INFINITELY MANY POSITIVE SOLUTIONS FOR AN ELLIPTIC PROBLEM WITH CRITICAL OR SUPER-CRITICAL GROWTH

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ABSTRACT. We prove that for some supercritical exponents \( p > \frac{N+2}{N-2} \) and for some smooth domains \( D \) in \( \mathbb{R}^N \) there are infinitely many (distinct) positive solutions to the following Lane-Emden-Fowler equation

\[
\begin{cases}
-\Delta u = u^p, & u > 0 \text{ in } D, \\
u = 0, & \text{on } \partial D.
\end{cases}
\]

This seems to be the first result for such type of equations.

1. INTRODUCTION

One of the earliest, and perhaps the simplest, nonlinear equations is the following Lane-Emden-Fowler equation,

\[
-\Delta u = u^p, \quad u > 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\]

where \( p > 1 \) and \( \Omega \) is a domain with smooth boundary in \( \mathbb{R}^N \).

It is well-known that the critical exponent \( p = \frac{N+2}{N-2} \) plays an important role in the solvability question. When \( 1 < p < \frac{N+2}{N-2} \), a solution can be found as an extremal for the best constant in the compact embedding of \( H^1_0(\Omega) \) into \( L^{p+1}(\Omega) \), namely a minimizer of the variational problem

\[
\inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2}{\left( \int_{\Omega} |u|^{p+1} \right)^{\frac{2}{p+1}}}. \tag{1.1}
\]

When \( p \geq \frac{N+2}{N-2} \), this minimization procedure fails, so does existence in general: Pohozaev [18] discovered that no solution exists in this case if the domain is strictly star-shaped. On the other hand Kazdan and Warner [12] observed that if \( \Omega \) is an annulus, \( \Omega = \{x : a < |x| < b\} \), compactness holds for any \( p > 1 \) within the class of radial functions, and a solution can again be found variationally without any constraint in \( p \).

In the classical paper [3], Brezis and Nirenberg considered the critical case \( p = \frac{N+2}{N-2} \) and proved that compactness, and hence solvability, is restored by the addition of a suitable linear term, that is, replacing \( u^{\frac{N+2}{N-2}} \) by \( u^{\frac{N+2}{N-2}} + \lambda u \). In case of pure nonlinearity \( u^{\frac{N+2}{N-2}} \), Coron
[4] used a variational approach to prove that (1.1) is solvable if $\Omega$ exhibits a small hole. Rey [19] established existence of multiple solutions if $\Omega$ exhibits several small holes. Bahri and Coron [1] established that solvability holds for $p = \frac{N+2}{N-2}$ whenever $\Omega$ has a non-trivial topology. On the other hand, examples in [5, 11] shows that when $p = \frac{N+2}{N-2}$ (1.1) can still have a solution on some domains whose topology is trivial. Thus both the topology and the shape of the domain can affect the existence of solution for (1.1) in the critical case. It is pointed out in [2] that Rabinowitz asked whether the non-triviality of the domain topology can guarantee the existence of at least one positive solution for solvability in the supercritical case $p > \frac{N+2}{N-2}$. This was answered negatively by Passaseo [16, 17] by means of an example for $N \geq 4$ and $p > \frac{N+1}{N-3}$. If $p$ is supercritical but close to critical, bubbling solutions are found, see [7, 8, 14, 15].

In the case of $p$ being purely supercritical, there are very few existence results. Variational machinery no longer applies, due to lack of Sobolev inequality. In [10], del Pino and Wei extended Coron’s result to supercritical problems (modulo some sequence of critical exponents) using perturbation methods. The role of the second critical exponent $p = \frac{N+1}{N-3}$, the Sobolev exponent in one dimension less, is investigated in the paper by del Pino, Musso and Pacard [9] in which they constructed solutions concentrating on a boundary geodesics for $p = \frac{N+1}{N-3} - \varepsilon$ with $\varepsilon \to 0^+$. But the results in [10, 9] are for problems with perturbations either on the nonlinearities, or on the domain. As far as we know, except the obvious radial solutions when the domain is an annulus ([12]), there is no existence result for (1.1) if there is no perturbation on the problem.

In this paper, we explore the role of lower-dimensional Sobolev exponents on the existence and multiplicity of solutions to (1.1). Namely we consider the following equation with super-critical growth:

$$
\begin{cases}
-\Delta u = u^{\frac{N-m+2}{N-m-2}}, & u > 0 \text{ in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
$$

(1.2)

where $m > 1$ is a positive integer, $\Omega$ is a bounded domain in $\mathbb{R}^N$, and $N \geq 3 + m$. Note that $\frac{N-m+2}{N-m-2}$ is the critical Sobolev exponent in $\mathbb{R}^{N-m}$.

By the results in [16, 17], it is not sufficient to just assume that $\Omega$ has non-trivial topology to obtain an existence result for (1.2). The aim of this paper is to investigate the conditions
on the domain $\Omega$ which ensure that (1.2) has infinitely many positive solutions. It is easy to see that (1.2) is equivalent to

(1.3)  
\[ \begin{align*} 
-\Delta u &= u^{\frac{N+2}{N-2}}, 
& \quad u > 0 \text{ in } \mathcal{D}, \\
& u = 0, 
& \quad \text{on } \partial \mathcal{D}, 
\end{align*} \]

where $m \geq 1$ is a positive integer, $\mathcal{D}$ is a bounded domain in $\mathbb{R}^{N+m}$, and $N \geq 3$.

To simplify (1.2), we impose a partial symmetry condition on $\mathcal{D}$:

(R): Write $y = (y^*, y^{**})$, $y^* \in \mathbb{R}^{N-1}$ and $y^{**} \in \mathbb{R}^{m+1}$. Then $y \in \mathcal{D}$ if and only if $(y^*, |y^{**}|, 0, \cdots, 0) \in \mathcal{D}$. For any $\mathcal{D}$ satisfying (R), we look for a solution of the form $u(y) = u(y^*, |y^{**}|)$ for (1.3).

Let

$$
\Omega = \{ (y^*, y_N) \in \mathbb{R}^N_+ : (y^*, y_N, 0, \cdots, 0) \in \mathcal{D} \},
$$

where $\mathbb{R}^N_+ = \{ y : y \in \mathbb{R}^N, y_N > 0 \}$. Then (1.3) is transformed to the following problem:

(1.4)  
\[ \begin{align*} 
-\text{div}(y_N^{m}Du) &= y_N^{m}u^{2^{*}-1}, 
& \quad y \in \Omega, \\
& u = 0, 
& \quad \text{on } \partial \Omega \cap \mathbb{R}^N_+. 
\end{align*} \]

Using the Pohozaev identity, we can easily find that (1.4) has no solution in some domains such as a half ball centered at the origin. On the other hand, if $\Omega \cap \mathbb{R}^N_+ \neq \emptyset$, (1.4) is degenerate. To avoid the difficulties caused by the possible degeneracy of (1.3), we impose that following condition on $\Omega$:

(\Omega_1): $\Omega \subset \subset \mathbb{R}^N_+$.

If $\Omega$ satisfies (\Omega_1), the corresponding domain $\mathcal{D}$ is a torus-like domain and thus it has non-trivial homology. Passaseo’s result suggests that more conditions on the domain be needed to obtain an existence result for (1.3).

Problem (1.4) is a critical problem in $\mathbb{R}^N$. Due to the non-compactness of this problem, it is not practical to use the variational techniques to obtain multiplicity result for (1.4). In this paper, we will prove that under some conditions on $\Omega$, (1.4) has infinitely many positive solutions by constructing solutions with many bubbles. To achieve this goal, we impose further the following conditions on $\Omega$.

\(\Omega_2\): For any $\theta \in (0, 2\pi)$, $(r \cos \theta, r \sin \theta, y_3, \cdots, y_N) \in \Omega$, if $(r, 0, y_3, \cdots, y_N) \in \Omega$.

\(\Omega_3\): $y \in \Omega$ if and only if $(y_1, y_2, y_3, \cdots, y_{i-1}, y_i, y_{i+1}, \cdots, y_N) \in \Omega$, $i = 3, \cdots, N - 1;
(Ω₄): there is \( x^* \in \partial \Omega \) with \( x^* = (r^*, 0, \cdots, 0, l^*) \) for some \( r^* > 0 \) and \( l^* > 0 \), such that

\[
\partial \Omega \cap \{ y_2 = \cdots = y_{N-1} = 0 \} \cap B_\delta(x^*) \\
= \{ y_N = \psi(y_1), y_2 = \cdots = y_{N-1} = 0 \} \cap B_\delta(x^*),
\]

and

\[
\Omega \cap \{ y_2 = \cdots = y_{N-1} = 0 \} \cap B_\delta(x^*) \\
= \{ y_N > \psi(y_1), y_2 = \cdots = y_{N-1} = 0 \} \cap B_\delta(x^*),
\]

for some \( C^2 \) function \( \psi \) and small \( \delta > 0 \). Moreover, \( r^* \) is either a strict local minimum point, or strict local maximum point of \( \psi \).

Our main result in this paper can be stated as follows:

**Theorem 1.1.** Suppose that \( N \geq 5 \). If \( \Omega \) satisfies (Ω₁), (Ω₂), (Ω₃), and (Ω₄), then problem (1.4) has infinitely many distinct positive solutions.

The domain in Figure 1 satisfies (Ω₁)-(Ω₄).

We make a comparison between our result and those in [9]. First we allow \( m \geq 2 \) while \( m = 1 \) in [9]. Now let \( m = 1 \). We prove the existence of solutions with large number of bubbles placed on a boundary geodesics for the purely critical exponent, while del Pino-Musso-Pacard [9] established the existence of a lower-dimensional bubble solution on a boundary geodesics for the slightly subcritical exponent.

As far as we know, the only other infinite multiplicity result is on Gelfand’s problem in a unit ball

\[
(1.5) \quad -\Delta u = \lambda(1 + u)^p, \quad u > 0 \text{ in } B_1, \quad u = 0 \text{ on } \partial B_1.
\]

Problem (1.5) can be reduced to an ODE problem by Gidas-Ni-Nirenberg theorem. Using ODE analysis, Joseph and Lundgren ([13]) showed that for some special values of \( \lambda = \lambda_p \) and for \( p \in \left( \frac{N+2}{N-2}, p_{JL} \right) \) there are infinitely many positive solutions to (1.5). (Here \( p_{JL} \) is the so-called Joseph-Lundgren exponent.) For the purely Lane-Emden-Fowler equation, theorem 1.1 is the first result of infinite multiplicity.

Before we close this introduction, let us outline the main idea in the proof of Theorem 1.1.

As we remarked earlier, our main idea is to glue bubbles together. Firstly, we construct an approximate solution, which is a bubble, for (1.4).
Denote $\frac{2^*}{N} = \frac{2N}{N-2}$. It is well-known that the functions

$$U_{x,\mu}(y) = \left(N(N-2)\right)^{\frac{N-2}{4}} \left(\frac{\mu}{1 + \mu^2|y-x|^2}\right)^{\frac{N-2}{2}}, \ \mu > 0, \ x \in \mathbb{R}^N$$

are the only solutions to the following problem

$$-\Delta u = u^{2^* - 1}, \ u > 0 \ \text{in} \ \mathbb{R}^N.$$

Since $U_{x,\mu}$ does not vanish on $\partial \Omega$, we define $PU_{x,\mu}$ as the solution of the following problem:

$$-\Delta PU_{x,\mu} = U_{x,\mu}^{2^* - 1}, \ \text{in} \ \Omega, \ PU_{x,\mu} = 0 \ \text{on} \ \partial \Omega.$$  \hspace{1cm} (1.6)

We use $PU_{x,\mu}$ as an approximate solution for (1.4). Our main task now is to determine the location $x$ of the bubbles, as well as the concentration rate $\mu$ of the bubbles. In the singular perturbation problems, such as those in [7, 8, 9], the parameter plays a crucial role in determining the concentration rate of the bubbles. Though (1.4) is not a singular
perturbation problem, it is well known now that we can use $k$, the number of bubbles, as our parameter, if $k$ is large. This idea was first introduced by us [20] in the study of prescribing scalar curvature problem on $S^N$

\[(1.7) \quad -\Delta_{S^N} u + \frac{N(N-2)}{2} u = K u^{\frac{N+2}{N-2}}, u > 0 \text{ on } S^N.\]

Let us fix a positive integer $k \geq k_0$, where $k_0$ is large, which is to be determined later.

