Homework Problems

(1) Exercises 2, 4, 8, 9 from Section 2.4 (Test) of the textbook.

(2) Let $V, W$ be real vector spaces; we say that a map $L: V \rightarrow W$ is a linear transformation if for all $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{R}$ we have that

$$L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2).$$

Prove that in this case $\ker(L) \triangleq \{v \in V \mid L(v) = 0\}$ is a subspace of $V$. Prove that $\text{Image}(L) \triangleq \{L(v) \mid v \in V\}$ is a subspace of $W$.

(3) Let $L: V \rightarrow V$ be the linear transformation on $\text{Functions}(\mathbb{Z} \rightarrow \mathbb{R})$ given by

$$(Lf)(n) = f(n + 2) - 2f(n + 1) + f(n).$$

Show that the functions (1) $f(n) = 1$ for all $n \in \mathbb{Z}$, and (2) $f(n) = n$ for all $n \in \mathbb{Z}$ are in the kernel of $L$. Show that given $f(0), f(1)$ one can give an exact formula for the function $f \in \text{Ker}(L)$ whose those values at 0, 1.

(4) Generalize the problem above to

$$(Lf)(n) = f(n + 3) - 3f(n + 2) + 3f(n + 1) - f(n):$$

Show that (1) any polynomial of degree at most 2 is in the kernel of $L$, and (2) given $f(0), f(1), f(2)$, one ca give an exact formula for the function $f \in \text{ker}(L)$ with those values at 0, 1, 2.

(5) Let $V = P_3 = \text{Poly} \leq 3(\mathbb{R})$. Which of the following subsets of $V$ are subspaces? Explain.

(a) $\{p \in V \mid p(3) - 4p(5) = 0\}$
(b) $\{p \in V \mid p(3) - 4p(5) = 3\}$
(c) $\{p \in V \mid p(3)p(4) - 4p(5) = 0\}$
(d) $\{p \in V \mid p(3)p(4) - 4p(5)p(6) = 0\}$
(e) $\{p \in V \mid p'(3) = 0\}$
(f) $\{p \in V \mid p'(3) + p(4) = 0\}$