(1) (a) Let
\[ \mathbf{v}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \quad \mathbf{\delta}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{\delta}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

(b) Consider the fact that \( \mathbf{v}_1 = 3\mathbf{\delta}_1 + 3\mathbf{\delta}_2 \): write down an analogous formula for \( \mathbf{v}_2 \). You don’t need to explain anything; just write down a formula.

(c) Organize the above two equations to get a set of equations that looks like
\[
\begin{align*}
\mathbf{v}_1 &= c_1 \mathbf{\delta}_1 + c_2 \mathbf{\delta}_2 \\
\mathbf{v}_2 &= c_3 \mathbf{\delta}_1 + c_4 \mathbf{\delta}_2
\end{align*}
\]
where \( c_1, c_2, c_3, c_4 \in \mathbb{R} \).

(d) Given that \( \mathbf{v}_1 = 3\mathbf{\delta}_1 + 3\mathbf{\delta}_2 \), solve for \( \mathbf{\delta}_1 \) in terms of \( \mathbf{v}_1 \) and \( \mathbf{\delta}_2 \).

(e) Using your answers to the above two parts, write a formula for \( \mathbf{v}_2 \) in term of \( \mathbf{v}_1 \) and \( \mathbf{\delta}_2 \).

(f) Combine your answers to the above to write a set of equations that looks like
\[
\begin{align*}
\mathbf{\delta}_1 &= c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 \\
\mathbf{v}_2 &= c_3 \mathbf{v}_1 + c_4 \mathbf{v}_2
\end{align*}
\]

(g) With a similar type of calculation, derive a table that looks like
\[
\begin{align*}
\mathbf{\delta}_1 &= c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 \\
\mathbf{\delta}_2 &= c_3 \mathbf{v}_1 + c_4 \mathbf{v}_2
\end{align*}
\]

(h) Write
\[
1984\mathbf{\delta}_1 + 2019\mathbf{\delta}_2
\]
in terms of \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \).
(i) Is there a connection between your answer to the previous part and the solution to the $2 \times 2$ system of equations

\[\begin{align*}
3x_1 + 5x_2 &= 1984 \\
3x_1 + 4x_2 &= 2019
\end{align*}\]

(j) Check your solution by hand or computer.

(2) Find a solution to the $2 \times 2$ system

\[\begin{align*}
3x_1 + 5x_2 &= b_1 \\
3x_1 + 4x_2 &= b_2
\end{align*}\]

(for general $b_1, b_2 \in \mathbb{R}$) using the method of basis exchange as in Problem 1.

(3) Write down explicit solutions (using \TeX{} or \LaTeX{} or by hand) to the following 2020 $2 \times 2$ systems of equations:

\[\begin{align*}
3x_1 + 5x_2 &= 1984 \\
3x_1 + 4x_2 &= 0
\end{align*}\]

and

\[\begin{align*}
3x_1 + 5x_2 &= 1984 \\
3x_1 + 4x_2 &= 1
\end{align*}\]

and

\[\begin{align*}
3x_1 + 5x_2 &= 1984 \\
3x_1 + 4x_2 &= 2
\end{align*}\]

and ... and

\[\begin{align*}
3x_1 + 5x_2 &= 1984 \\
3x_1 + 4x_2 &= 2019
\end{align*}\]

using the method of basis exchange as in Problem 1.

For additional problems, create your own $2 \times 2$ systems, $7 \times 7$ systems, etc.