Homework #2

1. Consider the LP:

   \[ \begin{align*}
   \text{max } & \quad 6x_1 + 7x_2, \\
   \text{s.t. } & \quad x_1 \leq 5, \quad x_2 \leq 8, \\
   & \quad x_1 + x_2 \leq 10, \quad x_1, x_2 \geq 0.
   \end{align*} \]

   (a) Solve this using the simplex method, starting with \( x_2 \) entering the basis after the first dictionary (i.e., \( x_1, x_2 \) will be non-basic in the first dictionary, and you should hold \( x_1 \) fixed at 0 and increase \( x_2 \)).

   (b) Solve this using the simplex method, starting with \( x_1 \) entering the basis after the first dictionary.

2. Consider a matrix game, \( A \). Let \( \mathbf{x} \) be a stochastic vector—a vector of non-negative components whose sum is 1—such that

   \[ \mathbf{x}^T A \geq [v \ v \ldots \ v], \tag{1} \]

   for some number \( v \) i.e., each entry of \( \mathbf{x}^T A \) is at least \( v \). [For example, if

   \[ A = \begin{bmatrix} 0 & 1 \\ 1/2 & 0 \end{bmatrix}, \]

   we know that the value of “Alice announces a mixed strategy” equals 1/3; by choosing \( \mathbf{x}^T = [1/2 \ 1/2] \) (for no particular reason) we have

   \[ \mathbf{x}^T A = [1/4 \ 1/2] \geq [v \ v] \]

   where \( v = 0.1 \) (or \( v = 0.2 \) or \( v \) can be anything \( \leq 1/4 \).] Explain why:

   (a) the value of “Alice announces a mixed strategy” is at least \( v \);

   (b) if \( \mathbf{y} \) is another stochastic vector, then explain why

   \[ \mathbf{x}^T A \mathbf{y} \geq v; \]
(c) similarly, if \( y \) is a stochastic vector such that

\[
Ay \leq \begin{bmatrix} w \\ \vdots \\ w \end{bmatrix},
\]

explain why the value of “Betty announces a mixed strategy” is at most \( w \), and why for any stochastic \( x \) we have

\[
x^T Ay \leq w.
\]

(d) Show that if \( x \) and \( y \) are stochastic vectors such that Equations 1 and 2 hold, then \( v \leq w \).

(e) If it turns out that for a matrix \( A \) we have

\[
\begin{bmatrix} 0.5 & 0.5 \end{bmatrix} A = \begin{bmatrix} 1.2 & 1.4 & 1.1 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 0.1 \\ 0.3 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 1.1 \\ 1.3 \end{bmatrix},
\]

what can you say about the value of the mixed strategy games for \( A \)?

(f) Is it possible that for some matrix \( A \) we have

\[
\begin{bmatrix} 0.6 & 0.4 \end{bmatrix} A = \begin{bmatrix} 1.2 & 1.4 & 1.1 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 0.1 \\ 0.3 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 0.8 \end{bmatrix}?
\]

3. Use the simplex method to find the value of “Alice announces a mixed strategy” and Alice’s optimal mixed strategy for the matrix game

\[
A = \begin{bmatrix} 1 & 3 \\ 8 & 1 \end{bmatrix}
\]

as discussed in class, i.e., maximizing \( v \) subject to \( [x_1 \ 1 - x_1] A \geq [v \ v] \), \( 1 - x_1 \geq 0 \) and \( x_1, v \geq 0 \).

Find the value of “Betty announces a mixed strategy” and Betty’s optimal mixed strategy for the above matrix game by the methods used in Homework #1. Do you see Betty’s optimal strategy somewhere in the final dictionary of the simplex method above?