The calculations in Appendix A suggest that we should make the scaling parameter satisfy

$$\mu \in \left[ \Lambda_0 k^{\frac{N-1}{N-2}}, \Lambda_1 k^{\frac{N-1}{N-2}} \right],$$

for some large constant $\Lambda_1 > 0$ and some small constant $\Lambda_0 > 0$.

Using the symmetry conditions $(\Omega_2)$ and $(\Omega_3)$, we introduce the following space:

$$H_s = \{ u : u \in H^1_0(\Omega), u \text{ is even in } y_h, h = 2, \cdots, N - 1, \}
\quad u(r \cos \theta, r \sin \theta, y') = u(r \cos(\theta + \frac{2\pi j}{k}), r \sin(\theta + \frac{2\pi j}{k}), y'), \}
$$

where $y = (y', y'')$, $y' \in \mathbb{R}^2$, $y'' \in \mathbb{R}^{N-2}$.

We will look for solutions in the space $H_s$. So, we put $k$ bubbles in an one-dimensional circle as follows: let

$$x_j = (r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0, \cdots, 0), \quad j = 1, \cdots, k,$$

and let

$$W_{r,l,\mu}(y) = \sum_{j=1}^{k} PU_{x_j,\mu}.$$

In this paper, we will prove that for any large $k$, (1.4) has a solution $u_k$ with

$$u_k \approx W_{r,l,\mu_k}.$$

Now, we discuss the location of the bubbles. Due the weight $y''_0$ in (1.4), the energy of the bubble $PU_{x_1,\mu}$ will increase as $x_{1,N}$ increases. On the other hand, due to the Dirichlet boundary condition, the energy of the bubble $PU_{x_1,\mu}$ will also increase as $x_1$ moves toward the boundary. So we see that in the vertical direction, the energy achieves its minimum at a point near the bottom part of the boundary. This property comes directly from the equation and the boundary condition. In order to obtain a balance in the horizontal directions, we need to impose $(\Omega_4)$ on the domain. From this discussion, we know we
should put $x_1$ close to the boundary point $x^*$. For such $x_1$, there is a unique $(h, d)$, such that

$$x_1 = (h, 0, \cdots, 0, \psi(h)) + d\nu,$$

where $\nu$ is the unit inward normal of $\partial\Omega$ at $(h, 0, \cdots, 0, \psi(h))$. In the following, we will use $h$ and $d$, instead of $r$ and $l$, as the coordinates for $x_1$. So we will use the notation:

$$W_{h,d,\mu}(y) = \sum_{j=1}^{k} PU_{x_j,\mu}.$$

Theorem 1.1 is a direct consequence of the following result:

**Theorem 1.2.** Suppose that $N \geq 5$. If $\Omega$ satisfies $(\Omega_1)-(\Omega_4)$, then there is an integer $k_0 > 0$, such that for any integer $k \geq k_0$, (1.4) has a solution $u_k$ of the form

$$u_k = W_{h_k,d_k,\mu_k}(y) + \omega_k,$$

where $\omega_k \in H_s$, and as $k \to +\infty$, $\mu_k^{-\frac{N-2}{2}}\|\omega_k\|_{L^\infty} \to 0$, $\mu_k \in [\Lambda_0 k^{\frac{N-2}{2}}, \Lambda_1 k^{\frac{N-2}{2}}]$, and $d_k \to 0$, $h_k \to r^*$.

Conditions $(\Omega_2)$ and $(\Omega_3)$ are symmetry conditions, which allow us to find a solution in the space $H_s$. It is condition $(\Omega_4)$ that makes the construction of the solution of the form (1.8) possible. We believe that these symmetry assumptions may be replaced by some kind of conditions on geodesics on the boundary. On the other hand, the weight $y^\alpha$ plays a crucial role in determining the location of the bubbles in the vertical direction. So the technique in this paper can not be used to obtain a multiplicity result for

$$-\Delta u = u^{\frac{N+2}{N-2}}, \quad u > 0 \text{ in } \Omega, \quad u = 0, \text{ on } \partial\Omega.$$

We will use a reduction argument to prove Theorem 1.2. In section 2, we will carry out the reduction in a weighted space. The main result Theorem 1.2 is proved in section 3, while all the technical estimates are put in the appendices.

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2. Finite-dimensional Reduction

In this section, we perform a finite-dimensional reduction. Since this part is similar to [20], we shall only give a sketch of the proofs.

Let

\[
\|u\|_* = \sup_{y \in \Omega} \left( \sum_{j=1}^{k} \frac{1}{(1 + \mu|y - x_j|)^{\frac{N-2}{2} + \tau}} \right)^{-1} \mu^{-\frac{N-2}{2}} |u(y)|,
\]

and

\[
\|f\|_{**} = \sup_{y \in \Omega} \left( \sum_{j=1}^{k} \frac{1}{(1 + \mu|y - x_j|)^{\frac{N+2}{2} + \tau}} \right)^{-1} \mu^{-\frac{N+2}{2}} |f(y)|,
\]

where \(\tau = \frac{N-2}{N-1}\). For this choice of \(\tau\), we find that

\[
\sum_{j=2}^{k} \frac{1}{|\mu x_j - \mu x_1|^\tau} \leq \frac{Ck^\tau}{\mu^\tau} \sum_{j=2}^{k} \frac{1}{j^\tau} \leq \frac{Ck}{\mu^\tau} \leq C'.
\]

We will use \(\partial_1, \partial_2\) and \(\partial_3\) to denote \(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\) and \(\frac{\partial}{\partial x_3}\) respectively. Let

\[
Z_{i,j} = -\text{div}(y_N^m \partial_j PU_{x_i, \mu}), \quad j = 1, 2, 3.
\]

Consider

\[
\begin{aligned}
&-\text{div}(y_N^m D\phi_k) - (2^* - 1)y_N^m W^{2^* - 2} \phi_k = \xi + \sum_{j=1}^{3} c_j \sum_{i=1}^{k} Z_{i,j}, \quad \text{in } \Omega, \\
&\phi_k \in H_s, \\
&<Z_{i,j}, \phi_k> = 0 \quad i = 1, \cdots, k, \quad j = 1, 2, 3
\end{aligned}
\]

for some numbers \(c_i\), where \(<u, v> = \int_{\Omega} uv\).

Before we proceed, we need the following lemma. The proof may be known but we can not find a reference so we give a proof in Appendix C.

**Lemma 2.1.** Let \(u\) be the solution of

\[
-\text{div}(y_N^m Du) = y_N^m f(y), \quad y \in \Omega, \quad u = 0, \quad \text{on } \partial \Omega.
\]

Then, there is a constant \(C > 0\), such that

\[
|u(x)| \leq C \int_{\Omega} \frac{|f(y)|}{|y - x|^{N-2}} dy.
\]
Now we have the following \textit{a priori} estimates.

\textbf{Lemma 2.2.} Assume that $\phi_k$ solves (2.3) for $\xi = \xi_k$. If $\|\xi_k\|_{**}$ goes to zero as $k$ goes to infinity, so does $\|\phi_k\|_s$.

\textit{Proof.} The proof of this lemma is similar to the proof of Lemma 2.1 in [20]. Thus, we just sketch it.

We argue by contradiction. Suppose that there are $k \to +\infty$, $\xi = \xi_k$, $h_k \to r^*$, $d_k \to 0$, $\mu_k \in \left[\Lambda \phi_k \frac{N-2}{N-2}, \Lambda_k \phi_k \frac{N-2}{N-2}\right]$, and $\phi_k$ solving (2.3) for $\xi = \xi_k$, $\mu = \mu_k$, $d = d_k$, $h = h_k$, with $\|\xi_k\|_{**} \to 0$, and $\|\phi_k\|_s \geq c' > 0$. We may assume that $\|\phi_k\|_s = 1$. For simplicity, we drop the subscript $k$.

Using Lemma 2.1, we obtain

\begin{equation}
|\phi(y)| \leq C \int_{\Omega} \frac{1}{|z - y|^{N-2}W_{h,d,\mu}^{2r-2}} |\phi(z)| \, dz \\
+ C \int_{\Omega} \frac{1}{|z - y|^{N-2}} \left( |\xi(z)| + \left| \sum_{j=1}^3 \sum_{i=1}^k c_j \sum_{i=1}^k Z_{i,j}(z) \right| \right) \, dz.
\end{equation}

(2.4)

Using Lemma B.3, we have

\begin{equation}
\int_{\Omega} \frac{1}{|z - y|^{N-2}W_{h,d,\mu}^{2r-2}} |\phi(z)| \, dz \\
\leq C \|\phi\|_s \mu^{\frac{N-2}{2}} \int_{\Omega} \frac{1}{|z - y|^{N-2}W_{h,d,\mu}^{2r-2}} \sum_{j=1}^k \frac{1}{(1 + \mu |z - x_j|)^{\frac{N-2}{2}+\tau}} \, dz \\
\leq C \|\phi\|_s \mu^{\frac{N-2}{2}} \sum_{j=1}^k \frac{1}{(1 + \mu |y - x_j|)^{\frac{N-2}{2}+\tau}}.
\end{equation}

(2.5)

It follows from Lemma 2.2 that

\begin{equation}
\int_{\Omega} \frac{1}{|z - y|^{N-2}} |\xi(z)| \, dz \leq C \|\xi\|_{**} \int_{\Omega} \frac{1}{|z - y|^{N-2}} \sum_{j=1}^k \frac{\mu^{\frac{N+2}{2}}}{(1 + \mu |z - x_j|)^{\frac{N-2}{2}+\tau}} \\
\leq C \|\xi\|_{**} \mu^{\frac{N-2}{2}} \sum_{j=1}^k \frac{1}{(1 + \mu |y - x_j|)^{\frac{N-2}{2}+\tau}},
\end{equation}

(2.6)

and
\[
\int_{\Omega} \frac{1}{|z - y|^{N-2}} \sum_{i=1}^{k} Z_{i,i}(z) \, dz \\
\leq C \int_{\mathbb{R}^N} \frac{1}{|z - \mu y|^{N-2}} \sum_{i=1}^{k} \mu^{\frac{N-2}{2} + m_i} \frac{1}{(1 + |z - \mu x_j|^2)^{\frac{N-2}{2} + \tau}} \, dz \\
\leq C \mu^{\frac{N-2}{2} + m_i} \sum_{i=1}^{k} \frac{1}{(1 + \mu |y - x_i|)^{\frac{N-2}{2} + \tau}},
\]
where \( m_i = 1 \) if \( i = 1, 2 \), and \( m_3 = -1 \).

