(1) Consider the constraint \( x_1 + x_2 \leq 1 \): for any feasible \( \vec{x} \) such that \( x_1 + x_2 < 1 \), it is always possible to increase \( x_1 \) (or \( x_2 \)) a bit and preserve the inequality \( x_1 + x_2 \leq 1 \). However, since the entries of \( A \) are all positive, any increase in \( x_1 \) will strictly increase \( v \). Hence \( \vec{x} \) cannot be an optimal solution for the objective \( z = v \) unless \( x_1 + x_2 = 1 \).

(2) The dual LP is

\[
\text{minimize} \quad w, \quad \text{subject to} \quad \\
11y_1 + 9y_2 \leq w, \\
8y_1 + 12y_2 \leq w, \\
y_1 + y_2 \geq 1 \\
\text{and} \quad y_1, y_2, w \geq 0.
\]

(We have chosen to call the third dual variable \( w \) instead of \( y_3 \) because the minimum value of \( w \) will be the value that will give an upper bound on the objective, \( z \).) Let \( y_3, y_4, y_5 \) be the slack variables for the first three inequalities,

\[
y_3 = w - 11y_1 - 9y_2, \quad y_4 = w - 8y_1 - 12y_2, \quad y_5 = -1 + y_1 + y_2,
\]

and let \( x_3, x_4, x_5 \) be the slack variables for the primal (as given by the solutions to Homework 3):

\[
x_3 = -v + 11x_1 + 8x_2, \quad x_4 = -v + 9x_1 + 12x_1, \quad x_5 = 1 - x_1 - x_2.
\]

We have the correspondence:

\[
x_1 \leftrightarrow y_3, \quad x_2 \leftrightarrow y_4, \quad v \leftrightarrow y_5, \quad x_3 \leftrightarrow y_1, \quad x_4 \leftrightarrow y_2, \quad x_5 \leftrightarrow w.
\]

(a) \( x_1 = x_2 = 1/2, \quad v = 19/2 \). From the solutions to Homework 3 we compute \( x_3 = 0, \quad x_4 = 1, \quad x_5 = 0 \). So \( x_1, x_2, x_4 \) are positive, which forces \( y_3 = y_4 = y_5 = y_2 = 0 \). \( y_3 = y_4 = y_5 = 0 \) give the three equations:

\[
11y_1 = w, \quad 8y_1 = w, \quad y_1 = 1
\]

(since \( y_2 = 0 \)). This system is inconsistent, since \( y_1 = 1 \) forces both \( w = 11 \) and \( w = 8 \). Hence the proposed solution is not optimal.
(b) $x_1 = 1/3, x_2 = 2/3, v = 9$; we have $x_3 = 0, x_4 = 2, x_5 = 0$. This forces $y_3 = y_4 = y_5 = y_2 = 0$, which we have seen in part (a) leads to an inconsistent system. Hence the proposed solution is not optimal.

(c) $x_1 = 2/3, x_2 = 1/3, v = 10$; we have $x_3 = 0, x_4 = 0, x_5 = 0$, which forces $y_3 = y_4 = y_5 = 0$. This gives the equations

$$0 = w - 11y_1 - 9y_2 = w - 8y_1 - 12y_2 = -1 + y_1 + y_2,$$

which we solve to find $y_1 = y_2 = 1/2, w = 10$, which is therefore an optimal solution (all the $x$'s and $y$'s are non-negative and satisfy complementary slackness).

(d) $x_1 = 1, x_2 = 0, v = 9$; we have $x_3 = 2, x_4 = 0, x_5 = 0$. This forces $y_3 = y_5 = y_1 = 0$. $y_3 = y_5 = 0$ gives the two equations

$$9y_2 = w, \quad y_2 = 1$$

(since $y_1 = 0$). This implies $y_2 = 1, w = 9$; the last variable to check is $y_4$, but

$$y_4 = w - 8y_1 - 12y_2 - v = 9 + 0 - 12 = -3,$$

which is negative. Hence the proposed solution is not optimal.

(e) $x_1 = 0, x_2 = 0, v = 0$; we have $x_3 = 0, x_4 = 0, x_5 = 1$. This forces $w = 0$. This leaves us with

$$y_3 = -11y_1 - 9y_2, \quad y_4 = -8y_1 - 12y_2, \quad y_5 = -1 + y_1 + y_2.$$

This set of three equations in five unknowns has many solutions, but it is not hard to see that none of these solutions can have $y_1, \ldots, y_5$ all non-negative. Indeed, the equation $y_3 = -11y_1 - 9y_2$ cannot have either $y_1$ or $y_2$ positive, or else $y_3$ would be negative; so $y_1 = y_2 = y_3 = 0$. The equation $y_5 = -1 + y_1 + y_2$ then forces $y_5 = -1$, which is negative. Hence the proposed solution is not optimal.

(3) The dual LP is already written above. If Betty plays columns 1 and 2 with frequencies $y_1$ and $y_2$, then Alice chooses

$$\max(11y_1 + 9y_2, 8y_1 + 12y_2),$$

which is the same thing as the smallest $v$ such that

$$11y_1 + 9y_2 \leq v \quad \text{and} \quad 8y_1 + 12y_2 \leq v.$$

Hence Betty will choose the non-negative $y_1, y_2$ subject to $y_1 + y_2 = 1$ such that $v$ is minimized. There is no harm in replacing $y_1 + y_2 = 1$ with $y_1 + y_2 \geq 1$ here, since if $y_1 + y_2 > 1$, then one cannot be at optimality since Betty can choose a slightly smaller $y_1$ (or $y_2$) and get a smaller $w$ value. Hence the dual LP above is precisely the LP that describes Betty’s best mixed strategy, with $w$ being the value of this game.