Consider the solution to the Markowitz utility of the model \( R = w_1 A + w_2 M \) subject to \( w_1, w_2 \geq 0 \) and \( w_1 + w_2 = 10 \).

(a) What is \( w_1^* \) for the values \( \mu = .05 \), \( \mu = 1 \), and \( \mu = 1000 \)?

(b) Explain intuitively—in terms of \( \mu \) representing risk aversion—why \( w_1 \) is very close to 0 for \( \mu = 1000 \).

(c) For the analogous solution for the model \( R = w_1 A + w_2 B \) (subject to the same conditions), what is the value of \( w_1^* \) for the value \( \mu = .05 \)?

(d) Explain intuitively—in terms of the difference between the models \( w_1 A + w_2 M \) and \( w_1 A + w_2 B \)—why for \( \mu = .05 \) the value of \( w_1 \) is 10 for one of the models and less than 10 for the other.

Solution:

(a) Respectively, \( w_1^* = 10, 5/4, 5/4000 \).

(b) With \( \mu = 1000 \) we are very averse to risk and hence want to invest mostly in \( M \) (which has zero variance and is therefore “without risk”).

(c) \( w_1^* = (1/8)(0.05)^{-1} = 1/(.4) = 2.5 \).

(d) The return on the riskless instruments \( M, B \) are respectively 0, 9, and hence higher for \( B \). Hence, for the same amount of risk aversion, we will spend more on \( B \) in a mix of \( A \) and \( B \) than in...
M in a mix of A and M. Hence it makes sense that for certain values of \( \mu \) we may invest everything in A and nothing in M in the optimal investment, and when we replace M by B it becomes more desirable to invest something in B.

(2) Compute the Markowitz Utility \( U(w_1, w_2, \mu) \) for the portfolio \( R = w_1A + w_2N \). [Note the formula \( \text{Corr}(A, N) = -1 \) in the Section 7, and note that the formulas in Section 8 imply that
\[
\text{Var}(w_1X + w_2Y) = w_1^2 \text{Var}(X) + 2w_1w_2 \text{Cov}(X, Y) + w_2^2 \text{Var}(Y).
\]
Then find the optimum feasible solution for this model under the conditions \( w_1, w_2 \geq 0 \) and \( w_1 + w_2 = 10 \).

**Solution:** We have
\[
\text{Cov}(A, N) = \text{Corr}(A, N) \sqrt{\text{Var}(A) \text{Var}(N)} = -4,
\]
and hence
\[
\text{Var}(w_1A + w_2N) = w_1^2 \text{Var}(A) + 2w_1w_2 \text{Cov}(A, N) + w_2^2 \text{Var}(N)
\]
\[
= w_1^2 4 - 2w_1w_24 + w_2^2 4.
\]
Under the condition \( w_2 = 10 - w_1 \) we therefore have
\[
\text{Var}(w_1A + w_2N) = w_1^2 4 - 2w_1(10 - w_1)4 + (10 - w_1)24 = 16w_1^2 - 160w_1 + 400.
\]
We also have
\[
w_1A + w_2N = 10w_1 + 10w_2 = 10w_1 + 10(10 - w_1) = 100.
\]
Hence the Markowitz utility is
\[
U(\mu; w_1A + w_2N) = w_1A + w_2N - \mu \text{Var}(w_1A + w_2N)
\]
\[
= 100 - \mu (16w_1^2 - 160w_1 + 400).
\]
Differentiating in \( w_1 \) we see that the maximum is attained when \( 32w_1 - 160 = 0 \) or \( w_1 = 5 \).

[This should make sense, since A, N have the same expected return, so \( w_1A + w_2N = 100 \) regardless of the investment, and \( A + N = 20 \) is “riskless,” and so \( 5A + 5N = 100 \) is also “riskless.”]

(3) Compute the Markowitz Utility \( U(w_1, w_2, \mu) \) for the portfolio \( R = w_1C + w_2P \). Then find all optimum feasible solutions for this model under the conditions \( w_1, w_2 \geq 0 \) and \( w_1 + w_2 = 10 \).

**Solution:** First solution: since \( C = P \), if \( w_1 + w_2 = 10 \), then \( w_1C + w_2P = w_1C + w_2C = 10C \). Hence any \( w_1^* \) with \( 0 \leq w_1^* \leq 10 \) gives the same (and therefore optimal) utility.

Second solution: We can also find this via computation: under the condition \( w_2 = 10 - w_1 \) we have:
\[
w_1C + w_2P = w_110 + w_210 = w_110 + (10 - w_1)10 = 100,
\]
and $\text{Cov}(C, P) = \text{Cov}(C, C) = 4$ (since $P = C$, or we can derive this using the fact that $\text{Corr}(P, C) = 1$ since they are the same instrument). Hence

$$\text{Var}(w_1C + w_2P) = w_1^24 + 2w_1w_24 + w_2^24$$

$$= w_1^24 + 2w_1(10 - w_1)4 + (10 - w_1)^24 = 400$$

and hence the Markowitz utility is $100 - \mu 400$, which is independent of $w_1$. Hence any $w_1^*$ with $0 \leq w_1^* \leq 10$ gives the optimal utility.