of size \( \sum_U \gamma_U \) denoted \( \{ v_U, 1, v_U, 2, \ldots, v_U, \gamma_U \} \cup V_K \).

For any \( S \) we let \( A^{-1}_{ij}(S) \) be those \( U \) such that \( U \cap A_{ij} = S \). For each \( S \subset A_{1t} \) the two sets

\[ \{ v_{U,i} \}_{U \in A^{-1}_{1t}(S)}, \quad \{ v_{U,i} \}_{U \in A^{-1}_{1h \ell_{1,th}}(S)} \]

(26)
are of the same size, by equation (24); choose an arbitrary pairing between them, and make each element of the first set the tail of an edge of colour 1 whose head is the pair (in the second set) of the tail. Do similarly for edges of colour 2. With these edges we get a bigraph, $K_1$. Now consider $H \subset K \times_{B_2} K_1$ where $H$ consists of the vertices $(u,v,i)$ ranging over all $v,U,i$ such that $u \in U$ and whose edges are all possible edges of $K \times_{B_2} K_1$ joining two such vertices. Then

$$-2\chi(H) = \sum_U \gamma_U (-2|U| + 2|U \cap A_{1t}| + 2|U \cap A_{2t}|).$$

Equations (24) and (25) now imply

$$-2\chi(H) = \sum_U \gamma_U \left(-2|U| + \sum_{ij} |U \cap A_{ij}|\right). \quad (27)$$

Now we wish to manicure $K_1$ and $H$ in a manner similar to the proofs of Theorem 1.3 and 1.6. First, let us form a new graph $\tilde{K}_1$ from $K_1$ by discarding from $K_1$ all vertices and edges whose fibre in $H$ is of size less than $\rho$ (as before, we could do this for size less than or equal to $\rho$, and nothing would change); we discard all $H$ vertices and edges above the discarded $K_1$ vertices and edges to get a graph $\tilde{H} \subset K \times_{B_2} \tilde{K}_1$. We have

$$-2\chi(\tilde{H}) = \sum_{|U| \geq \rho} \gamma_U \left((I_U - 2)\rho(K) - 2|U| \rho(K) + \sum_{ij} |U \cap A_{ij}| \rho(K)\right),$$

where $I_U$ is the number of $(i,j)$ with $|U \cap A_{ij}| \geq \rho$,

$$= \sum_{|U| \geq \rho} \gamma_U (I_U - 2)\rho(K) + \sum_{|U| \geq \rho} \gamma_U \left(-2|U| \rho(K) + \sum_{ij} |U \cap A_{ij}| \rho(K)\right),$$

$$= -2\chi(\tilde{K}_1) \rho(K) + \sum_{|U| \geq \rho} \gamma_U \left(-2|U| \rho(K) + \sum_{ij} |U \cap A_{ij}| \rho(K)\right).$$

Next let prune all the leaves and then discard the isolated (i.e., degree 0) vertices of $\tilde{K}_1$, and discard what lies above in $\tilde{H}$, leaving $H'$ and $K'_1$ respectively. Notice that the pruning leaves $\chi$ invariant, both in $\tilde{K}_1$ and $\tilde{H}$. Furthermore,
discarding an isolated vertex in $\tilde{K}_1$ discards at least $\rho$ vertices in $\tilde{H}$. Hence we get an $H' \subset K \times B_2 K_1'$ with

$$-2\chi(H') \geq -2\chi(\tilde{K}_1)\rho(K) + 2D\rho(K) + \sum_{|U| \geq \rho} \gamma_U \left( -2|U|_{\rho(K)} + \sum_{ij} |U \cap A_{ij}|_{\rho(K)} \right),$$

where $D$ is the number of discarded vertices; since $\chi(K_1') = \chi(\tilde{K}_1) - D$, we have

$$-2\chi(H') \geq -2\chi(K_1')\rho(K) + \sum_{|U| \geq \rho} \gamma_U \left( -2|U|_{\rho(K)} + \sum_{ij} |U \cap A_{ij}|_{\rho(K)} \right).$$

