WRITTEN HOMEWORK 6 (SOLUTIONS), MATH 200, FALL 2015

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Final 2013WT2, Problem 3

We have
\[ \nabla f = (f_x, f_y) = (6kxy - 6x, 3kx^2 + 3y^2 - 6y). \]

Solving for \( \nabla f = (0, 0) \), we have that \( f_x = 0 \) implies that \( 6x(ky - 1) = 0 \), which gives two cases:

(1) Case \( x = 0 \): then \( f_y = 3y^2 - 6y \) so \( f_y = 0 \) implies that \( 3y(y - 2) = 0 \), i.e., either \( y = 0 \) or \( y = 2 \). So this case yields the critical points \((0, 0)\) and \((0, 2)\).

For the second derivative test we have
\[
\begin{align*}
    f_{xx} &= (6kxy - 6x)_{xx} = 6ky - 6, \\
    f_{xy} &= (6kxy - 6x)_{xy} = 6kx, \\
    f_{yy} &= (3kx^2 + 3y^2 - 6y)_{yy} = 6y - 6,
\end{align*}
\]

and therefore
\[
D = f_{xx}f_{yy} - (f_{xy})^2 = (6ky - 6)(6y - 6) - (6kx^2) = 36[(ky - 1)(y - 1) - k^2x^2].
\]

Hence

(a) At \((0, 0)\) we have \( f_{xx} = -6 \) and \( D = 36 \), so \((0, 0)\) is a local maximum; and

(b) at \((0, 2)\) we have \( f_{xx} = 12k - 6 \) and \( D = 36(2k - 1) \), so when \( k = 1/2 \) then \( f_{xx} = D = 0 \) so the critical point is indeterminate; when \( k > 1/2 \)

then \( f_{xx} > 0 \) and \( D > 0 \) so the point is a local maximum; and when \( k < 1/2 \) then \( D < 0 \) so the point is a saddle.

(2) Case \( ky = 1 \), i.e., \( y = 1/k \): here we have
\[
f_y = 3kx^2 + 3y^2 - 6y = 3kx^2 + 3/k^2 - 6/k,
\]

so \( f_y = 0 \) implies that \( 3kx^2 = 6/k - 3/k^2 \) so \( x^2 = 2/k^2 - 1/k^3 \) so
\[
x = \pm \sqrt{2/k^2 - 1/k^3} = \pm \sqrt{2 - 1/k}.\]

So if \( k = 1/2 \) we have \( x = 0, y = 2 \), which was covered above. If \( k < 1/2 \), then there are no real values of \( x \). If \( k > 1/2 \), then there are the two \( x \) values above to check. Since
\[
f_{xx} = (6kxy - 6x)_x = 6ky - 6, \quad f_{xy} = (6kxy - 6x)_y = 6kx, \quad f_{yy} = (3kx^2 + 3y^2 - 6y)_y = 6y - 6,
\]
for \( ky = 1 \) we have \( f_{xx} = 0 \). So
\[
D = f_{xx}f_{yy} - (f_{xy})^2 = -36k^2x^2 < 0,
\]
so \( y = 1/k \) and \( x = \pm \sqrt{2 - 1/k}/k \) are saddles.

**Final 2012WT1, Problem 3**

By the chain rule we have
\[
G_t = \frac{\partial G}{\partial t} = \frac{\partial}{\partial t} F(\gamma + s, \gamma - s, At)
\]
\[
= F_x(\gamma + s)_t + F_y(\gamma - s)_t + F_z(At)_t = F_x0 + F_y0 + F_zA = AF_z = AF_z(\gamma + s, \gamma - s, At).
\]

(It is important to remember that \( F, F_z \), etc. are being evaluated at the point \((x, y, z) = (\gamma + s, \gamma - s, At)\); it is a bit cumbersome to put this everywhere.) Similarly,
\[
G_\gamma = F_x(\gamma + s)_\gamma + F_y(\gamma - s)_\gamma + F_z(At)_\gamma = F_x + F_y = F_x(\gamma + s, \gamma - s, At) + F_y(\gamma + s, \gamma - s, At).
\]

Similarly
\[
G_{\gamma\gamma} = (G_\gamma)_\gamma = [F_x(\gamma + s, \gamma - s, At)]_\gamma + [F_y(\gamma + s, \gamma - s, At)]_\gamma = [F_{xx} + F_{xy}] + [F_{yx} + F_{yy}]
\]
Similarly
\[
G_s = F_x - F_y, \quad G_{ss} = F_{xx} - F_{xy} - F_{yx} + F_{yy}.
\]

Hence
\[
G_{\gamma\gamma} + G_{ss} = 2F_{xx} + 2F_{yy} = 2F_z,
\]
by the equation for \( F \). Since \( G_t = AF_z \), we have \( G_t = G_{\gamma\gamma} + G_{ss} \) iff \( 2F_z = AF_z \), which holds if \( A = 2 \).

**Problem 34, Section 14.7**

We need to find the min/max of \( f(x, y) = xy^2 \) in the region \( D \) described by the inequalities \( x \geq 0, y \geq 0, \) and \( x^2 + y^2 \leq 3 \).

We have
\[
\nabla f = (f_x, f_y) = (y^2, 2xy).
\]
So if \( \nabla f = (0, 0) \) we have \( f_x = 0 \) and therefore \( y = 0 \). But \( y \) is never 0 in the interior of \( D \) (i.e., for \( x > 0, y > 0, \) and \( x^2 + y^2 < 3 \), where \( y \) must be positive). Hence \( \nabla f \) is never zero in the interior of \( D \), and it suffices to check the values of \( f \) on the boundary of \( D \).

On the boundary where \( x = 0 \) or \( y = 0 \) we have that \( f = xy^2 = 0 \). On the boundary where \( x^2 + y^2 = 3 \) (and both \( x \) and \( y \) are non-negative) we have
\[
f = xy^2 = x(3 - x^2),
\]
and \( x \) ranges from 0 to \( \sqrt{3} \). So aside from the values where \( f = 0 \), the only other possible min/max values of \( f \) occur for the function \( g(x) = x(3 - x^2) \) with \( x \in (0, \sqrt{3}) \): since
\[
g'(x) = (3x - x^3)' = 3 - 3x^2,
\]
we have \( g'(x) = 0 \) for \( x = \pm 1 \); since we only are considering \( x \in (0, \sqrt{3}) \), \( g' = 0 \) there only for \( x = 1 \); furthermore \( g(1) = 1 \cdot (3 - 1^2) = 2 \).

So the only possibly min/max values of \( f(x, y) \) in \( D \) are the values 0 and 2; hence 0 is the minimum value of \( f \), and 2 is the maximum value of \( f \).