Homework Problems

3.2  (a) We say that a polynomial \( p(x) = a_0 + a_1 x + a_3 x^2 + a_3 x^3 \) is odd if \( p(-x) = -p(x) \). For which \( a_0, a_1, a_2, a_3 \) is \( p \) odd?

Solution: If \( p(-x) = -p(x) \) then
\[
a_0 + a_1(-x) + a_3(-x)^2 + a_3(-x)^3 = -(a_0 + a_1 x + a_3 x^2 + a_3 x^3)
\]
expanding we see that the above equality is equivalent to
\[
2a_0 + 2a_2 x^2 = 0
\]
(as polynomials). Since the LHS (left-hand-side) equals the zero polynomial, we have that \( a_0 = 0 \) and \( a_2 = 0 \). Hence \( p \) is odd for arbitrary \( a_0, \ldots, a_3 \) for which \( a_0 = 0 \) and \( a_2 = 0 \), i.e., \( p(x) = a_1 x + a_3 x^3 \).

(b) Show that if \( p(x) = a_0 + a_1 x + a_3 x^2 + a_3 x^3 \) is odd, then \( p(0) = 0 \).

Solution: By the last part \( p(x) = a_1 x + a_3 x^3 \), so \( p(0) = a_1 \cdot 0 + a_3 \cdot 0 = 0 \).

(c) If \( p(x) = a_0 + a_1 x + a_3 x^2 + a_3 x^3 \), and \( q(x) = p(x - 1/2) \) is odd, what can you say about \( q(-1/2) \)? How does this relate to the discussion in this subsection?

Solution: [The problem as written had a error; the correction is given in red.] Since \( q \) is odd, \( q(0) = 0 \), and hence \( p(-1/2) = q(0) = 0 \). This relates to the above, since \( p_2(n) \) is a polynomial of degree 3 and has \( p_2(-1 - n) = -p(n) \) for infinitely many \( n \); it follows that \( p(x) \) defined as \( p_2(x - 1/2) \) has \( p(-x) = -p(x) \) for infinitely value of \( x \), and hence \( p(-x) = -p(x) \) as polynomials.

[This last point was explained in class: the point is that \( p(-x) + p(x) \) is a polynomial, and since it has infinitely many roots it must be the zero polynomial; hence \( p(-x) = -p(x) \) as polynomials.]

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(d) We say that a polynomial \( p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \) is even if \( p(-x) = p(x) \). For which \( a_0, a_1, a_2, a_3 \) is \( p \) even?

**Solution:** Similarly to the above, \( p \) is even iff \( a_1 = a_3 = 0 \), i.e., iff \( p(x) = a_0 + a_2 x^2 \).

(e) Show that if \( p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \) is even, then \( p'(0) = 0 \) where \( p' \) shorthand for the derivative \( dp/dx \).

**Solution:** From the previous part we have \( p'(x) = 2a_2 x \); hence \( p'(0) = 2a_2 \cdot 0 = 0 \).

3.3 If \( f: \mathbb{Z} \to \mathbb{R} \) or \( f: \mathbb{R} \to \mathbb{R} \), we say that

(a) \( f \) is odd if \( f(-x) = -f(x) \) for all \( x \) (in the domain of \( f \)).
(b) \( f \) is even if \( f(-x) = f(x) \) for all \( x \) (in the domain of \( f \)).

(a) Show that if \( f: \mathbb{Z} \to \mathbb{R} \) or \( f: \mathbb{R} \to \mathbb{R} \), then

\[
f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}
\]

expresses \( f \) as the sum of an even plus an odd function; in other words, show that the first expression on the RHS (right-hand-side) is an even function, and second expression on the RHS is an odd function, and that the above equation is correct.

**Solution:** Setting \( g(x) = \frac{f(x) + f(-x)}{2} \)

then

\[
g(-x) = \frac{f(-x) + f(-(-x))}{2} = \frac{f(-x) + f(x)}{2} = g(x).
\]

Similarly setting \( g(x) = (f(x) - f(-x))/2 \) we see that \( g(-x) = -g(x) \). We easily verify the last part.

(b) If \( f: \mathbb{Z} \to \mathbb{Z} \), is

\[
f(x) + f(-x)
\]

always a function \( \mathbb{Z} \to \mathbb{Z} \)? Either (1) show that it is, or (2) give a counterexample or show that it isn’t always.