But since

$$\sum_{|U| \geq \rho} \gamma_U \left( -2|U|_{\rho(K)} + \sum_{ij} |U \cap A_{ij}|_{\rho(K)} \right)$$

$$= \sum_{U} \gamma_U \left( -2|U|_{\rho(K)} + \sum_{ij} |U \cap A_{ij}|_{\rho(K)} \right) > 0,$$

this means that $-2\chi(H') > -2\chi(K_1')\rho(K)$. Hence

$$\rho(K \times B_2 K_1') > -\chi(K_1')\rho(K).$$

Since $K_1'$ has no vertices of degree zero or one, we have $-\chi(K_1') = \rho(K_1')$, and hence $K_1'$ is a counterexample to the SHNC, which cannot exist since $K$ is universal for the SHNC.

We now modify in the argument in the presence of $\alpha_{S,T,i}$ and/or $\beta_{S,T,i}$ that do not vanish. Again, we may assume these variables and the $\gamma_U$ are non-negative integers.

We need to slightly modify our construction of $K_1$ and of $H \subset K \times B_2 K_1$. Let $\pi: K \times B_2 K_1 \rightarrow K$ be the projection onto the first component, as before (which isn’t explicit until once we have spelled out $K$ and $K_1$). Let $V_{K_1}$ be as before (with the same $v_{U,i}$ notation). For each $S \subset A_{1t}$ let

$$\Delta_S = \left| \{v_{U,i}\}_{U \in A_{1t}^{-1}(S)} \right| - \left| \{v_{U,i}\}_{U \in A_{1t}^{-1,1,1t}(S)} \right|.$$

Intuitively, a vertex, $v_{U,i}$, in $K_1$ will “ideally” have edges in $H$ whose colour 1 tails are mapped by $\pi$ to $S = U \cap A_{1t}$, and similarly for colour 1 heads; this was the case before, when the two sets in equation (26) were of equal
size and $\Delta_S = 0$; when $\Delta_S \neq 0$ this “ideal” situation is impossible, since each colour $1$ tail in $H$ of a given $\pi$ value has to be matched to a colour $1$ head of the appropriate $\pi$ value; $\Delta_S$ measures the tails to heads imbalance. Of course, equation (22) shows that

$$\Delta_S = \sum_{T;i} \alpha_{S,T;i}(-1)^i - \sum_{R;i} \alpha_{R,S;i}(-1)^i,$$

and any non-zero $\alpha_{S,T;i}$ and/or $\alpha_{R,S;i}$ that causes $\Delta_S$ to be non-zero and gives a tail to heads imbalance is compensated for in equation (21). Let us describe our $K_1$ modification to exploit this compensation.

To explain the $K_1$ modification, let $Q$ be the directed graph with $V_Q$ consisting of the subsets, $S \subset A_1$. For each $S \subset T$, let $E_Q$ have $\alpha(S,T,1)$ edges from $S$ to $T$ and $\alpha(S,T,2)$ edges from $T$ to $S$. Then $\Delta_S$ is the outdegree of $S$ in $Q$ minus its indegree. Now we inductively construct a collection of edge disjoint Eulerian paths, $p_1, \ldots, p_r$, as follows (a type of Fleury algorithm): take any $S$ with $\Delta_S > 0$, and start a walk at $S$ and stop when we reach a vertex $T$ with $\Delta_T < 0$ (if our path encounters a $T$ with $\Delta_T \leq 0$ we can always continue the path through $T$); then discard the edges on that path from $Q$ and repeat; stop when there are no more vertices with $\Delta_S < 0$. At that point all $\Delta_S = 0$ (in $Q$ with all path edges discarded), since the sum of the outdegree minus the indegree over all vertices of a directed graph must vanish. Hence the origins of these paths consist of each $S$ with $\Delta_S > 0$ exactly $\Delta_S$ times, and the terminals of paths consist of each $S$ with $\Delta_S < 0$ exactly $-\Delta_S$ times. Notice also that if any path, $p$, of $Q$ originates in $S$ and terminates in $T$ and consists of the vertices $S = S_0, S_1, \ldots, S_r = T$ in order, then

$$d(T, S) \leq \sum_{i=1}^{r} d(S_i, S_{i-1}),$$

where $d(, )$ denotes the symmetric difference of sets. So if we weight all the edges of $Q$ joining $S$ to $T$ with weight

$$|T \setminus S| = d(T, S),$$

then

$$\sum_{i} d(\text{origin}(p_i), \text{terminus}(p_i)) \leq \sum_{e \in E_Q} \text{weight}(e) = W,$$