Next, we estimate \( c_i \), \( l = 1, 2, 3 \). Multiplying (2.3) by \( Z_{i,i} \) and integrating, we see that \( c_i \) satisfies

\[
(2.8) \quad \sum_{i=1}^{3} \sum_{j=1}^{k} \langle Z_{j,i}, \partial_i U_{x_i;\mu} \rangle c_i = \langle -\text{div}(y_N^m D\phi) - (2^* - 1)y_N^m W_{h,d;\mu}^{2^* - 2} \phi, \partial_i U_{x_i;\mu} \rangle - \langle h, \partial_i U_{x_i;\mu} \rangle.
\]

It follows from Lemma B.1 that

\[
|\langle h, \partial_i U_{x_i;\mu} \rangle| \leq C\|h\|_{m_i} \int_{\Omega} \frac{\mu^{\frac{N-2}{2} + m_i}}{(1 + \mu |z - x_i|)^{N-2}} \sum_{j=1}^{k} \mu^{\frac{N-2}{2}} \frac{1}{(1 + \mu |z - x_j|)^{\frac{N-2}{2} + \tau}} \, dz \\
\leq C\|h\|_{m_i}.
\]

On the other hand, using Lemma B.3, we can prove

\[
(2.9) \quad \langle -\text{div}(y_N^m D\phi) - (2^* - 1)y_N^m W_{h,d;\mu}^{2^* - 2} \phi, \partial_i U_{x_i;\mu} \rangle = \langle -\text{div}(y_N^m D\partial_i U_{x_i;\mu}) - (2^* - 1)y_N^m W_{h,d;\mu}^{2^* - 2} \partial_i U_{x_i;\mu}, \phi \rangle = o(\|\phi\|_{m_i})\mu^{m_i}.
\]

Thus we obtain from (2.8) that

\[
(2.10) \quad c_i = \frac{1}{\mu^{m_i}} \left( o(\|\phi\|_{m_i}) + O(\|h\|_{m_i}) \right).
\]

So,

\[
(2.11) \quad \|\phi\|_{m_i} \leq \left( o(1) + \|h_k\|_{m_i} + \frac{\sum_{j=1}^{k} \frac{1}{(1 + \mu |y - x_j|)^{\frac{N-2}{2} + \tau + \theta}}}{\sum_{j=1}^{k} \frac{1}{(1 + \mu |y - x_j|)^{\frac{N-2}{2} + \tau}}} \right).
\]

Since \( \|\phi\|_{m_i} = 1 \), we obtain from (2.11) that there is \( R > 0 \), such that
\[(2.12) \quad \|\mu^{-\frac{N-2}{2}}\phi(y)\|_{B_{\mu^{-1}}(x_i)} \geq a > 0,\]

for some \(i\). But \(\bar{\phi}(y) = \mu^{-\frac{N-2}{2}}\phi(\mu(y - x_i))\) converges uniformly in any compact set to a solution \(u\) of

\[(2.13) \quad -\Delta u - (2^* - 1)u^{2^* - 2}u = 0, \quad \text{in} \ \mathbb{R}^N,\]

for some \(\Lambda \in [\Lambda_1, \Lambda_2]\), and \(u\) is perpendicular to the kernel of \(2.13\). So, \(u = 0\). This is a contradiction to \((2.12)\).

\[\square\]

From Lemma 2.2, using the same argument as in the proof of Proposition 4.1 in [7], we can prove the following result:

**Proposition 2.3.** There exists \(k_0 > 0\) and a constant \(C > 0\), independent of \(k\), such that for all \(k \geq k_0\) and all \(h \in L^\infty(\mathbb{R}^N)\), problem \((2.3)\) has a unique solution \(\phi \equiv L_k(h)\). Besides,

\[(2.14) \quad \|L_k(h)\|_* \leq C\|h\|_\ast.\]

Now, we consider

\[(2.15) \quad \begin{cases} -\text{div} \left( y_N^m D(W_{h,d,\mu} + \phi) \right) = y_N^m (W_{r,\mu} + \phi)^{2^* - 1} + \sum_{i=1}^{3} c_i \sum_{i=1}^{k} Z_{i,l}, \quad \text{in} \ \Omega, \\ \phi \in H_s, \\ < Z_{i,l}, \phi > = 0, \\ i = 1, \cdots, k, \ l = 1, 2, 3. \end{cases}\]

The main result of this section is the following:

**Proposition 2.4.** There is an integer \(k_0 > 0\), such that for each \(k \geq k_0\), \((h, d)\) close to \((r^*, 0)\), \(\mu \in [\Lambda_0 k^{\frac{N-2}{2}}, \Lambda_1 k^{\frac{N-2}{2}}]\), \((2.15)\) has a unique solution \(\phi = \phi(h, d, \mu)\), satisfying

\[\|\phi\|_* \leq C \left(\frac{1}{\mu}\right)^{\frac{1}{2} + \sigma},\]

if \(N \geq 5\), where \(\sigma > 0\) is a small constant.

To prove Proposition 2.4, we need to prove two lemmas first. Rewrite \((2.15)\) as
\[
\begin{aligned}
-\text{div}(y_N^m D\phi) - (2^*-1)y_N^m W_h^{2^*-2} \phi &= N(\phi) + l_k + \sum_{t=1}^{3} c_i \sum_{i=1}^{k} Z_{i,t}, \text{ in } \Omega, \\
\phi \in H_s, \\
< Z_{i,t}, \phi > &= 0, \\
i = 1, \cdots, k, \quad l = 1, 2,
\end{aligned}
\]

where

\[
N(\phi) = y_N^m \left((W_{h,d,\mu} + \phi)^{2^*-1} - W_{h,d,\mu}^{2^*-1} - (2^*-1)W_{h,d,\mu}^{2^*-2}\phi\right),
\]

and

\[
l_k = y_N^m \left(W_{h,d,\mu}^{2^*-1} - \sum_{j=1}^{k} U_{x,j,\mu}^{2^*-1}\right) - my_N^{m-1} \sum_{j=1}^{k} \frac{\partial P U_{x,j,\mu}}{\partial y_N}.
\]

In order to use the contraction mapping theorem to prove that (2.16) is uniquely solvable in the set that \(\|\phi\|^s\) is small, we need to estimate \(N(\phi)\) and \(l_k\).

**Lemma 2.5.** If \(N \geq 4\), then

\[
\|N(\phi)\|^s \leq C\|\phi\|^{\min(2^*-1,2)}.
\]

**Proof.** We have

\[
|N(\phi)| \leq \begin{cases} 
C|\phi|^{2^*-1}, & N \geq 6; \\
C\left(W_{h,d,\mu}^{\frac{N-2}{N}}\phi^2 + |\phi|^{2^*-1}\right), & N = 4, 5.
\end{cases}
\]

Firstly, we consider \(N \geq 6\).

Using

\[
\sum_{j=1}^{k} a_j b_j \leq \left(\sum_{j=1}^{k} a_j^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{k} b_j^q\right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad a_j, b_j \geq 0,
\]

we obtain
\[
|N(\phi)| \leq C \|\phi\|_{s, k}^{2^{*-1}} \left( \sum_{j=1}^{k} \frac{\mu \frac{N-2}{2}}{(1 + \mu |y - x_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^{*-1}} \\
\leq C \|\phi\|_{s, k}^{2^{*-1}} \mu \frac{N-2}{2} \sum_{j=1}^{k} \frac{1}{(1 + \mu |y - x_j|)^{\frac{N-2}{2} + \tau}} \left( \sum_{j=1}^{k} \frac{1}{(1 + \mu |y - x_j|)^{\frac{N-2}{2} + \tau}} \right)^{\frac{N}{N-2}} \\
\leq C \|\phi\|_{s, k}^{2^{*-1}} \mu \frac{N-2}{2} \sum_{j=1}^{k} \frac{1}{(1 + \mu |y - x_j|)^{\frac{N-2}{2} + \tau}}.
\]

Thus, the result follows.

Suppose that \( N = 4, 5 \). Noting that \( N - 2 \geq \frac{N-2}{2} + \tau \), we find

\[
|N(\phi)| \leq C \|\phi\|_{s, k}^{2^{*-1}} \mu \frac{N-2}{2} \left( \sum_{i=1}^{k} \frac{\mu \frac{N-2}{2}}{(1 + \mu |y - x_i|)^{N-2}} \right)^{\frac{N}{N-2}} \left( \sum_{j=1}^{k} \frac{1}{(1 + \mu |y - x_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^{*-1}} \\
+ C \|\phi\|_{s, k}^{2^{*-1}} \mu \frac{N-2}{2} \sum_{j=1}^{k} \frac{1}{(1 + \mu |y - x_j|)^{\frac{N-2}{2} + \tau}} \\
\leq C \|\phi\|_{s, k}^{2^{*-1}} \mu \frac{N-2}{2} \left( \sum_{j=1}^{k} \frac{1}{(1 + \mu |y - x_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^{*-1}} \\
+ C \|\phi\|_{s, k}^{2^{*-1}} \mu \frac{N-2}{2} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2}{2} + \tau}} \\
= C \|\phi\|_{s, k}^{2^{*-1}} \mu \frac{N-2}{2} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{\frac{N-2}{2} + \tau}}.
\]

So, we have proved that for \( N \geq 4 \),

\[
\|N(\phi)\|_{\ast} \leq C \|\phi\|_{s}^{\min(2^{2^{*-1}})}.
\]

\[\square\]

Next, we estimate \( l_k \).

**Lemma 2.6.** If \( N \geq 5 \), then

\[
\|l_k\|_{\ast} \leq C \left( \frac{1}{\mu} \right)^{\frac{1}{4} + \sigma},
\]

where \( \sigma > 0 \) is a small constant.
Proof. Define

$$\Omega_j = \{ y : y \in \Omega, \left\langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|} \right\rangle \geq \cos \frac{\pi}{k} \}, \quad y' = (y_1, y_2, 0, \cdots, 0).$$

We have

$$l_k = y_N^m \left(W_{k,d}^{2,-1} - \sum_{j=1}^{k} (PU_{x_j,\mu})^{2^{x,-1}} \right) + y_N^m \left(\sum_{j=1}^{k} (PU_{x_j,\mu})^{2^{x,-1}} - \sum_{j=1}^{k} U_{x_j,\mu}^{2^{x,-1}} \right)
- my_N^{m-1} \sum_{j=1}^{k} \frac{\partial PU_{x_j,\mu}}{\partial y_N}$$

$$=: J_0 + J_1 + J_2.$$ From the symmetry, we can assume that $y \in \Omega_1$. Then,

$$|y - x_j| \geq |y - x_1|, \quad \forall y \in \Omega_1.$$ Firstly, we claim

$$1 \leq \frac{1}{1 + |y - x_j|} \leq \frac{C}{|x_j - x_1|}, \quad \forall y \in \Omega_1, j \neq 1.$$ In fact, if $|y - x_1| \leq \frac{1}{2}|x_1 - x_j|$, then $|y - x_j| \geq \frac{1}{2}|x_1 - x_j|$. If $|y - x_1| \geq \frac{1}{2}|x_1 - x_j|$, then $|y - x_j| \geq |y - x_1| \geq \frac{1}{2}|x_1 - x_j|$, since $y \in \Omega_1$.

