(1) Use branch and bound to solve the integer linear program max $\mathbf{c}^T \mathbf{x}$ subject to $A\mathbf{x} \leq \mathbf{b}$ and $\mathbf{x} \geq 0$ and $\mathbf{x} \in \mathbb{Z}^2$ (i.e., $x_1, x_2$ must be integers)

$$\mathbf{c} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 29.8 \\ 7.3 \\ 8.2 \end{bmatrix}.$$

Do not make use of the specific properties of $A, \mathbf{b}, \mathbf{c}$ in this problem (i.e., that they all have non-negative entries/coefficients). Specifically:

(a) Enter the corresponding LP into your LP software; you should find that the optimum solution is $x_1 = 0.9, x_2 = 6.4$, and $z = 22.8.$

(b) Try the following branches: $x_2 \leq 6$ and $x_2 \geq 7$. If you need to explore the $x_2 \leq 6$ branch further, divide this branch into $x_2 \leq 5$ and $x_2 = 6$; if you need to explore the $x_2 \geq 7$ branch further, divide this branch into $x_2 = 7$ and $x_2 \geq 8$. (You should find that the branch $x_2 \geq 8$ is infeasible.)

(c) When you reach a branch with $x_2$ fixed, branch on $x_1$ in a similar fashion (solve the relaxed LP, and round up and down).

(d) Complete the branch and bound, and make a diagram of the result.
Solution: First LP solution: \((x_1, x_2, z) = (0.9, 6.4, 22.8)\). Branching according to the instructions on the homework, the branch tree that we will explore looks like this:

(a) Branch 1: \(x_2 \leq 6\).
   (i) Sub-branch 1.1: \(x_2 = 6\).
   (ii) Sub-branch 1.2: \(x_2 \leq 5\).

(b) Branch 2: \(x_2 \geq 7\).
   (i) Sub-branch 2.1: \(x_2 \geq 8\).
   (ii) Sub-branch 2.2: \(x_2 = 7\).

Now we have to start exploring the sub-branches. Since we want to quickly find a feasible integral solution, we should explore sub-branch 1.1 (\(x_2 = 6\)) or 2.2: (\(x_2 = 7\)).

Let’s begin exploring sub-branch 2.2: we solve the LP with the constraint \(x_2 = 7\); the optimal solution turns out to be \((x_1, x_2, z) = (0.3, 7, 22.2)\). Hence we have two further sub-branches to explore:

(a) Sub-sub-branch 2.2.1: \(x_1 \geq 1\).

(b) Sub-sub-branch 2.2.2: \(x_1 \leq 0\).

Solving both these LPs (since will to in order to fully explore this branch) gives the optimal solutions:

(a) Sub-sub-branch 2.2.1: \(x_1 \geq 1\). Infeasible.

(b) Sub-sub-branch 2.2.2: \(x_1 \leq 0\). \((x_1, x_2, z) = (0, 7, 21)\) (which is an integral feasible solution!).

So far our tree looks like the following:

Root: \((x_1, x_2, z) = (0.9, 6.4, 22.8)\)

(a) Branch 1: \(x_2 \leq 6\). Not yet explored
   (i) Sub-branch 1.1: \(x_2 = 6\). Not yet explored
   (ii) Sub-branch 1.2: \(x_2 \leq 5\). Not yet explored

(b) Branch 2: \(x_2 \geq 7\). \((x_1, x_2, z) = (0.3, 7, 22.2)\)
   (i) Sub-branch 2.1: \(x_2 \geq 8\). Not yet explored
   (ii) Sub-branch 2.2: \(x_2 = 7\). \((x_1, x_2, z) = (0.3, 7, 22.2)\)
      (A) Sub-sub-branch 2.2.1: \(x_1 \geq 1\). Infeasible.
      (B) Sub-sub-branch 2.2.2: \(x_1 \leq 0\). \((x_1, x_2, z) = (0, 7, 21)\)

And the best feasible integral solution that we have found has \(z = 21\).

Now we try to eliminate some other branches. Solving Sub-branch 2.1 \((x_2 \geq 8)\) yields: infeasible, which completes all of branch 2.

So all that is left is to descend and search branch 1. We solve \(x_2 \leq 6\) and get the solution \((x_1, x_2, z) = (1.1, 6, 22.4)\). Since the \(z\) value here is greater than 21, which is our current best, we have to descend the tree further. We solve both LP’s on sub-branches 1.1 and 1.2 which gives us:

Root: \((x_1, x_2, z) = (0.9, 6.4, 22.8)\)

(a) Branch 1: \(x_2 \leq 6\). \((x_1, x_2, z) = (1.1, 6, 22.4)\)
   (i) Sub-branch 1.1: \(x_2 = 6\). \((x_1, x_2, z) = (1.1, 6, 22.4)\)
   (ii) Sub-branch 1.2: \(x_2 \leq 5\). \((x_1, x_2, z) = (1.6, 5, 21.4)\)