(28)
with
\[ W = \sum_{S,T,i} |T \setminus S| \alpha_{S,T,i}. \]

In particular, equations (21) then shows
\[
\sum_{U} \gamma_{U} \left( -2|U|_{\rho(K)} + \sum_{i=1}^{2} \sum_{j=K_{H}} |U \cap A_{ij}|_{\rho(K)} \right) > W + W', \tag{29}
\]

where \( W' \) is sum of weights of the similarly constructed graph, \( Q' \) for the colour 2.

So consider the paths \( p_{1}, \ldots, p_{r} \) constructed in \( Q \). Let the colour 1 edges of \( K_{1} \) be constructed as follows: for each \( S \), pair as many \( v_{U,i} \) with \( U \in A_{1}^{-1}(S) \) to those with \( U \in A_{1}^{-1}(S) \); this leaves \( \Delta_{S} \) unpaired for each \( S \). For \( i = 1, \ldots, r \) (in sequence), if \( p_{i} \) has origin \( S \) and terminus \( S' \), pair one currently unpaired \( v_{U,i} \) with \( U \in A_{1}^{-1}(S) \) with one currently unpaired \( v_{U',j} \) with \( U' \in A_{1}^{-1}(S) \). This pairs all the \( K_{1} \) vertices with one tail and one head of colour 1. Do similarly for colour 2. Again, let \( H \) consist of those vertices \( u, v_{U,i} \) with \( u \in U \). The edges of colour 1 in \( H \) are defined as follows: for each edge, \( e \in E_{K_{1}} \), of colour 1 with tail \( v_{U,j} \) and head \( v_{U'j} \), let \( H \) consist of those edges above \( e \) whose tail has \( \pi \) image in
\[ (U \cap A_{1t}) \cap t_{1,ht}(U' \cap A_{1h}). \]

The number of such edges can be written as
\[
(1/2)|U \cap A_{1t}| + (1/2)|U' \cap A_{1h}| - (1/2)d(U \cap A_{1t}, t_{1,ht}(U' \cap A_{1h})).
\]

So the total number of edges of colour 1 is
\[
-(D_{1}/2) + (1/2) \sum_{U} \gamma_{U} \sum_{j} |U \cap A_{1j}|,
\]

where
\[
D_{1} = \sum_{(v_{U,i}, v_{U'j}) \in E_{K_{1},1}} d(U \cap A_{1t}, t_{1,ht}(U' \cap A_{1h})).
\]

According to equation (28), summing over all colour 1 edges \((v_{U,i}, v_{U'j})\) we have
\[ D_{1} \leq W. \]
Similarly for colour 2 and $W'$. So we conclude, similarly to equation (27)

\[-2\chi(H) = -D_1 - D_2 + \sum_U \gamma_U \left(-2|U| + \sum_{ij} |U \cap A_{ij}|\right)\]

\[\geq -W - W' + \sum_U \gamma_U \left(-2|U| + \sum_{ij} |U \cap A_{ij}|\right).\]

Hence, performing the same manicuring as before we obtain $K'_1$ and $H'$ with $H' \subset K \times_{B_2} K'_1$ and

\[-2\chi(H') \geq -W - W' + 2\rho(K)\rho(K'_1) + \sum_U \gamma_U \left(-2|U|_{\rho(K)} + \sum_{ij} |U \cap A_{ij}|_{\rho(K)}\right).\]

By equation (29) we have

\[-2\chi(H') > 2\rho(K)\rho(K'_1).\]

and so, once again,

\[\rho(K \times_{B_2} K'_1) > \rho(K)\rho(K'_1).\]

\[\square\]

### 3.3 Self-Loops and Cycles

In this subsection we prove Theorem 1.9. Let us begin with the following special case: let $K$ be a graph with $v$ having a self-loop $e$ of colour 1. Then we claim if there is a balancing function for $K$, then there is a function supported on $K_0$ which is $K$ minus the edge $e$.