For the estimate of $J_0$, we have

$$|J_0| \leq \frac{C \mu^2}{(1 + \mu |y - x_1|)^4} \sum_{j=2}^{k} \frac{\mu^{N-2}}{(1 + \mu |y - x_j|)^{N-2}} + \frac{C \mu^{N-2}}{(1 + \mu |y - x_j|)^{N-2}} \right) x^{2-1}.$$ Using (2.18), taking $1 < \alpha \leq N - 2$, we obtain for any $y \in \Omega_1$,

$$\frac{1}{(1 + \mu |y - x_1|)^4} \frac{1}{(1 + \mu |y - x_j|)^{N-2}} \leq C \frac{1}{(1 + \mu |y - x_1|)^{N+2-\alpha}} \frac{1}{|\mu x_j - \mu x_1|^{\alpha}}, \quad \forall j > 1.$$

Take $\alpha > \max\left(\frac{N-1}{2}, 1\right)$ satisfying $N + 2 - \alpha \geq \frac{N+2}{2} + \tau$. Then
\[
\sum_{j=2}^{k} \frac{1}{(1 + \mu|y - x_j|)^{N-2}} \leq \frac{C}{(1 + \mu|y - x_1|)^{N+2-\alpha}} \left( \frac{k}{\mu} \right)^{\alpha} = \frac{C}{(1 + \mu|y - x_1|)^{N+2-\alpha}} \mu^{-\frac{\alpha}{\alpha-1}} \\
\leq \frac{C}{(1 + \mu|y - x_1|)^{N+2-\tau}} \left( \frac{1}{\mu} \right)^{\frac{1}{2} + \sigma}.
\]

Using the Hölder inequality, we obtain

\[
\left( \sum_{j=2}^{k} \frac{1}{(1 + \mu|y - x_j|)^{N-2}} \right)^{2^* - 1} \leq \sum_{j=2}^{k} \frac{1}{(1 + \mu|y - x_j|)^{N+2-\tau}} \left( \sum_{j=2}^{k} \frac{1}{(1 + \mu|y - x_j|)^{N+2-\tau}} \right)^{\frac{4}{N+2-\tau}}.
\]

Noting that $\frac{N+2}{4}(\frac{N-2}{2} - \frac{\tau}{N+2}) > 1$ if $N \geq 4$, we obtain

\[
\left( \sum_{j=2}^{k} \frac{1}{(1 + \mu|y - x_j|)^{N-2}} \right)^{2^* - 1} \leq C \left( \sum_{j=2}^{k} \frac{1}{(1 + \mu|y - x_j|)^{N+2-\tau}} \right)^{\frac{4}{N+2-\tau}} \sum_{j=1}^{k} \frac{1}{(1 + \mu|y - x_j|)^{N+2-\tau}}
\]

(2.22)

\[
\leq C \left( \frac{k}{\mu} \right)^{\frac{N+2}{4}(\frac{N-2}{2} - \frac{\tau}{N+2})} \sum_{j=1}^{k} \frac{1}{(1 + \mu|y - x_j|)^{N+2-\tau}}
\]

\[
= C \left( \frac{k}{\mu} \right)^{\frac{N+2}{N+1}(\frac{1}{2} - \frac{\tau}{N+2})} \sum_{j=1}^{k} \frac{1}{(1 + \mu|y - x_j|)^{N+2-\tau}}
\]

\[
= C \left( \frac{1}{\mu} \right)^{\frac{1}{2} + \sigma} \sum_{j=1}^{k} \frac{1}{(1 + \mu|y - x_j|)^{N+2-\tau}},
\]

since $\frac{N+2}{N+1}(\frac{1}{2} - \frac{\tau}{N+2}) > \frac{1}{2}$. Thus, we have proved that if $N \geq 4$,

\[
\|J_0\|_{**} \leq C \left( \frac{1}{\mu} \right)^{\frac{1}{2} + \sigma}.
\]
Now, we estimate $J_1$. Let $H(y, x)$ be the regular part of the Green function for $-\Delta$ in $\Omega$ with the zero boundary condition. Let $x_j^*$ be the reflection point of $x_j$ with respect to $\partial \Omega$. Then

$$
(2.23) \quad \frac{H(y, x_j)}{\mu^{N-2}} = \frac{C}{\mu^{N-2}|y - x_j^*|^{N-2}} \leq \frac{C}{(1 + \mu|y - x_j|)^{N-2}},
$$

since $\mu|y - x_j^*| \geq \mu d \to +\infty$. Using (A.1), we find

$$
|J_1| \leq \sum_{j=1}^{k} \frac{C \mu^2}{(1 + \mu|y - x_j|)^{t}} \frac{H(y, x_j)}{\mu^{N-2}}
$$

$$
\leq C \mu^{\frac{N+2}{2}} \sum_{j=1}^{k} \frac{1}{(1 + \mu|y - x_j|)^{4 + ((1-t)(N-2))}} \left( \frac{H(y, x_j)}{\mu^{N-2}} \right)^t
$$

$$
\leq C \left( \frac{1}{\mu d} \right)^{(N-2)} \sum_{j=1}^{k} \frac{1}{(1 + |y - x_j|)^{4 + t(N-2)}}
$$

$$
\leq C \mu^{\frac{N+2}{2}} \left( \frac{1}{\mu} \right)^{1 + \sigma} \sum_{j=1}^{k} \frac{1}{(1 + \mu|y - x_j|)^{3 + (1-t)(N-2)}}
$$

$$
\leq C \mu^{\frac{N+2}{2}} \left( \frac{1}{\mu} \right)^{\frac{1}{2} + \sigma} \sum_{j=1}^{k} \frac{1}{(1 + \mu|y - x_j|)^{\frac{N+2}{2} + \tau}},
$$

if we take $t \frac{N-2}{N-1} = \frac{1}{2} + \sigma$ for some small $\sigma > 0$. For such $t$, we can check that $4 + (1 - t)(N - 2) \geq \frac{N+2}{2} + \tau$. Here, we have used $d \geq \frac{r_0}{k}$.

Finally, we estimate $J_2$. Similar to (2.23), we have

$$
\frac{\partial P U_{x_j, \lambda}}{\partial y_N} = \frac{\partial U_{x_j, \lambda}}{\partial y_N} + O \left( \frac{1}{\mu^{N-2}} \left| \frac{\partial H(y, x_j)}{\partial y_N} \right| \right)
$$

$$
= \frac{\partial U_{x_j, \lambda}}{\partial y_N} + O \left( \frac{1}{\mu^{N-2}} |y - x_j^*|^{N-1} \right)
$$

$$
= O \left( \frac{\mu^{rac{N}{2}}}{(1 + \mu|y - x_j|)^{N-1}} \right) = \mu^{\frac{N+2}{2}} O \left( \frac{1}{\mu^{\frac{1}{2} + \sigma} (1 + \mu|y - x_j|)^{N-\frac{1}{2} - \sigma}} \right),
$$

where $x_j^*$ is the reflection point of $x_j$ with respect to $\partial \Omega$.

If $N \geq 5$ and $\sigma > 0$ is small, then we have $N - \frac{1}{2} - \sigma > \frac{N+2}{2} + \tau$. As a result
\[ |J_2| \leq \frac{C}{\mu^{\frac{1}{2} + \sigma}} \sum_{j=1}^{k} \left( \frac{\mu^{\frac{N+2}{2}}}{1 + \mu|y - x_j|^{\frac{N+2}{2} + \tau}} \right). \]

Now, we are ready to prove Proposition 2.4.

**Proof of Proposition 2.4.** Let

\[ E = \{ u : u \in C(\Omega) \cap H_s, \|u\|_s \leq \left( \frac{1}{k} \right)^{\frac{1}{2}}, \int_{\Omega} Z_{i,l} \phi = 0, i = 1, \ldots, k, l = 1, 2, 3 \}. \]

Then, (2.16) is equivalent to

\[ \phi = A(\phi) =: L_k(N(\phi)) + L_k(l_k), \]

where \( L_k \) is defined in Proposition 2.3. We will prove that \( A \) is a contraction map from \( E \) to \( E \).

We have

\[ \|A(\phi)\|_s \leq C\|N(\phi)\|_{ss} + C\|l_k\|_{ss} \]

(2.26)

\[ \leq C\|\phi\|_{s \min(2^*,1.2)} + C\|l_k\|_{ss} \leq \frac{C}{k^{\frac{1}{2} + \sigma}} \leq \frac{1}{k^{\frac{1}{2}}}. \]

Thus, \( A \) maps \( E \) to \( E \).

On the other hand,

\[ \|A(\phi_1) - A(\phi_2)\|_s = \|L_k(N(\phi_1)) - L_k(N(\phi_2))\|_s \leq C\|N(\phi_1) - N(\phi_2)\|_{ss}. \]

If \( N \geq 6 \), then

\[ |N'(t)| \leq C|t|^{2^* - 2}. \]

As a result,

\[ |N(\phi_1) - N(\phi_2)| \leq C(|\phi_1|^{2^* - 2} + |\phi_2|^{2^* - 2})|\phi_1 - \phi_2| \]

\[ \leq C(\|\phi_1\|_{s}^{2^* - 2} + \|\phi_2\|_{s}^{2^* - 2})\|\phi_1 - \phi_2\|_s \left( \sum_{j=1}^{k} \frac{\mu^{\frac{N+2}{2}}}{1 + \mu|y - x_j|^{\frac{N+2}{2} + \tau}} \right)^{2^* - 1} \]

As before, we have
\[
\left( \sum_{j=1}^{k} \frac{1}{(1 + \mu |y - x_j|)^{\frac{N-2}{2} + \tau}} \right)^{2^{*} - 1} \leq C \sum_{j=1}^{k} \frac{1}{(1 + \mu |y - x_j|)^{\frac{N+2}{2} + \tau}}.
\]

So,
\[
\|A(\phi_1) - A(\phi_2)\|_* \leq C\|N(\phi_1) - N(\phi_2)\|_*
\leq C\left(\|\phi_1\|_*^{2^{*} - 2} + \|\phi_2\|_*^{2^{*} - 2}\right)\|\phi_1 - \phi_2\|_* \leq \frac{1}{2}\|\phi_1 - \phi_2\|_*.
\]

Thus, \( A \) is a contraction map.

For \( N = 5 \),
\[
|N'(t)| \leq C W_{h,d,\Lambda}^\frac{6-N}{2} |t| + C |t|^{2^{*} - 2}.
\]

So,
\[
\|N(\phi_1) - N(\phi_2)\|_* \leq C\left(\|\phi_1\|_*^{2^{*} - 2} + \|\phi_2\|_*^{2^{*} - 2}\right)\|\phi_1 - \phi_2\|_* \leq C\left(\|\phi_1\|_* + \|\phi_2\|_*\right)\|\phi_1 - \phi_2\|_* \sum_{j=1}^{k} \frac{\mu^{\frac{N-2}{2}}}{(1 + \mu |y - x_j|)^{\frac{N+2}{2} + \tau}}.
\]

Thus, \( A \) is a contraction map.

It follows from the contraction mapping theorem that there is a unique \( \phi \in E \), such that
\[
\phi = A(\phi).
\]

Moreover, it follows from Proposition 2.3 that
\[
\|\phi\|_* \leq C\|l_k\|_* + C\|N(\phi)\|_* \leq C\|l_k\|_* + C\|\phi\|_*^{\min(2^{*} - 1,2)},
\]
which gives
\[
\|\phi\|_* \leq C\left(\frac{1}{\mu}\right)^{\frac{1}{2^{*} + \sigma}},
\]
if $N \geq 5$. \hfill $\square$

3. Proof of Theorem 1.2

In this section, we will choose $h, d$ and $\mu$, such that the corresponding $c_i$ is zero. For this purpose, we only need to solve the following problem:

$$\langle I'(W_{h,d,\mu} + \phi), \partial_i P U_{x_1,\mu} \rangle = 0, \quad i = 1, 2, 3,$$

where we use $\partial_1, \partial_2$ and $\partial_3$ to denote $\frac{\partial}{\partial h}, \frac{\partial}{\partial d}$ and $\frac{\partial}{\partial \mu}$ respectively.

We first prove the following result:

**Proposition 3.1.** We have

$$\langle I'(W_{h,d,\mu} + \phi), \frac{\partial P U_{x_1,\mu}}{\partial \mu} \rangle = - \frac{B_1}{\mu^{N-1} d^{N-2} \mu} + \frac{B_4 k^{N-2}}{\mu^{N-1}} + O\left(\frac{1}{\mu^{2+\sigma}}\right),$$

(3.1)

$$\langle I'(W_{h,d,\mu} + \phi), \frac{\partial P U_{x_1,\mu}}{\partial h} \rangle = B_2 \psi'(h) + O\left(\frac{1}{\mu^{\sigma}}\right),$$

(3.2)

and

$$\langle I'(W_{h,d,\mu} + \phi), \frac{\partial P U_{x_1,\mu}}{\partial d} \rangle = B_3 - \frac{B_1}{\mu^{N-2} d^{N-1}} + O\left(\frac{1}{\mu^{\sigma}}\right).$$

(3.3)

The proof of Proposition 3.1 is similar to that of Lemma 2.6 and is quite technical. We leave it to the end of this section.