(c) Show that if \( f \) is odd, then \( f(0) = 0 \).

**Solution:** The equation \( f(-x) = -f(x) \) with \( x = 0 \) yields \( f(0) = -f(0) \) and hence \( 2f(0) = 0 \).

(d) Show that if \( f: \mathbb{R} \to \mathbb{R} \) is odd and differentiable, then \( f' = df/dx \) is even; show the same with “odd” and “even” exchanged.

**Solution:** The chain rule shows that if \( g(x) = f(-x) \) then \( g'(x) = -f'(-x) \). Hence if \( f \) is odd, i.e., \( f(-x) = -f(x) \),
then applying $d/dx$ to both sides yields $f'(-x) = -f'(x)$, i.e., $f'(-x) = f'(x)$, i.e., $f'$ is even. Similarly $f$ is even implies that $f'$ is odd.

**Solution:**

Then apply $d/dx$ to both sides yields $-f'(-x) = -f'(x)$, i.e., $f'(-x) = f'(x)$, i.e., $f'$ is even. Similarly $f$ is even implies that $f'$ is odd.

(e) Show that if $f$ is odd and infinitely differentiable (i.e., has derivatives to all orders), then $f(0), f''(0), f'''(0), \ldots$ are zero. Similarly show that if $f: \mathbb{R} \to \mathbb{R}$ is even and infinitely differentiable, then $f'(0), f''(0), \ldots$ are zero.

**Solution:**

If $f$ can be expressed as $g_1 + h_1$ and as $g_2 + h_2$ where $g_1, g_2$ are even and $h_1, h_2$ are even, then $g = g_1 - g_2$ is even and $h = h_2 - h_1$ is odd and $g = h$. Since $g$ is even, we have $g(x) = g(-x)$, and since $g = h$ we have $h(x) = h(-x)$; but since $h$ is odd we have $h(x) = -h(-x)$. It follows $h(-x) = h(x) = -h(-x)$ so $h = -h$ so $2h = 0$ (i.e. the zero function) so $h = 0$ (the zero function). Since $0 = h = h_2 - h_1$, we have $h_1 = h_2$. Since $g = h$ we have $g = h = 0$ and hence $g_1 = g_2$.

(f) Show any function $Z \to \mathbb{R}$ or $\mathbb{R} \to \mathbb{R}$ can be expressed *uniquely* as a sum of an even plus an odd function.

**Solution:**

If $f$ can be expressed as $g_1 + h_1$ and as $g_2 + h_2$ where $g_1, g_2$ are even and $h_1, h_2$ are even, then $g = g_1 - g_2$ is even and $h = h_2 - h_1$ is odd and $g = h$. Since $g$ is even, we have $g(x) = g(-x)$, and since $g = h$ we have $h(x) = h(-x)$; but since $h$ is odd we have $h(x) = -h(-x)$. It follows $h(-x) = h(x) = -h(-x)$ so $h = -h$ so $2h = 0$ (i.e. the zero function) so $h = 0$ (the zero function). Since $0 = h = h_2 - h_1$, we have $h_1 = h_2$. Since $g = h$ we have $g = h = 0$ and hence $g_1 = g_2$.

3.5 The binomial theorem (??) for $n = 4$ says that

$$(x + y)^4 = (x + y)(x + y)(x + y)(x + y) = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$  

Notice that there are four strings with three $x$’s and one $y$:

<table>
<thead>
<tr>
<th>xxx, xx, xy, yx</th>
</tr>
</thead>
</table>

and six strings with two $x$’s and two $y$’s:

| xx, yy, xy, yx, yyy, yyy |

Notice that in both cases above we have listed the strings in lexicographical order, meaning the order they would appear in a dictionary (if they were words).

(a) List all strings of one $x$ and three $y$’s in lexicographical order.

**Solution:**

| xyy, yxyy, yyxy, yyyx |

(b) List all strings of one $x$ and four $y$’s in lexicographical order.