So our overall strategy is to take the linear program for a balancing function, namely equations (16)–(20), and (1) describe how to modify it and the dual for a balancing function definable away from $e$, and (2) show that one can still build a counterexample to the SHNC for this modified linear program and dual.

So $v \in A_{1t}$ and is paired with itself as an element $v \in A_{1h}$ (via $t_{1,h}$). The linear program for a balancing function of $K$ which is supported on $K'$ is that same as that for $K$ (described at the beginning of Subsection 3.2), except that we add the constraint $a_S = a_{S \setminus \{v\}}$ for every $S$ (of course, we
may as well just consider $S$ with $v \in S$). Then we get the same dual linear program as in equation (21)–(23), provided that we set

$$\tilde{A}_{ij}^{-1}(S) = A_{ij}^{-1}(S) \cup A_{ij}^{-1}(S \cup \{v\})$$

for $j = h, t$, letting $S$ range over subsets of $A_{1t} \setminus \{v\}$, and use

$$U \in \tilde{A}_{1t}^{-1}(S) \quad \text{and} \quad U \in \tilde{A}_{1h}^{-1,th}(S)$$

instead of

$$U \cap A_{1t} = S \quad \text{and} \quad U \cap A_{1h} = A_{1t,h}(S)$$

in equation (22).

Next note that if for every $U$ we set $\tilde{U} = U \cup \{v\}$, then

$$-2|\tilde{U}| + \sum_j |\tilde{U} \cap A_{1j}| \geq -2|U| + \sum_j |U \cap A_{1j}|,$$

since $\tilde{U}$ can only differ from $U$ by adding $\{v\}$, and $\{v\}$ occurs in both of the $A_{1j}$. Hence

$$\sum_U \gamma_U \left(-2|\tilde{U}| + \sum_j |\tilde{U} \cap A_{1j}| + \sum_j |U \cap A_{2j}|\right) \geq \sum_U \gamma_U \left(-2|U| + \sum_{ij} |U \cap A_{ij}|\right).$$

Now we construct $K_1$ as before, and construct $H \subset K \times_{B_2} K_1$ as before with the following modification: $H$ consist of pairs $(u, v_{U,i})$ with $u \in \tilde{U}$; in other words, for any $U, i$ we add the vertex $(v, v_{U,i})$ regardless of whether or not $v \in U$; for colour 1, we connect $(u, v_{U,i})$ with colour 1 heads and tails, respectively, for all $u \in \tilde{U}$ according to the $\tilde{A}_{1t}^{-1}(S) \tilde{A}_{1h}^{-1,th}(S)$, respectively; we do similarly for colour 2, but only for those $u \in U$ (as opposed to those $u \in \tilde{U}$); in other words, if $v \notin U$, then $(v, v_{U,i})$ will have no edges of colour 2.

We get

$$-2\chi(H) = \sum_U \gamma_U \left(-2|\tilde{U}| + \sum_j |\tilde{U} \cap A_{1j}| + \sum_j |U \cap A_{2j}|\right) - W - W'$$

$$\geq \sum_U \gamma_U \left(-2|U| + \sum_{ij} |U \cap A_{ij}|\right) - W - W',$$

and, as before, we get a manicured $K_1'$ with $\rho(K \times_{B_2} K_1') \leq \rho(K)\rho(K_1')$. 

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Hence if there is no balancing function for $K$ supported away from the self-loop at $v$, then $K$ is not universal for the SHNC and so there is no balancing function for $K$ at all.

We argue similarly if we have a cycle of colour 1 as in the hypothesis of the theorem, letting $\tilde{U}$ be $U$ plus those vertices, and defining $A_{ij}^{-1}(S)$ to be the union of $A_{ij}^{-1}$ over sets which are unions of $S$ with vertices on the cycle. Since adding each vertex $(u, v_{U,i})$ with $u \in \tilde{U} \setminus U$ is offset by membership in $A_{ij}$ after adding colour 1 edges, we conclude equation (30) and finish as before.