**Proof of Theorem 1.2.** Define

$$d = \frac{D}{k}, \quad \mu = \Lambda k^{\frac{N-1}{N-2}}.$$

Then, (3.1), (3.2) and (3.3) are equivalent to

$$- \frac{B_1}{\Lambda^{N-1} D^{N-2}} + \frac{B_4}{\Lambda^{N-1}} = o(1),$$

(3.4)

$$\psi'(h) = o(1),$$

(3.5)

and
(3.6) \[ B_3 = \frac{B_1}{\Lambda^{N-2} D^{N-1}} = o(1), \]
respectively.

Let
\[ f_1(D, \Lambda) = -\frac{B_1}{\Lambda^{N-1} D^{N-2}} + \frac{B_4}{\Lambda^{N-1}}, \]
and
\[ f_2(D, \Lambda) = B_3 - \frac{B_1}{\Lambda^{N-2} D^{N-1}}. \]

Then, \( f_1 = 0 \) and \( f_2 = 0 \) have a unique solution
\[ D_0 = \left( \frac{B_1}{B_3} \right)^{\frac{1}{N-2}}, \quad \Lambda_0 = \left( \frac{B_1}{B_3 D_0^{N-1}} \right)^{\frac{1}{N-2}}. \]

On the other hand, it is easy to see that
\[ \frac{\partial f_1(D_0, \Lambda_0)}{\partial \Lambda} = 0, \quad \frac{\partial f_2(D_0, \Lambda_0)}{\partial D} > 0, \]
and
\[ \frac{\partial f_1(D_0, \Lambda_0)}{\partial D} = \frac{\partial f_2(D_0, \Lambda_0)}{\partial \Lambda} > 0. \]

Thus
\[ \text{deg}(f^*, 0, B^*) \neq 0, \]
where \( f^*(D, \Lambda) = (f_1(D, \Lambda), f_2(D, \Lambda)) \), and \( B^* = \{(D, \Lambda) : |D - D_0| + |\Lambda - \Lambda_0| < \delta\} \).

On the other hand, since \( r^* \) is either a strict local minimum point, or strict local maximum point of \( \psi \), we know that for \( \delta > 0 \) small,
\[ \text{deg}(\psi', 0, (r^* - \delta, r^* + \delta)) \neq 0. \]

Let \( \bar{f}((h, D, \Lambda) = (\psi(h), f^*(D, \Lambda)) \), and let \( \bar{B} = (r^* - \delta, r^* + \delta) \times B^* \). Then
\[ \text{deg}(\bar{f}, 0, \bar{B}) \neq 0, \]
from which, we know that (3.4), (3.5) and (3.6) have a solution near \((r^*, D_0, \Lambda_0)\). \( \square \)
Proof of Proposition 3.1. We have

\[
\langle I'(W_{h,d,\mu} + \phi), \partial_t PU_{x_1,\mu} \rangle \\
= \langle I'(W_{h,d,\mu}), \partial_t PU_{x_1,\mu} \rangle + \int_{\Omega} y_N^m \left( (W_{h,d,\mu} + \phi)^{2^*-1} - W_{h,d,\mu}^{2^*-1} - (2^* - 1)U_{x_1,\mu}^{2^*-2} \phi \right) \partial_t PU_{x_1,\mu} \\
- m \int_{\Omega} y_N^{m-1} \frac{\partial (\partial_t PU_{x_1,\mu})}{\partial y_N} \phi.
\]

In view of Proposition A.1, we need to prove

\[
\int_{\Omega} y_N^m \left( (W_{h,d,\mu} + \phi)^{2^*-1} - W_{h,d,\mu}^{2^*-1} - (2^* - 1)U_{x_1,\mu}^{2^*-2} \phi \right) \partial_t PU_{x_1,\mu} \\
- m \int_{\Omega} y_N^{m-1} \frac{\partial (\partial_t PU_{x_1,\mu})}{\partial y_N} \phi
\]

\[
= \mu^m \frac{1}{(\mu^1+\sigma)},
\]

where \( m_i = 1 \) if \( i = 1, 2 \), and \( m_3 = -1 \).

The proof of (3.7) is similar to that of Lemma 2.6. Write

\[
\int_{\Omega} y_N^m \left( (W_{h,d,\mu} + \phi)^{2^*-1} - W_{h,d,\mu}^{2^*-1} - (2^* - 1)U_{x_1,\mu}^{2^*-2} \phi \right) \partial_t PU_{x_1,\mu} \\
= \int_{\Omega} y_N^m \left( (W_{h,d,\mu} + \phi)^{2^*-1} - W_{h,d,\mu}^{2^*-1} - (2^* - 1)W_{h,d,\mu}^{2^*-2} \phi \right) \partial_t PU_{x_1,\mu} \\
+ (2^* - 1) \int_{\Omega} y_N^m \left( W_{h,d,\mu}^{2^*-2} - U_{x_1,\mu}^{2^*-2} \phi \right) \partial_t PU_{x_1,\mu}.
\]

If \( N \geq 6 \), then \( 2^* - 1 \leq 2 \). In this case, we have the formula

\[
(1 + t)^{2^*-1} - 1 - (2^* - 1)t = O(t^2).
\]
As a result,
\[
\left| \int_{\Omega} y_\Omega^m \left( (W_{h,d,\mu} + \phi)^{2^*-1} - W_{h,d,\mu}^{2^*-1} - (2^*-1)W_{h,d,\mu}^{2^*-2}\phi \right) \partial_t PU_{x_1}\mu \right| 
\leq C \int_{\Omega} W_{h,d,\mu}^{2^*-3} |\partial_t PU_{x_1}\mu|^2 \leq C \mu^{m_i} \int_{\tilde{\Omega}_{x_1}} U_{x_1,\mu}^{2^*-2} |\phi|^2 \quad \text{(since } 2^*-3 \leq 0) 
\leq C \mu^{N+m_i} \|\phi\|^2 \int_{\Omega} \sum_{j=1}^k \frac{1}{(1 + \mu |y - x_j|)^{N/2+2}} \frac{1}{(1 + \mu |y - x_1|)^4} 
= C \mu^{m_i} \|\phi\|^2 \int_{\Omega} \sum_{j=1}^k \frac{1}{(1 + |y - \mu x_j|)^{N/2+2}} \frac{1}{(1 + |y - \mu x_1|)^4}, 
\]
where \( \tilde{\Omega} = \{ y : \mu^{-1} y \in \Omega \} \).

Recall
\[
\Omega_j = \{ y : y \in \Omega, \langle y', x_j \rangle \geq \cos \frac{\pi}{k} \}, \quad y' = (y_1, y_2, 0, \cdots, 0).
\]

For any \( y \in \tilde{\Omega}_n = \{ y : \mu^{-1} y \in \Omega_n \} \),
\[
\sum_{j=1}^k \frac{1}{(1 + |y - \mu x_j|)^{N/2+2}} \leq \frac{1}{(1 + |y - \mu x_n|)^{N/2+2}} + \frac{1}{(1 + |y - \mu x_1|)^{N/2+2}} \sum_{j \neq n} \frac{C}{|\mu x_n - \mu x_j|^2} 
\leq \frac{C}{(1 + |y - \mu x_n|)^{N/2}}.
\]

So,
\[
\int_{\tilde{\Omega}_n} \sum_{j=1}^k \frac{1}{(1 + |y - \mu x_j|)^{N/2+2}} \frac{1}{(1 + |y - \mu x_1|)^4} \leq C \int_{\Omega_n} \frac{1}{(1 + |y - \mu x_n|)^N} \frac{1}{(1 + |y - \mu x_1|)^4} \leq C \frac{1}{|\mu x_n - \mu x_1|^2}. 
\]

So, for \( N \geq 6 \),
\[
\left| \int_{\Omega} y_\Omega^m \left( (W_{h,d,\mu} + \phi)^{2^*-1} - W_{h,d,\mu}^{2^*-1} - (2^*-1)W_{h,d,\mu}^{2^*-2}\phi \right) \partial_t PU_{x_1}\mu \right| 
\leq C \mu^{m_i} \|\phi\|^2 \leq \frac{C \mu^{m_i}}{\mu^{1+\sigma}}.
\]
If \( N = 5 \), then

\[
\left| \int_{\Omega} y_N^{m} \left( (W_{h,d,\mu} + \phi)^{2^*-1} - W_{h,d,\mu}^{2^*-1} - (2^* - 1)W_{h,d,\mu}^{2^*-2}\phi \right) \partial_t PU_{x_1,\mu} \right| \\
\leq C \int_{\Omega} \left( W_{h,d,\mu}^{2^*-3} \left| \phi \right|^2 + \left| \phi \right|^{2^*-1} \right) \left| \partial_t PU_{x_1,\mu} \right| \\
\leq C \mu^{m_i} \left( \left\| \phi \right\|_{*}^2 + \left\| \phi \right\|_{*}^{2^*-1} \right) \int_{\Omega} \sum_{j=1}^{k} \left( \frac{1}{1 + \left| y - \mu x_j \right|^{N-2}} \right)^{2^*-1} \frac{1}{(1 + \left| y - \mu x_1 \right|)^{N-2}} \\
\leq C \mu^{m_i} \left( \left\| \phi \right\|_{*}^2 + \left\| \phi \right\|_{*}^{2^*-1} \right) \leq \frac{C \mu^{m_i}}{\mu^{1+\sigma}}.
\]

So, we have proved

\[
\int_{\Omega} y_N^{m} \left( (W_{h,d,\mu} + \phi)^{2^*-1} - W_{h,d,\mu}^{2^*-1} - (2^* - 1)W_{h,d,\mu}^{2^*-2}\phi \right) \partial_t PU_{x_1,\mu} = \mu^{m} O \left( \frac{1}{\mu^{1+\sigma}} \right).
\]

Now, we estimate the second term in the right hand side of (3.8):

Write

\[
\int_{\Omega} y_N^{m} \left( W_{h,d,\mu}^{2^*-2} - U_{x_1,\mu}^{2^*-2} \right) \phi \partial_t PU_{x_1,\mu} \\
= \int_{\Omega} y_N^{m} \left( W_{h,d,\mu}^{2^*-2} - PU_{x_1,\mu}^{2^*-2} \right) \phi \partial_t PU_{x_1,\mu} + \int_{\Omega} y_N^{m} \left( PU_{x_1,\mu}^{2^*-2} - U_{x_1,\mu}^{2^*-2} \right) \phi \partial_t PU_{x_1,\mu}
\]

If \( N \geq 6 \), then using (2.21),

\[
\left| \int_{\Omega} y_N^{m} \left( W_{h,d,\mu}^{2^*-2} - PU_{x_1,\mu}^{2^*-2} \right) \phi \partial_t PU_{x_1,\mu} \right| \leq C \mu^{m_i} \int_{\Omega} \sum_{j=2}^{k} U_{j,\mu} U_{x_1,\mu}^{2^*-2} \left| \phi \right| \\
\leq C \mu^{m_i} \left\| \phi \right\|_{*} \int_{\Omega} \sum_{j=2}^{k} \left( \frac{1}{1 + \left| y - \mu x_j \right|^{N-2}} \right)^{2^*-1} \frac{1}{(1 + \left| y - \mu x_1 \right|)^{N-2}} \\
\leq C \mu^{m_i} \frac{1}{\mu^{1+\sigma}} \int_{\Omega} \sum_{i=1}^{k} \left( \frac{1}{1 + \left| y - \mu x_i \right|^{N-2}} \right)^{2^*-1} \frac{1}{(1 + \left| y - \mu x_1 \right|)^{N+2+\tau}} \\
\leq C \mu^{m_i} \frac{1}{\mu^{1+\sigma}}.
\]

