Part of the homework will use the inclusion-exclusion principle; a special case of this principle says that if $A_1, A_2$ are subsets of a finite set, then
\begin{equation}
|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|
\end{equation}
where use the notation $|B|$ to denote the cardinality (size) of a set $B$. Similarly, if $A_1, A_2, A_3$ are subsets of a finite set, then
\begin{equation}
|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|
\end{equation}
the principle generalizes in many ways, such as to any number of subsets of a set whose elements are weighted with appropriate finiteness conditions on the weights.

The textbook proves that for finite dimensional subspaces $V_1, V_2$ of a vector space $V$, we have
\begin{equation}
\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2),
\end{equation}
where $V_1 + V_2$ is defined to be
\begin{equation}
V_1 + V_2 = \{v_1 + v_2 \in V \mid v_1 \in V_1 \text{ and } v_2 \in V_2\}
\end{equation}
(one easily checks that this set is a subspace of $V$).

The following theorem from class and the textbook is used to prove everything in Sections 3.1–3.4, including (2).

**Theorem 0.1 (Basis Extension Theorem).** Let $V$ be a finite dimensional (real) vector space, i.e., one spanned by a finite set of vectors. Let $v_1, \ldots, v_s \in V$ be linearly independent, and $u_1, \ldots, u_t \in V$ such that
\begin{equation}
V = \text{Span}(v_1, \ldots, v_s, u_1, \ldots, u_t).
\end{equation}
Then $V$ has a basis consisting of $v_1, \ldots, v_s$ and some elements of $\{u_1, \ldots, u_t\}$. 

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Homework Problems

(1) Let $V_1, V_2$ be subspaces of a finite dimension vector space $V$. The point of this exercise is to prove (2) in a bit more detail than is done in the textbook.

(a) Explain in 20–50 words, and using the Basis Extension Theorem, why any basis, $v_1, \ldots, v_r$, of $V_1 \cap V_2$ can be extended to a basis, $v_1, \ldots, v_r, u_1, \ldots, u_s$, of $V_1$.

(b) Explain in 6–15 words why given your solution to part (a), any basis, $v_1, \ldots, v_r, V_1 \cap V_2$ can be extended to a basis, $v_1, \ldots, v_r, w_1, \ldots, w_t$, of $V_2$.

(c) Explain in 8–20 words what is the dimension of $V_1 \cap V_2$ in terms of $r, s, t$, using facts shown above.

(d) Explain briefly what are the dimensions of $V_1$ and of $V_2$ in terms of $r, s, t$, using facts shown above.

(e) With notation in parts (a) and (b), let $B = \{v_1, \ldots, v_r, u_1, \ldots, u_s, w_1, \ldots, w_t\}$.

(i) Explain in 6–15 words why each $u_i$ ($i = 1, \ldots, s$) lies in $V_1 + V_2$ (i.e., can be written as some element of $V_1$ plus some element of $V_2$).

(ii) Explain in 6–15 words why each $w_i$ ($i = 1, \ldots, t$) lies in $V_1 + V_2$ (i.e., can be written as some element of $V_1$ plus some element of $V_2$).

(iii) Explain in 9–25 words why each $u_i$ ($i = 1, \ldots, r$) lies in $V_1 + V_2$ (i.e., can be written as some element of $V_1$ plus some element of $V_2$) then Span($B$) is contained in $V_1 + V_2$.

(iv) Prove that the elements of $B$ are linearly independent by showing that if

$$\sum_{i=1}^{r} \alpha_i v_i + \sum_{j=1}^{s} \beta_j u_j + \sum_{k=1}^{t} \gamma_k w_t = 0$$

then all the $\alpha_i, \beta_j, \gamma_k$ must be 0. [Hint: one way of doing this is to write

$$\sum_{i=1}^{r} \alpha_i v_i + \sum_{j=1}^{s} \beta_j u_j = -\sum_{k=1}^{t} \gamma_k w_t$$

and to explain in 20–60 words why $u = 0$ and hence all the $\beta_j$ must be 0. Then one can finish in another 10–25 words.]

(f) Briefly explain why $B$ spans all of $V_1 + V_2$.

(g) Using the above, briefly explain why $B$ is a basis for $V_1 + V_2$.

(h) Using the above, briefly explain what is the dimension of $V_1 + V_2$ in terms of $r, s, t$, using facts shown above.

(2) Let

$$V_1 = \{f: \mathbb{Z} \to \mathbb{R} \mid f(n+4) = f(n)\}, \quad V_2 = \{f: \mathbb{Z} \to \mathbb{R} \mid f(n+5) = f(n)\}.$$

(a) Briefly explain: what are the dimensions of $V_1$ and $V_2$?

(b) Briefly explain: what is a simple description of $V_1 \cap V_2$?

(c) Briefly explain: what is the dimension of $V_1 + V_2$?
(3) Same question as the previous exercise, except with

\[ V_1 = \{ f : \mathbb{Z} \to \mathbb{R} \mid f(n + 8) = f(n) \}, \quad V_2 = \{ f : \mathbb{Z} \to \mathbb{R} \mid f(n + 10) = f(n) \} \].

(4) Fix an \( n \in \mathbb{Z} \), and let \( e_1, \ldots, e_n \) denote the standard basis of \( \mathbb{R}^n \). For \( A \subset \{1, \ldots, n\} \), let \( E_A \) denote the span of all \( e_a \) with \( a \in A \). Let \( A_1, A_2 \) be two subsets of \( \{1, \ldots, n\} \).

(a) What is the dimension of \( E_A \) in terms of \( |A| \)? Briefly justify your answer.

(b) Describe \( E_{A_1} \cap E_{A_2} \), and briefly justify your answer. What is its dimension?

(c) Describe \( E_{A_1} + E_{A_2} \), and briefly justify your answer. What is its dimension?

(d) If \( V_1 = E_{A_1} \) and \( V_2 = E_{A_2} \), use your answers to give an expression for

\[ \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2) \]

an expression for

\[ \dim(V_1 + V_2) \]

and explain why inclusion-exclusion implies that these are equal.

(e) Let \( A_1, A_2, A_3 \) be subsets of \( \{1, \ldots, n\} \), and set \( V_i = E_{A_i} \) for \( i = 1, 2, 3 \).

Show that

\[ \dim(V_1 + V_2 + V_3) = \dim(V_1) + \dim(V_2) + \dim(V_3) - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3) \].

(f) If \( V_1, V_2, V_3 \) are subspaces of \( \mathbb{R}^n \), is it always true that

\[ \dim(V_1 + V_2 + V_3) = \dim(V_1) + \dim(V_2) + \dim(V_3) - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3) \]?