Consider the solution (??) to the Markowitz utility of the model $R = w_1 A + w_2 M$ subject to $w_1, w_2 \geq 0$ and $w_1 + w_2 = 10$.

(a) What is $w_1^*$ for the values $\mu = 0.05$, $\mu = 1$, and $\mu = 1000$?

(b) Explain intuitively—in terms of $\mu$ representing risk aversion—why $w_1$ is very close to 0 for $\mu = 1000$.

(c) For the analogous solution (??) for the model $R = w_1 A + w_2 B$ (subject to the same conditions), what is the value of $w_1^*$ for the value $\mu = 0.05$?

(d) Explain intuitively—in terms of the difference between the models $w_1 A + w_2 M$ and $w_1 A + w_2 B$—why for $\mu = 0.05$ the value of $w_1$ is 10 for one of the models and less than 10 for the other.

Solution:

(a) Respectively, $w_1^* = 10, 5/4, 5/4000$.

(b) With $\mu = 1000$ we are very averse to risk and hence want to invest mostly in $M$ (which has zero variance and is therefore “without risk”).

(c) Respectively, $w_1^* = 5, 1/8, 1/8000$.

(d) The return on the riskless instruments $M, B$ are respectively 0, 9, and hence higher for $B$. Hence, for the same amount of risk aversion, we will spend more on $B$ in a mix of $A$ and $B$ than in...
$M$ in a mix of $A$ and $M$. Hence it makes sense that for certain values of $\mu$ we may invest everything in $A$ and nothing in $M$ in the optimal investment, and when we replace $M$ by $B$ it becomes more desirable to invest something in $B$.

(2) Compute the Markowitz Utility $U(w_1, w_2, \mu)$ for the portfolio $R = w_1A + w_2N$. [Note the formula $\text{Corr}(A, N) = -1$ in the Section 7, and note that the formulas in Section 8 imply that $\text{Var}(w_1X + w_2Y) = w_1^2 \text{Var}(X) + 2w_1w_2 \text{Cov}(X, Y) + w_2^2 \text{Var}(Y)$.]

Then find the optimum feasible solution for this model under the conditions $w_1, w_2 \geq 0$ and $w_1 + w_2 = 10$.

Solution: We have

\[ \text{Cov}(A, N) = \text{Corr}(A, N) \sqrt{\text{Var}(A) \text{Var}(N)} = -4, \]

and hence

\[ \text{Var}(w_1A + w_2N) = w_1^2 \text{Var}(A) + 2w_1w_2 \text{Cov}(A, N) + w_2^2 \text{Var}(N) \]
\[ = w_1^2 - 2w_1w_2 + w_2^2. \]

Under the condition $w_2 = 10 - w_1$ we therefore have

\[ \text{Var}(w_1A + w_2N) = w_1^2 - 2w_1(10 - w_1) + (10-w_1)^2 = 16w_1^2 - 160w_1 + 400. \]

We also have

\[ w_1A + w_2N = 10w_1 + 10w_2 = 10w_1 + 10(10 - w_1) = 100. \]

Hence the Markowitz utility is

\[ U(\mu; w_1A + w_2N) = w_1A + w_2N - \mu \text{Var}(w_1A + w_2N) \]
\[ = 100 - \mu(16w_1^2 - 160w_1 + 400). \]

Differentiating in $w_1$ we see that the maximum is attained when $32w_1 - 160 = 0$ or $w_1 = 5$.

[This should make sense, since $A, N$ have the same expected return, so $w_1A + w_2N = 100$ regardless of the investment, and $A + N = 20$ is “riskless,” and so $5A + 5N = 100$ is also “riskless.”]

(3) Compute the Markowitz Utility $U(w_1, w_2, \mu)$ for the portfolio $R = w_1C + w_2P$. Then find all optimum feasible solutions for this model under the conditions $w_1, w_2 \geq 0$ and $w_1 + w_2 = 10$.

Solution: First solution: since $C = P$, if $w_1 + w_2 = 10$, then $w_1C + w_2P = w_1C + w_2C = 10C$. Hence any $w_1^* = w_1$ with $0 \leq w_1^* \leq 10$ gives the same (and therefore optimal) utility.

Second solution: We can also find this via computation: under the condition $w_2 = 10 - w_1$ we have:

\[ w_1^*C + w_2^*P = w_110 + w_210 = w_110 + (10 - w_1)10 = 100. \]
and $\text{Cov}(C, P) = \text{Cov}(C, C) = 4$ (since $P = C$, or we can derive this using the fact that $\text{Corr}(P, C) = 1$ since they are the same instrument). Hence

$$\text{Var}(w_1C + w_2P) = w_1^24 + 2w_1w_24 + w_2^24$$

$$= w_1^24 + 2w_1(10 - w_1)4 + (10 - w_1)^24 = 400$$

and hence the Markowitz utility is $100 - \mu 400$, which is independent of $w_1$. Hence any $w_1^*$ with $0 \leq w_1^* \leq 10$ gives the optimal utility.