(b) Branch 2: \(x_2 \geq 7\). \((x_1, x_2, z) = (0.3, 7, 22.2)\)
(i) Sub-branch 2.1: \( x_2 \geq 8 \). Not yet explored
(ii) Sub-branch 2.2: \( x_2 = 7 \). \((x_1, x_2, z) = (0.3, 7, 22.2)\)
   (A) Sub-sub-branch 2.2.1: \( x_1 \geq 1 \). Infeasible.
   (B) Sub-sub-branch 2.2.2: \( x_1 \leq 0 \). \((x_1, x_2, z) = (0, 7, 21)\)

Still nothing in branch 1, i.e., sub-branches 1.1 or 1.2, is eliminated since our best current feasible solution is 21 (from sub-sub-branch 2.2.2). Let’s now descend sub-branch \( x_2 = 6 \), bounding on \( x_1 \leq 1 \) or \( x_2 \geq 2 \):
   (a) Sub-sub-branch 1.1.1: \( x_2 = 6, x_1 \geq 2 \). Infeasible
   (b) Sub-sub-branch 1.1.2: \( x_2 = 6, x_1 \leq 1 \). \((x_1, x_2, z) = (1, 6, 22)\), which is a feasible integral solution with higher objective value than the best so far.

The new value \( z = 22 \) is the best integral feasible solution so far. This solution is better than \( x_1 \leq 5 \), so we eliminate this sub-branch. This gives the complete branch and bound tree below:

Root: \((x_1, x_2, z) = (0.9, 6.4, 22.8)\)
   (a) Branch 1: \( x_2 \leq 6 \). \((x_1, x_2, z) = (1.1, 6, 22.4)\)
      (i) Sub-branch 1.1: \( x_2 = 6 \). \((x_1, x_2, z) = (1.1, 6, 22.4)\)
         (A) Sub-sub-branch 1.1.1: \( x_2 = 6, x_1 \geq 2 \). Infeasible
         (B) Sub-sub-branch 1.1.2: \( x_2 = 6, x_1 \leq 1 \). \((x_1, x_2, z) = (1, 6, 22)\).
      (ii) Sub-branch 1.2: \( x_2 \leq 5 \). \((x_1, x_2, z) = (1.6, 5, 21.4)\) which was eliminated because of \( z = 22 \) in sub-sub-branch 1.1.2.
   (b) Branch 2: \( x_2 \geq 7 \). \((x_1, x_2, z) = (0.3, 7, 22.2)\)
      (i) Sub-branch 2.1: \( x_2 \geq 8 \). Infeasible
      (ii) Sub-branch 2.2: \( x_2 = 7 \). \((x_1, x_2, z) = (0.3, 7, 22.2)\)
         (A) Sub-sub-branch 2.2.1: \( x_1 \geq 1 \). Infeasible.
         (B) Sub-sub-branch 2.2.2: \( x_1 \leq 0 \). \((x_1, x_2, z) = (0, 7, 21)\)

(2) Try the above branch and bound method on the integer program with
\[
\vec{c} = \begin{bmatrix} 1 \\ 500 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 100 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2030 \end{bmatrix}.
\]

Specifically, try branch and bound by searching the possible values of \( x_2 \) based on the LP relaxation, branching on \( x_1 \) values on branches where the \( x_2 \) value has been fixed. Then do the same where you first branch on \( x_1 \) values, then \( x_2 \) values. Is there a significant difference? Explain.

**Solution:** The original LP has optimal solution \((x_1, x_2, z) = (0, 20.3, 10150)\). Say we branch on \( x_2 \leq 20 \) and \( x_2 \geq 21 \). We get
   (a) Branch 1: \( x_2 \leq 20 \). Optimal solution is \((x_1, x_2, z) = (30, 20, 10030)\), which is integral.
   (b) Branch 2: \( x_2 \geq 21 \). Infeasible.
Now say we branch on $x_1$ in the way described in the exercise. We get

(a) Branch 1: $x_1 = 0$. Optimal: $(x_1, x_2, z) = (0, 20, 10000)$.

(b) Branch 2: $x_1 \geq 1$. Optimal: $(x_1, x_2, z) = (1, 20.29, 10146)$.

(i) Sub-branch 2.1: $x_1 = 1$. Optimal $(x_1, x_2, z) = (1, 20.29, 10146)$.

(A) Sub-sub-branch 2.1.1: $x_2 \leq 20$. Not yet explored.

(B) Sub-sub-branch 2.1.1: $x_2 \leq 21$. Not yet explored.

(ii) Sub-branch 2.2: $x_1 \geq 2$. Optimal $(x_1, x_2, z) = (2, 20.28, 10146)$.

(A) Sub-sub-branch 2.2.1: $x_1 = 2$. Not yet explored.

(B) Sub-sub-branch 2.2.2: $x_1 \geq 3$. Not yet explored.

We are similarly going to have to expand the $x_1$ search until we get to $x_1 \geq 30$. Even if we know don’t explore sub-sub-branch 2.2.1 $x_1 = 2$, sub-sub-sub-branch 2.2.2.1 $x_1 = 3$, sub-sub-sub-sub-branch 2.2.2.2.1 $x_1 = 4$, etc., and wait for $x_1 \geq 30$, we still solve a lot of LP’s that are not very interesting. The problem is that $x_1$ is much less influential on the objective, $z$, (given the constraints), and the root LP has $x_1 = 0$, which is far from the optimal $x_1 = 30$ value.