**Solution:**

| xyxyy, yxyyy, yyxyy, yyxyy, yyyx |

(c) List all strings of two $x$’s and three $y$’s in lexicographical order.
Solution:

\[ \text{xxyyy, xyxyy, xyyxy, xyyyx, yxxyy, yxyxy, yxyyx, yyxxy, yyxyx, yyyxx} \]

(d) Using your answer to the last part, describe—IN 15 WORDS OR FEWER—an algorithm to list all strings of three x’s and two y’s in lexicographical order; i.e., do not produce this list, but instead describe how you would take the list you wrote in the last part as input and then output a list of all strings of three x’s and two y’s.

Solution: List the strings in reverse order and exchange the x’s and y’s.

(e) Explain how the number of elements in some of your lists above relate to the binomial theorem

\[ (x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5. \]

Solution: When you write \((x + y)^5\) as

\[ (x + y)(x + y)(x + y)(x + y)(x + y) \]

and expand naively (i.e., without collecting terms), you get 32 strings of x’s and y’s; when you collect terms you see the binomial coefficients.

3.7 Prove (??) (i.e., \( \binom{1}{k} + \ldots + \binom{n}{k} = \binom{n+1}{k+1} \)) directly, by noting that its right-hand-size represents the number of strings of \(n - k\) x’s and \(k + 1\) y’s, and using the fact that each such string begins with some number of x’s before it encounters its first y.

Solution: Each string, s, with \(n - k\) x’s and \(k + 1\) y’s must have at least one y (we assume \(k \geq 0\) so \(k + 1 \geq 1\); the string s must begin with some number of x’s, say m of them \((m = 0, 1, \ldots)\), followed by a y, then there are \(n + 1 - (m + 1) = n - m\) letters left, k of which must be y’s. Hence

\[ \binom{n+1}{k+1} = \sum_{m=0,1,\ldots} \binom{n-m}{k} = \binom{n}{k} + \binom{n-1}{k} + \ldots + \binom{k}{k}; \]

we can add \(\binom{k-1}{k}, \binom{k-2}{k}, \ldots, \binom{1}{k}\) to the RHS (right-hand-side) since they are zero.

3.9 Compute the function \((Df)(n)\) for all \(n \in \mathbb{N}\):

(a) \(f(n) = (n - 1)^2;\)

Solution:

\[ f(n + 1) - f(n) = n^2 - (n - 1)^2 = 2n - 1. \]

(b) \(f(n) = (n - 1)n(2n - 1)/6;\)
Solution:
\[ f(n+1) - f(n) = \frac{n(n+1)(2n+1)}{6} - \frac{(n-1)(2n-1)}{6} = \frac{n}{6} (n+1)(2n+1) - (n-1)(2n-1) \]
\[ = \frac{6n}{6} = n^2 \]

(c) \( f(n) = \binom{n}{2} \overset{\text{def}}{=} n(n-1)(n-2)(n-3)/24; \)

Solution:
\[ f(n+1) - f(n) = \frac{(n+1)n(n-1)(n-2)}{24} - \frac{n(n-1)(n-2)(n-3)}{24} = n(n-1)(n-2) \frac{4}{24} = \binom{n}{3} \]

(d) \( f(n) = -(1/3)^{n-1}/2 \) and simplify your answer.

Solution:
\[ f(n+1) - f(n) = \frac{-(1/3)^n - (1/3)^{n-1}}{2} = -(1/3)^n \frac{1 - 3}{2} = (1/3)^n. \]

(e) Show how \( (??) \) (i.e., \( (SDF)(n) = f(n+1) - f(1) \)) and the above computations yield the following formulas:

\[ 1 + 3 + 5 + \cdots + (2n-1) = n^2, \]
\[ 1 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6, \]
\[ \binom{1}{3} + \binom{2}{3} + \cdots + \binom{n}{3} = \binom{n+1}{4}, \]
\[ (1/3)^1 + (1/3)^2 + \cdots + (1/3)^n = \frac{1 - (1/3)^n}{2}, \]

Solution: They all arise from the formula
\[ (SDF)(n) = f(n+1) - f(1); \]
for example, in the first case \( f(n) = (n-1)^2 \) we have
\[ f(n+1) - f(1) = n^2 \]
while
\[ (SDF)(n) = \sum_{m=1}^{n} (Df)(n) = \sum_{m=1}^{n} (2m-1) = 1+3+5+\cdots+(2n-1), \]
and similarly for the other formulas.
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