Self-loops and cycles of colour 2 are handled similarly.

\[ \square \]

### 3.4 Coverings of Balanced Graphs

In this subsection we briefly indicate why one can directly show that the cover of a balanced graph is balanced, and why it seems to be harder to show that the cover of a balanceable graph is balanceable.

**Definition 3.3** Let $\phi: M \to K$ be a morphism of sets. By a slicing of $\phi$ we mean a disjoint union of subsets of $K$, $M_1, M_2, \ldots$ such that

$$\forall k \in K \quad |\phi^{-1}(k)| = |\{i \text{ s.t. } k \in M_i\}|.$$  

We define the symmetrization, $\text{Symm}(\phi)$, of $\phi$ to be the disjoint union of the sets $M_1, M_2, \ldots$, where $M_i$ is the subset of $K$ given by those $x \in K$ such that $|\phi^{-1}(x)| \geq i$. $M_i$ will be called the $i$-th component of the symmetrization.

The symmetrization is an example of a slicing.

Let $\phi: K' \to K$ be the covering map, and let $f$ be a balancing function for $K$. We’d like to directly construct a balancing function on $K'$; one natural strategy is for each $U \subseteq V_K$, take a slicing $U_1, U_2, \ldots$ of $\phi^{-1}(U) \to V_{K'}$, and try to show that

$$f'(U) = f(U_1) + f(U_2) + \cdots$$

is a balancing function for $K'$, by applying the balancing condition on $K'$ to $U_1, U_2, \ldots$ individually and combining the result.

One serious issue is to show that

$$\sum_k \left( \sum_{i,j} |U_k \cap A_{ij}|_{\rho} - 2|U_k|_{\rho} \right) \geq \sum_{i,j} |U \cap A_{ij}(K')|_{d\rho} - 2|U|_{d\rho} \quad (31)$$

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for the slicing. Another serious issue is to show that $f'$ is a borrowing function; $f'$ will be Lipschitz provided that $f$ is and that the splitting $U \mapsto (U_1, \ldots)$ is Lipschitz in that adding one vertex to $U$ adds exactly one vertex to exactly one of the $U_i$.

Notice that if $f = 0$ we can take $f' = 0$, giving us the conditions of being a borrowing function and of being Lipschitz automatically. Then we can split $U$ as equally as possible into $d$ sets $U_1, \ldots, U_d$, and we have

$$\sum_i |U_i|_\rho = |U|_{dp}.$$ 

Since $|S|_n + |T|_m \geq |S \amalg T|_{n+m}$ for any $S, T, n, m$ we conclude

$$\sum_k \sum_{i,j} |U_k \cap A_{ij}|_\rho \geq \sum_{i,j} |U \cap A_{ij}(K')|_{dp}.$$ 

We conclude equation (31), and easily see that in the case where $f = 0$, i.e., where $K$ is balanced, that $K'$ is balanced.

Notice that a splitting like the symmetrization gives $f'$ that is Lipschitz and a borrowing function; however the symmetrization can violate equation (31) (even for $d = 2, \rho = 1$, $|U| = |U_1| = 2$, with $U$ being two degree two vertices of opposite “type”).

### 4 Conclusion of the Proof of the SHNC

In this section we use canonical paths to finish the proof of the SHNC (i.e., Theorem 1.1).

We know it suffices to prove Theorem 1.1 for JKM graphs. Hence, by Theorem 1.6, it suffices to prove that for any JKM graph, $K$, is balanceable.

So let $K$ be a JKM graph. Let $p_1, \ldots, p_{\rho(K)}$ be canonical paths in $K$, and for each path $p_i$ let $w_i$ be the word over $\{a, a^{-1}, b, b^{-1}\}$ giving the colour pattern of $p_i$, and let $M$ be the disjoint union of the $M(w_i)$. By the result of Tardos we know that each $M(w_i)$ is universal for the SHNC, and hence so is $M$. Hence $M$ is balanceable with a Lipschitz balancing function, $f$, which is definable on $M'$, the graph $M$ with its self-loops removed. The inclusion of the canonical paths in $K$ gives rise to a morphism $\phi: M' \rightarrow K$. Consider

$$f'(U) = f(\phi^{-1}(U)).$$
Let $T$ be the $2\rho(K)$ vertices of degree three, and let $T'$ be those vertices of $T$ that occur as an intermediate point in a canonical path; consider the superweight on $V_K$