If \( N = 5 \), then \( 2^* - 2 = \frac{4}{3} \) and \( \tau = \frac{3}{4} \). We have
(3.12) 
\[
\left| \int_{\Omega} y_{N}^{m} \left( W_{h, d, \mu}^{x_{1, \mu}} - P U_{x_{1, \mu}}^{2^*-2} \right) \phi \partial_{t} P U_{x_{1, \mu}} \right|
\]
\[
\leq C \mu^{m_{\iota}} \int_{\Omega} \left( \sum_{j=2}^{k} U_{x_{j, \mu}}^{2^*-2} + \sum_{j=2}^{k} U_{x_{j, \mu}}^{2^*-2} U_{x_{1, \mu}} \right) |\phi| 
\]
\[
\leq C \mu^{m_{\iota}} \frac{1}{\mu^{1+\sigma}} + C \mu^{m_{\iota}} \int_{\Omega} \left( \sum_{j=2}^{k} U_{x_{j, \mu}}^{2^*-2} U_{x_{1, \mu}} \right) |\phi| 
\]
\[
= C \mu^{m_{\iota}} \frac{1}{\mu^{1+\sigma}} 
\]
\[
+ \mu^{m_{\iota}} \|\phi\|_{*} \int_{\Omega} \left( \sum_{j=2}^{k} \frac{1}{1 + |y - \mu x_{j}|^{N-2}} \right)^{2^*-2} \frac{1}{(1 + |y - \mu x_{1}|)^{N-2}} \sum_{i=1}^{k} \frac{1}{1 + |y - \mu x_{i}|^{N-2} + \tau}. 
\]
But
\[
\sum_{n=2}^{k} \int_{\Omega_{n}} \left( \sum_{j=2}^{k} \frac{1}{1 + |y - \mu x_{j}|^{N-2}} \right)^{2^*-2} \frac{1}{(1 + |y - \mu x_{1}|)^{N-2}} \sum_{i=1}^{k} \frac{1}{1 + |y - \mu x_{i}|^{N-2} + \tau}
\]
\[
\leq \sum_{n=2}^{k} \int_{\Omega_{n}} \frac{1}{1 + |y - \mu x_{n}|^{4 - \frac{4}{N-2} + \frac{4}{N-2}}} \frac{1}{(1 + |y - \mu x_{1}|)^{N-2}} 
\]
\[
\leq \ln \mu \sum_{n=2}^{k} \frac{1}{|\mu x_{n} - \mu x_{1}|^{\frac{4}{N-2} - \frac{4}{N-2}}} \leq \frac{C}{\mu^{1+\sigma}}, 
\]
and
\[
\int_{\Omega_{1}} \left( \sum_{j=2}^{k} \frac{1}{1 + |y - \mu x_{j}|^{N-2}} \right)^{2^*-2} \frac{1}{(1 + |y - \mu x_{1}|)^{N-2}} \sum_{i=1}^{k} \frac{1}{1 + |y - \mu x_{i}|^{N-2} + \tau}
\]
\[
\leq C \mu^{4} \int_{\Omega_{1}} \frac{1}{(1 + |y - \mu x_{1}|)^{N-2}} \sum_{i=1}^{k} \frac{1}{1 + |y - \mu x_{i}|^{N-2} + \tau}
\]
\[
\leq \frac{C}{\mu^{1+\sigma}}. 
\]
So, we also prove that for $N = 5$,

(3.13) 
\[
\left| \int_{\Omega} y_{N}^{m} \left( W_{h, d, \mu}^{x_{1, \mu}} - P U_{x_{1, \mu}}^{2^*-2} \right) \phi \partial_{t} P U_{x_{1, \mu}} \right| \leq \frac{C \mu^{m_{\iota}}}{\mu^{1+\sigma}}
\]
Using (2.23), similar to (2.24), we can prove

$$\left| \int_{\Omega} y_N^{m_l} \left( P_{x_{1, \mu}} - U_{x_{1, \mu}}^{2^*-2} \right) \phi \partial_t P_{x_{1, \mu}} \right|$$

$$\leq C \mu^{m_l} \int_{\Omega} \frac{H(y, x_1)}{\mu^{n/2}} U_{x_{1, \mu}}^{2^*-2} |\phi|$$

$$\leq C \mu^{m_l} \mu^{-\frac{n+2}{2}} \int_{\Omega} \frac{1}{\mu^{2+\sigma}} \left( 1 + \mu |y - x_1| \right)^{\frac{n+2}{2}+\sigma} |\phi|$$

$$\leq C \mu^{m_l} \frac{1}{\mu^{2+\sigma}} \|\phi\|_* \leq C \mu^{m_l} \mu^{1+\sigma}.$$  

(3.14)

Combining (3.10), (3.11), (3.13) and (3.14), we obtain that for $N \geq 5$,

$$\int_{\Omega} y_N^{m_l} \left( W_{i, \mu}^{2^*-2} - \lambda U_{x_{1, \mu}}^{2^*-2} \right) \phi \partial_t P_{x_{1, \mu}} = \mu^{m_l} O \left( \frac{k\|\phi\|_*}{\mu^N} \right).$$

(3.15)

Finally,

$$\left| \int_{\Omega} y_N^{m_l-1} \frac{\partial(\partial_t P_{x_{1, \mu}})}{\partial y_N} \phi \right|$$

$$\leq C \mu^{-1+\sigma} \|\phi\|_* \int_{\Omega} \sum_{j=1}^{k} \frac{1}{(1 + |y - \mu x_j|)^{\frac{n+2}{2}+\sigma}} \frac{1}{(1 + |y - \mu x_1|)^{N-1}}$$

$$\leq C \mu^{-1+\sigma} \|\phi\|_* \leq C \mu^{m_l} \frac{1}{\mu^{\frac{n+2}{2}+\sigma}}.$$  

(3.16)

So, (3.7) follows from (3.9), (3.15) and (3.16).

\[\square\]

**Appendix A. Energy Expansion**

In the appendix, we always assume that

$$x_j = (r \cos \frac{2(j-1)\pi}{k}, r \sin \frac{2(j-1)\pi}{k}, 0, l), \quad j = 1, \cdots, k.$$  

where $0$ is the zero vector in $\mathbb{R}^{N-3}$, and $(r, l)$ is close to $(r^*, l^*)$. Recall that we write

$$x_1 = (h, 0, \cdots, 0, \psi(h)) + \nu,$$

where $\nu$ is the inward unit normal of $\partial \Omega$ at $(h, 0, \cdots, 0, \psi(h))$. 


Let $G(y, z)$ be the Green function of $-\Delta$ in $\Omega$ with the Dirichlet boundary condition. Let $H(y, z)$ be the regular part of the Green function.

Let recall that

$$\mu \in [\Lambda_0 k^{\frac{N-1}{N-2}}, \Lambda_1 k^{\frac{N-1}{N-2}}],$$

where $\Lambda_0 > 0$ is a small constant, and $\Lambda_1 > 0$ is a large constant.

Define

$$I(u) = \frac{1}{2} \int_\Omega y_N^n |Du|^2 - \frac{1}{2^*} \int_\Omega y_N^n |u|^{2^*},$$

$$U_{x_j, \mu}(y) = \left(N(N - 2)\right)^{\frac{N-2}{4}} \frac{\mu^{\frac{N-2}{2}}}{(1 + \mu^2 |y - x_j|^2)^{\frac{N-2}{2}}},$$

and

$$W_{h,d, \mu}(y) = \sum_{j=1}^{k} PU_{x_j, \mu}(y),$$

where $PU_{x, \mu}$ is the solution of (1.6). It is well known that

$$U_{x_j, \mu}(y) - PU_{x_j, \mu}(y) = \frac{\tilde{B}H(y, x)}{\mu^{\frac{N-2}{2}}} + O\left(\frac{1}{d N \mu^{\frac{N-2}{2}}}\right),$$

where $\tilde{B} > 0$ is a constant.

The main result of this section is the following estimates.

**Proposition A.1.** We have

$$\langle I'(W_{h,d, \mu}), \frac{\partial PU_{x_1, \mu}}{\partial \mu} \rangle = - \frac{B_1}{\mu^{N-1} d^{N-2}} + \frac{B_4 k^{N-2}}{\mu^{N-1}} + O\left(\frac{1}{\mu^{2+\sigma}}\right),$$

$$\langle I'(W_{h,d, \mu}), \frac{\partial PU_{x_1, \mu}}{\partial h} \rangle = B_2 \psi'(h) + O\left(\frac{1}{\mu^\sigma}\right),$$

and

$$\langle I'(W_{h,d, \mu}), \frac{\partial PU_{x_1, \mu}}{\partial d} \rangle = B_3 - \frac{B_1}{\mu^{N-2} d^{N-1}} + O\left(\frac{1}{\mu^\sigma}\right),$$

where $B_1, B_2, B_3$ and $B_4$ are some positive constants, $\sigma > 0$ is a small constant.
Proof. We use \( \partial_1, \partial_2, \) and \( \partial_3 \) to denote \( \frac{\partial}{\partial n}, \frac{\partial}{\partial a}, \) and \( \frac{\partial}{\partial \mu} \) respectively. Then

\[
(A.5) \quad \langle I'(W_{h,d,\mu}), \partial_i PU_{x_1,\mu} \rangle = \int_{\Omega} y_{N}^{m} \left( \sum_{j=1}^{k} U_{x_j,\mu}^{2r-1} - W_{h,d,\mu}^{2r-1} \right) \partial_i PU_{x_1,\mu} - \int_{\Omega} y_{N}^{m-1} \frac{\partial PU_{x_j,\mu}}{\partial y_{N}} \partial_i PU_{x_1,\mu} \nonumber \\
= m \int_{\Omega} y_{N}^{m} \left( \sum_{j=1}^{k} U_{x_j,\mu}^{2r-1} - W_{h,d,\mu}^{2r-1} \right) \partial_i PU_{x_1,\mu} + \int_{\Omega} (y_{N}^{m} - l) \left( \sum_{j=1}^{k} U_{x_j,\mu}^{2r-1} - W_{h,d,\mu}^{2r-1} \right) \partial_i PU_{x_1,\mu} 
- m \sum_{j=1}^{k} \int_{\Omega} y_{N}^{m-1} \frac{\partial PU_{x_j,\mu}}{\partial y_{N}} \partial_i PU_{x_1,\mu}.
\]

On the other hand,

\[
(A.6) \quad \int_{\Omega} \left( \sum_{j=1}^{k} U_{x_j,\mu}^{2r-1} - W_{h,d,\mu}^{2r-1} \right) \partial_i U_{x_1,\mu} = \sum_{n=1}^{k} \int_{\Omega} \left( \sum_{j=1}^{k} U_{x_j,\mu}^{2r-1} - W_{h,d,\mu}^{2r-1} \right) \partial_i PU_{x_1,\mu},
\]

where

\[
\Omega_j = \{ y : y \in \Omega, \left\langle \frac{y'}{|y'|}, \frac{x_j}{|x_j|} \right\rangle \geq \cos \frac{\pi}{k} \}, \quad y' = (y_1, y_2, 0, \cdots, 0).
\]

Let \( m_i = 1 \) if \( i = 1, 2 \) and let \( m_3 = -1 \). Then

\[
|\partial_i PU_{x_1,\mu}| \leq C m_i U_{x_1,\mu}.
\]