$$\|U\| = |U| + |U \cap T'|.$$  

We claim that $f'$ is a balancing function for this superweight. Indeed, let $f = f_1 + f_2$ with $f_i$ a Lipschitz borrowing arrangement through colour $i$, and let

$$f'_i(U) = f_i(\phi^{-1}(U)).$$

So we finish the proof of the SHNC by showing (1) each $f'_i$ is a borrowing function through colour $i$, (2) each $f'_i$ is Lipschitz, and (3) equation (13) holds with $f'$ replacing $f$, i.e.,

$$f'(U) + \sum_j \|U \cap A_{1j}\|_{\rho} + \sum_j |U \cap A_{2j}|_{\rho} \leq 2\|U\|_{\rho}.$$

We start by showing that $f'_i$ is a borrowing function through colour $i$; the reader will note that this is a general result true of any morphism $\phi: M' \to K$ of arbitrary graphs. For $i = 1, 2$, let $h_i: \mathcal{P}(A_{1i}(M')) \to \mathbb{R}$ be such that for all $W \subset V_{M'}$ we have

$$f_i(W) = h_i(W \cap A_{1i}(M')) - h_i(\iota_{1,ht,M'}(W \cap A_{ih}(M'))).$$

So

$$f'_i(U) = h_i(\phi^{-1}U \cap A_{1i}(M')) - h_i(\iota_{1,ht,M'}(\phi^{-1}U \cap A_{ih}(M'))).$$

Let $h'_i: \mathcal{P}(A_{1i}(M')) \to \mathbb{R}$ be the function given by

$$h'_i(U) = h_i(\phi^{-1}(U) \cap A_{1i}(M')).$$

To show that $f'_i$ is borrowing through colour $i$ it suffices to show

$$h_i(\phi^{-1}U \cap A_{1i}(M')) = h'_i(U \cap A_{1i}(K)), \quad (32)$$

and that

$$h_i(\iota_{1,ht,M'}(\phi^{-1}U \cap A_{ih}(M'))) = h'_i(\iota_{1,ht,K}U \cap A_{ih}(K)). \quad (33)$$

Since for any $U \subset V_K$ we have

$$(\phi^{-1}U) \cap A_{1i}(M') = \phi^{-1}(U \cap A_{1i}(K)) \cap A_{1i}(M'),$$

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we have equation (32). Since $\phi$ is a graph homomorphism we have
\begin{equation*}
\phi \iota_{1,ht,M'} = \iota_{1,ht,K}\phi,
\end{equation*}
and hence for any $U \subset V_K$ we have
\begin{equation*}
\iota_{i,ht,M'}(\phi^{-1}U \cap A_{ih}(M')) = \phi^{-1}(\iota_{i,ht,K}(U \cap A_{ih}(K))) \cap A_{ih}(M').
\end{equation*}
Hence equation (33) follows.