So we have

\[
\left| \int_{\Omega} \left( \sum_{j=1}^{k} U_{x_j,\mu}^{2r-1} - W_{h,d,\mu}^{2r-1} \right) \partial_i PU_{x_1,\mu} \right| \leq C m_i \int_{\Omega} \left| \sum_{j=1}^{k} U_{x_j,\mu}^{2r-1} - W_{h,d,\mu}^{2r-1} \right| U_{x_1,\mu} 
\leq C m_i \sum_{j=1}^{k} \{ \sum_{j=1}^{k} U_{x_j,\mu}^{2r-1} \} \sum_{j \neq n} \left( 1 + \frac{y - \mu x_j}{(1 + |y - \mu x_j|)^{N-2}} \right)^{2r-1} U_{x_1,\mu} 
\leq C m_i \sum_{j=1}^{k} \{ \sum_{j \neq n} \left( 1 + \frac{y - \mu x_j}{(1 + |y - \mu x_j|)^{N-2}} \right)^{2r-1} \}
\]

where \( \tilde{\Omega}_n = \{ y : \mu^{-1} y \in \Omega_n \} \).
It is easy to check that for any \( y \in \hat{\Omega}_n, \ n \neq j, \ |y - \mu x_j| \geq \frac{1}{2}|\mu x_j - \mu x_n| \). As a result,

\[
\sum_{j \neq n} \frac{1}{(1 + |y - \mu x_n|)^{N-2}} \leq C \sum_{j \neq n} \frac{1}{|\mu x_j - \mu x_n|^{N-2}} \leq \frac{C k^{N-2}}{\mu^{N-2}}.
\]

So, we obtain

\[
\int_{\hat{\Omega}_n} \frac{1}{(1 + |y - \mu x_n|)^4} \sum_{j \neq n} \frac{1}{(1 + |y - \mu x_j|)^{N-2}(1 + |y - \mu x_1|)^{N-2}} \leq \frac{C k^{N-2}}{\mu^{N-2}} \int_{\hat{\Omega}_n} \frac{1}{\mu^2|x_n - x_1|^2} \leq \frac{C k^{N-2}}{\mu^{N-2}} \frac{\ln \mu}{\mu^2|x_n - x_1|^2}.
\]

(A.8)

On the other hand, let \( t > \max(1, \frac{2(N-2)}{N+2}) \). Then, for \( y \in \hat{\Omega}_n, \)

\[
\sum_{j \neq n} \frac{1}{(1 + |y - \mu x_j|)^{N-2}} \leq C \sum_{j \neq n} \frac{1}{|\mu x_j - \mu x_n|^t(1 + |y - \mu x_n|)^{N-2-t}} \leq \frac{C k^t}{\mu^t} \frac{1}{(1 + |y - \mu x_n|)^{N-2-t}}.
\]

As a result,

\[
\int_{\hat{\Omega}_n} (\sum_{j \neq n} \frac{1}{(1 + |y - \mu x_j|)^{N-2}})^{2^t-1} \frac{1}{(1 + |y - \mu x_1|)^{N-2}} \leq \frac{C k^{2^t-1}}{\mu^{2^t-1}} \int_{\hat{\Omega}_n} \frac{1}{\mu^{N-2-t}(2^t-1)} \frac{1}{(1 + |y - \mu x_n|)^{N-2-t}} \leq \frac{C k^{2^t-1}}{\mu^{N-2-t}} \frac{1}{\mu|x_n - x_1|^{N-2-t}} \int_{\hat{\Omega}_n} \frac{1}{(1 + |y - \mu x_n|)^{(N-2-t)(2^t-1)+N-2-(N-t(2^t-1))}} \leq \frac{C k^{2^t-1}}{\mu^{N-2-t}} \frac{\ln \mu}{\mu|x_n - x_1|^{N-2-t}} \cdot
\]

(A.9)

Combining (A.7), (A.8) and (A.9), we find
\[\left| \sum_{n=2}^{k} \int_{\Omega, n} \left( \sum_{j=1}^{k} U_{x,j,\mu}^{2^*-1} - W_{h,d,\mu}^{2^*-1} \right) \partial_t PU_{x,1,\mu} \right| \]

(A.10)

\[\leq \mu^{m_i} \frac{C k^{N-2}}{\mu^{N-2}} \sum_{n=2}^{k} \frac{\ln \mu}{\mu^2|x_n - x_1|^2} + \mu^{m_i} \frac{C k^{(2^*-1)}}{\mu^t(2^*-1)} \sum_{n=2}^{k} \frac{\ln \mu}{|\mu x_n - \mu x_1|^{N-2(2^*-1)}}\]

\[\leq \mu^{m_i} \frac{C k^{N} \ln \mu}{\mu^N} .\]

So, (A.6) and (A.10) yield

\[\int_{\Omega} \left( \sum_{j=1}^{k} U_{x,j,\mu}^{2^*-1} - W_{h,d,\mu}^{2^*-1} \right) \partial_t PU_{x,1,\mu} \]

(A.11)

\[= \int_{\Omega_1} \left( \sum_{j=1}^{k} U_{x,j,\mu}^{2^*-1} - W_{h,d,\mu}^{2^*-1} \right) \partial_t PU_{x,1,\mu} + O\left( \mu^{m_i} \frac{k^N \ln \mu}{\mu^N} \right).\]

It is standard to show that

\[\int_{\Omega_1} \left( \sum_{j=1}^{k} U_{x,j,\mu}^{2^*-1} - W_{h,d,\mu}^{2^*-1} \right) \partial_t PU_{x,1,\mu} \]

(A.12)

\[= \int_{\Omega_1} (U_{x,1,\mu}^{2^*-1} - (PU_{x,1,\mu})^{2^*-1}) \partial_t PU_{x,1,\mu} + \int_{\Omega_1} (PU_{x,1,\mu})^{2^*-2} \left( \sum_{j=2}^{k} PU_{x,j,\mu} \right) \partial_t PU_{x,1,\mu} + O\left( \mu^{m_i} \frac{k^N \ln \mu}{\mu^N} \right)\]

\[= \partial_t \left( \frac{\bar{B}_1 H(x_1, x_1)}{\mu^{N-2}} - \sum_{i=2}^{k} \frac{\bar{B}_1 G(x_1, x_i)}{\mu^{N-2}} \right) + O\left( \mu^{m_i} \frac{k^N \ln \mu}{\mu^N} \right),\]

where \(\bar{B}_1 > 0\) is a constant.

We now estimate the second term in (A.5). Similar to the above discussion, we have
\[
\int_{\Omega} (y_N^m - l^m) (\sum_{j=1}^{k} U_{x,j,\mu}^{2^*-1} - W_{h,i,\mu}^{2^*-1}) \partial_t PU_{x,1,\mu} \\
= \int_{\Omega_1} (y_N^m - l^m) (\sum_{j=1}^{k} U_{x,j,\mu}^{2^*-1} - W_{h,i,\mu}^{2^*-1}) \partial_t PU_{x,1,\mu} + O\left(\mu^m \frac{k^N \ln \mu}{\mu^N}\right)
\]
(A.13)
\[
= \int_{\Omega_1} (y_N^m - l^m) (U_{x,1,\mu}^{2^*-1} -(PU_{x,1,\mu})^{2^*-1}) \partial_t PU_{x,1,\mu} \\
+ \int_{\Omega_1} (y_N^m - l^m) (PU_{x,1,\mu})^{2^*-2} (\sum_{j=2}^{k} PU_{x,j,\mu}) \partial_t PU_{x,1,\mu} + O\left(\mu^m \frac{k^N \ln \mu}{\mu^N}\right)
\]
\[
= O\left(\frac{1}{\mu} |\partial_t \frac{H(x_1,x_1)}{\mu^N-2}| + \frac{1}{\mu} \sum_{j=2}^{k} |\partial_t \frac{G(x_j,x_1)}{\mu^N-2}| + \frac{k^N \ln \mu}{\mu^N}\right) \mu^m.
\]

Finally, we estimate the last terms in (A.5).

If \(i = 3\), then

\[
\left| \sum_{j=2}^{k} \int_{\Omega} y_N^{m-1} \frac{\partial PU_{x,j,\mu}}{\partial y_N} \partial_3 PU_{x,1,\mu} \right|
\leq C \frac{1}{\mu^2} \sum_{j=2}^{k} \int_{\mathbb{R}^N} \left(\frac{1}{1 + |y - \mu x_j|^N-1}\right) \left(\frac{1}{1 + |y - \mu x_j|^N-1}\right)
\leq C \frac{1}{\mu^2} \sum_{j=2}^{k} \frac{1}{|\mu x_1 - \mu x_j|^N-3} = O\left(\frac{1}{\mu^2} \frac{k^{N-3} \ln \mu}{\mu^{N-3}}\right).
\]

and

\[
\left| \int_{\Omega} y_N^{m-1} \frac{\partial PU_{x,1,\mu}}{\partial y_N} \partial_3 PU_{x,1,\mu} \right|
\leq C \int_{\Omega} \frac{\mu^{N-1}}{(1 + \mu |y - x_1|)^{N-1} \mu (1 + \mu |y - x_1|)^{N-2}} = O\left(\frac{1}{\mu^{2+\sigma}}\right).
\]

So, we obtain

\[
\sum_{j=1}^{k} \int_{\Omega} y_N^{m-1} \frac{\partial PU_{x,j,\mu}}{\partial y_N} \partial_3 PU_{x,1,\mu} = O\left(\frac{1}{\mu^{2+\sigma}}\right).
\]

Thus, we have proved
(A.14)
\[
\langle f(W_{h,d,m}), \frac{\partial PU_{x_1,\mu}}{\partial \mu} \rangle = -\frac{(N-2)\tilde{B}_1 H(x_1, x_1)}{\mu^{N-1}} + \sum_{i=2}^{k} \frac{(N-2)\tilde{B}_1 G(x_1, x_i)}{\mu^{N-1}} + O\left(\frac{1}{\mu^{2+\sigma}}\right),
\]
for some $\tilde{B}_1 > 0$.

Let $T$ be the unit tangent vector of $\partial \Omega$ at $(h,0,\cdots,0,\psi(h))$. Note that
\[
T = \frac{\{1,0,\cdots,0,\psi'(h)\}}{\sqrt{1 + |\psi'(h)|^2}}, \quad \nu = \frac{\{-\psi'(h),0,\cdots,0,1\}}{\sqrt{1 + |\psi'(h)|^2}}.
\]
As a result,
\[
\partial_1 PU_{x_1,\mu} = \frac{\partial PU_{x_1,\mu}}{\partial x_{1,1}} \frac{1}{\sqrt{1 + |\psi'(h)|^2}} + \frac{\partial PU_{x_1,\mu}}{\partial x_{1,N}} \frac{\psi'(h)}{\sqrt{1 + |\psi'(h)|^2}}
\]
Noting that
\[
\left| \frac{\partial H(y,x_1)}{\partial x_{1,1}} \right| \leq \frac{C}{|y - x_1^*|^{N-1}},
\]
where $x_1^*$ is the reflection point of $x_1$ with respect to $\partial \Omega$, we can prove
\[
\sum_{j=1}^{k} \int_{\Omega} \frac{\partial PU_{x_1,\mu}}{\partial y_N} \frac{\partial PU_{x_1,\mu}}{\partial x_{1,1}} \frac{1}{\sqrt{1 + |\psi'(h)|^2}}
\]
\[
= \int_{\Omega} \frac{\partial U_{x_1,\mu}}{\partial y_N} \frac{\partial U_{x_1,\mu}}{\partial x_{1,1}} \frac{1}{\sqrt{1 + |\psi'(h)|^2}} + O\left(\frac{k^{N-2} \ln \mu}{\mu^{N-2}}\right)
\]
\[
= \int_{\Omega} \frac{\partial U_{x_1,\mu}}{\partial y_N} \frac{\partial U_{x_1,\mu}}{\partial x_{1,1}} \frac{1}{\sqrt{1 + |\psi'(h)|^2}}
\]
\[
+ O\left(\int_{\Omega} (1 + \mu |y - x_1|^{N-1}) \frac{1}{\mu^{N-2}} |y - x_1^*|^{N-1} \right) + O\left(\frac{k^{N-2} \ln \mu}{\mu^{N-2}}\right)
\]
\[
= \int_{\Omega} \frac{\partial U_{x_1,\mu}}{\partial y_N} \frac{\partial U_{x_1,\mu}}{\partial x_{1,1}} \frac{1}{\sqrt{1 + |\psi'(h)|^2}}
\]
\[
+ O\left(\frac{1}{\mu^{N-2}} \int_{\Omega} \frac{1}{|y - x_1|^{N-\theta}} + \frac{1}{\mu^{N-2}}\right)
\]
\[
= O\left(\frac{1}{\mu^{N-2-\theta}} \right) = O\left(\frac{1}{\mu^{\sigma}}\right),
\]
where $\theta > 0$ is any small constant and $\sigma > 0$ is a small constant. As a result,
\[
\sum_{j=1}^{k} \int_{\Omega} y_{N}^{n-1} \frac{\partial P U_{x_{j}, \mu}}{\partial y_{N}} \partial_{1} P U_{x_{1}, \mu} \nabla_{x} \psi(h) \sqrt{1 + \psi(h)^{2}} + O\left(\frac{1}{\mu^{\sigma}}\right) \\
= \sum_{j=1}^{k} \int_{\Omega} y_{N}^{n-1} \frac{\partial P U_{x_{j}, \mu}}{\partial y_{N}} \frac{\partial P U_{x_{1}, \mu}}{\partial x_{1, N}} \psi(h) \sqrt{1 + \psi(h)^{2}} + O\left(\frac{1}{\mu^{\sigma}}\right) \\
= \int_{\Omega} y_{N}^{n-1} \frac{\partial U_{x_{1}, \mu}}{\partial y_{N}} \frac{\partial U_{x_{1}, \mu}}{\partial x_{1, N}} \psi(h) \sqrt{1 + \psi(h)^{2}} + O\left(\frac{1}{\mu^{\sigma}}\right) \\
= -B'\psi(h) + O\left(\frac{1}{\mu^{\sigma}}\right).
\]