Next we show that each $f^0_i$ is Lipschitz. Note that $\phi$ is injective on $C_i = A_{it}(M') \cap A_{ih}(M')$ (basically since $M'$ does not contain the colour 1 self-loops). Now $f^0_i(U) = f_i(\phi^{-1}(U) \cap C_i)$, and so for $S \subset T \subset V_K$ we have
\begin{equation*}
|f^0_i(T) - f^0_i(S)| \leq |(\phi^{-1}(T) \cap C_i) \setminus (\phi^{-1}(S) \cap C_i)| \leq |T \setminus S|.
\end{equation*}
It remains to see that equation (9) holds with $f'$ replacing $f$. For any $U \subset V_K$ we have $f'(U) = f(\phi^{-1}U)$, and so
\begin{equation*}
f'(U) + \sum_{i,j} |(\phi^{-1}U) \cap A_{ij}(M)|_{\rho} \leq 2|\phi^{-1}U|_{\rho},
\end{equation*}
where $\rho = \rho(M) = \rho(K)$. Letting $U' = U \cap \phi(V_{M'})$, we have
\begin{equation*}
|\phi^{-1}(U)| = |U'| + |U' \cap T'|,
\end{equation*}
since $\phi$ is one-to-one on vertices except that it two-to-one regarding $T'$ and its preimage. Similarly
\begin{equation*}
|(\phi^{-1}U) \cap A_{ij}(M)| = \begin{cases} 
|U' \cap A_{ij}(K)| + |U' \cap T'| & \text{if } i = 1, \\
|U' \cap A_{ij}(K)| & \text{if } i = 2
\end{cases}
\end{equation*}
This establishes equation (13) for $U'$ replacing $U$, i.e.,
\begin{equation*}
f'(U') + \sum_{j} |U' \cap A_{1j}|_{\rho} + \sum_{j} |U' \cap A_{2j}|_{\rho} \leq 2\|U'\|_{\rho}.
\end{equation*}
Now we claim that this implies the same equation with $U$ replacing $U'$. First note that $f'(U) = f'(U')$. So we can ignore $f'$ and focus on how replacing $U'$ by $U$ affects the other terms. We may obtain $U$ from $U'$ by adding a succession of vertices outside the image of $\phi$. Hence, by induction, it suffices to show that if the above inequality holds for any $U' \subset V_K$, and if
\( U = U' \cap \{u\} \) with \( u \) outside the image of \( \phi \), then the inequality holds for \( U \) replacing \( U' \). But then \( u \) is of degree two and outside of \( T' \); so if \( \|U'\| \geq \rho \), \( \|U'\|_\rho \) is increased by one upon adding \( u \), and the sum of the \( \|U' \cap A_{1j}\|_\rho \) and \( |U' \cap A_{2j}|_\rho \) is increased by at most two, and the inequality is maintained; otherwise, if \( \|U'\| < \rho \), adding \( u \) gives \( |U| \leq \|U\| \leq \rho \), and so for \( W = U \) or \( W = U' \) we have
\[
\|W\|_\rho = \|W \cap A_{1j}\|_\rho = |W \cap A_{2j}|_\rho = 0,
\]
and the inequality is preserved.

\( \square \)

5 Concluding Remarks

Earlier versions of this work contained Galois theory for graphs and sheaf and cohomology; these will be discussed\(^7\) in the sequel to this paper ([Frib]). The sheaf theory inspired our notions of balanced and balancing functions. The Galois theory seems necessary or at least helpful for the sheaf approach, but it was not needed here.

The JKM transformation simplifies things, although we can do without it. Namely, the JKM transformation teaches us to view a vertex of degree four as two of degree three; the canonical paths were defined on an étale bigraph, \( K \), even if some of the vertices have degree four based on this, based on this view; we just need to choose a majority colour for each degree four vertex. If we always choose colour 1 as the majority colour, then we can then use balancing functions along these paths to give a balancing function for \( K \) as before. If not we need to generalize our discussion of superweights to allow each of \( V_K \) and the \( A_{ij}(K) \) to have their own superweights, as discussed after Theorem 3.2. This eliminates the JKM transformation, but uses the ideas we learned from their work.

At present we have very little knowledge about the explicit construction of balancing functions or how they behave under basic operations. Their study may yield simplified proofs of the SHNC, especially since we might only need the relatively simple Theorem 1.6, and not the duality theory and more difficult Theorems 1.7 and 1.9.

\(^7\)These were also discussed in [Fria].
We mention that one can borrow across a collection of paths of the same
colour pattern, not just the edges of colour 1 or 2. But this “borrowing
through paths” can be written as a succession of borrowing across edges of
colour 1 or 2, so we get no new balancing functions.

Finally, the reader will notice that while we know every étale bigraph has
a Lipschitz borrowing function, our converse only requires a pseudo-Lipschitz
borrowing function, and in fact we only apply equation (6) for $U$’s in which
$|U| \geq \rho(K)$, or really $|U| > \rho(K)$ if we like (of course, for some of these $U$’s,
some of the $|U \cap A_{ij}|$ may be less than $\rho(K)$). So we are free to “forget”
about $f(U)$ and equation (6) for $|U| \leq \rho(K)$. However, it may be useful in the
future to know that there exist a balancing function satisfying equation (6)
for all $U$ and which is Lipschitz, not merely pseudo-Lipschitz.

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