So we obtain

\[
\langle I'(W_{h,d,\mu}), \frac{\partial P U_{x_{1}, \mu}}{\partial h} \rangle = B_2\psi(h) + \frac{B_1}{\mu^{N-2}} \frac{\partial H(x_{1, x_1})}{\partial h} - \frac{B_1}{\mu^{N-2}} \sum_{i=2}^{k} \frac{\partial G(x_{i, x_1})}{\partial h} + O\left(\frac{1}{\mu^{\sigma}}\right),
\]

where \(B_2 > 0\) is a constant.

From

\[
\partial_{2} P U_{x_{j}, \mu} = -\frac{\partial P U_{x_{j}, \mu}}{\partial x_{1, 1}} \frac{\psi(h)}{\sqrt{1 + \psi(h)^{2}}} + \frac{\partial P U_{x_{j}, \mu}}{\partial x_{1, N}} \frac{1}{\sqrt{1 + \psi(h)^{2}}}
\]

we can prove

\[
\sum_{j=1}^{k} \int_{\Omega} y_{N}^{n-1} \frac{\partial P U_{x_{j}, \mu}}{\partial y_{N}} \partial_{2} P U_{x_{1}, \mu} \nabla_{x} \psi(h) \sqrt{1 + \psi(h)^{2}} + O\left(\frac{1}{\mu^{\sigma}}\right) \\
= \sum_{j=1}^{k} \int_{\Omega} y_{N}^{n-1} \frac{\partial P U_{x_{j}, \mu}}{\partial y_{N}} \frac{\partial P U_{x_{1}, \mu}}{\partial x_{1, N}} \frac{1}{\sqrt{1 + \psi(h)^{2}}} + O\left(\frac{1}{\mu^{\sigma}}\right) \\
= \int_{\Omega} y_{N}^{n-1} \frac{\partial U_{x_{1}, \mu}}{\partial y_{N}} \frac{\partial U_{x_{1}, \mu}}{\partial x_{1, N}} \frac{1}{\sqrt{1 + \psi(h)^{2}}} + O\left(\frac{1}{\mu^{\sigma}}\right) \\
= -B'' + O\left(\frac{1}{\mu^{\sigma}}\right),
\]

where \(B'' > 0\) is a constant. As a result,
\[
\langle I'(W_{h,d,\mu}), \frac{\partial PU_{x_1,\mu}}{\partial d} \rangle
\]
(A.16)
\[
= B_3 + \frac{B_1}{\mu^{N-2}} \frac{\partial H(x_1, x_1)}{\partial d} - \frac{B_1}{\mu^{N-2}} \sum_{i=2}^{k} \frac{\partial G(x_i, x_1)}{\partial d} + O\left(\frac{1}{\mu^\sigma}\right),
\]
where \(B_3 > 0\) is a constant.

To finish the proof of Proposition A.1, we need estimate the Green function.

Recall that
\[
x_1 = (h, 0, \cdots, 0, \psi(h)) + d\nu = (h - \frac{d\psi'(h)}{\sqrt{1 + |\psi'(h)|^2}}, 0, \cdots, 0, \psi(h) + \frac{d}{\sqrt{1 + |\psi'(h)|^2}}).
\]

Then, we have
\[
x_j = ((h - \frac{d\psi'(h)}{\sqrt{1 + |\psi'(h)|^2}}) \cos \frac{2(j - 1)\pi}{k}, (h - \frac{d\psi'(h)}{\sqrt{1 + |\psi'(h)|^2}}) \sin \frac{2(j - 1)\pi}{k}, 0, \cdots, 0, \psi(h) + \frac{d}{\sqrt{1 + |\psi'(h)|^2}}).
\]

We have
\[
|x_j - x_1|^2 = \left(h - \frac{d\psi'(h)}{\sqrt{1 + |\psi'(h)|^2}}\right)^2 (2 - 2 \cos \frac{2(j - 1)\pi}{k}) = \left(h - \frac{d\psi'(h)}{\sqrt{1 + |\psi'(h)|^2}}\right)^2 \sin^2 \frac{(j - 1)\pi}{k}.
\]

For the Green function \(G(y, x_1)\),
\[
G(y, x_1) = \frac{c_0}{|y - x_1|^{N-2}} - H(y, x_1),
\]
where \(c_0 > 0\) is a constant, and \(H(y, x_1)\) is the regular part.

Let \(x_1^*\) be the reflection of \(x_1\) with respect to \(\partial \Omega\). Then
\[
x_1^* = (h, 0, \cdots, 0, \psi(h)) - d\nu,
\]
and
\[
H(y, x_1) = \frac{c_0}{|y - x_1^*|^{N-2}} (1 + O(d)).
\]

So, we obtain
\[ H(x_1, x_1) = \frac{c_0}{2^{N-2}d^{N-2}} (1 + O(d)). \]

On the other hand,

\[
x_j - x_1^* = ((h - \frac{d\psi'(h)}{\sqrt{1 + |\psi'(h)|^2}})(1 - \cos \frac{2(j - 1)\pi}{k}) + \frac{2d}{\sqrt{1 + |\psi'(h)|^2}})(h - \frac{d\psi'(h)}{\sqrt{1 + |\psi'(h)|^2}}) \sin \frac{2(j - 1)\pi}{k}, 0, \ldots, 0, \frac{2d}{\sqrt{1 + |\psi'(h)|^2}}).
\]

So

\[
| x_j - x_1^* |^2 = (h - \frac{d\psi'(h)}{\sqrt{1 + |\psi'(h)|^2}})^2 (2 - 2 \cos \frac{2(j - 1)\pi}{k}) + 2(h - \frac{d\psi'(h)}{\sqrt{1 + |\psi'(h)|^2}})(1 - \cos \frac{2(j - 1)\pi}{k}) \frac{2d}{\sqrt{1 + |\psi'(h)|^2}} + \frac{8d^2}{1 + |\psi'(h)|^2} = |x_j - x_1|^2 + 2(h - \frac{d\psi'(h)}{\sqrt{1 + |\psi'(h)|^2}}) \sin \frac{(j - 1)\pi}{k}) \frac{d}{\sqrt{1 + |\psi'(h)|^2}} + \frac{8d^2}{1 + |\psi'(h)|^2}.
\]

(A.18)

Since \( dk \to c > 0 \) and

\[
0 < c' \leq \frac{\sin \frac{(j-1)\pi}{k}}{\frac{(j-1)\pi}{k}} \leq c'', \quad j = 2, \ldots, \lceil \frac{k}{2} \rceil,
\]

we can deduce

\[
a_0 \frac{j^2}{j^2} \leq \frac{1}{|x_j - x_1|^2} \left( 2(h - \frac{d\psi'(h)}{\sqrt{1 + |\psi'(h)|^2}}) \sin \frac{(j - 1)\pi}{k}) \frac{d}{\sqrt{1 + |\psi'(h)|^2}} + \frac{8d^2}{1 + |\psi'(h)|^2} \right) \leq \frac{a_1}{j^2}
\]

for some constant \( a_1 \geq a_0 > 0 \). From

\[
G(x_j, x_1) = \frac{c_0}{|x_j - x_1|^{N-2}} (1 - \frac{c_0 |x_j - x_1|^{N-2}}{|x_j - x_1^*|^{N-2}} (1 + O(d))),
\]

we obtain
\[
\sum_{j=2}^{k} G(x_j, x_1) = \frac{k^{N-2}}{(h - \frac{\psi'(h)}{\sqrt{1+|\psi'(h)|^2}})^{N-2}} \left( B'_4 + O(d) \right) = B_4 k^{N-2} + O(k^{N-2}d),
\]

where \( B'_4 \) and \( B_4 \) are some positive constants.

So, (A.14) can be rewritten to (A.2).

From (A.17), we find

\[
\frac{\partial |x_j - x_1|}{\partial h} = O\left( \frac{dk}{j} \right), \quad \frac{\partial |x_j - x_1|}{\partial d} = O\left( \frac{|\psi'(h)|k}{j} \right),
\]

while from (A.18), we find

\[
\frac{\partial |x_j - x^*_1|}{\partial h} = O\left( d + |\psi'(h)| \frac{k}{j} \right), \quad \frac{\partial |x_j - x^*_1|}{\partial d} = O\left( d + |\psi'(h)| \frac{k}{j} \right).
\]

So, we can prove

\[
\sum_{j=2}^{k} \frac{1}{\mu^{N-2}} \frac{\partial G(x_j, x_1)}{\partial h}, \quad \sum_{j=2}^{k} \frac{1}{\mu^{N-2}} \frac{\partial G(x_j, x_1)}{\partial d} = O\left( \frac{1}{\mu^{N-2}d^{N-2}} \right).
\]

Thus, (A.15) and (A.16) are equivalent to (A.3) and (A.4) respectively.

\[\square\]

**Appendix B. Basic Estimates**

In this section, we list some lemmas, whose proof can be found in [20].

For each fixed \( i \) and \( j, i \neq j \), consider the following function

\begin{equation}
\tag{B.19}
g_{ij}(y) = \frac{1}{(1 + |y - x_j|^\alpha)(1 + |y - x_i|^\beta)},
\end{equation}

where \( \alpha \geq 1 \) and \( \beta \geq 1 \) are two constants.

**Lemma B.1.** For any constant \( 0 < \sigma \leq \min(\alpha, \beta) \), there is a constant \( C > 0 \), such that

\[
g_{ij}(y) \leq \frac{C}{|x_i - x_j|^\sigma} \left( \frac{1}{(1 + |y - x_j|)^{\alpha + \beta - \sigma}} + \frac{1}{(1 + |y - x_j|)^{\alpha + \beta - \sigma}} \right).
\]

**Lemma B.2.** For any constant \( 0 < \sigma < N - 2 \), there is a constant \( C > 0 \), such that

\[
\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} \left( \frac{1}{(1 + |z|)^{2+\sigma}} \right) dz \leq \frac{C}{(1 + |y|)^{\sigma}}.
\]
Let recall that
\[ W_{h,d,\mu}(y) = \sum_{j=1}^{k} P \mu_{x_j,\mu}. \]

**Lemma B.3.** Suppose that $N \geq 4$. Then there is a small $\theta > 0$, such that
\[
\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} W_{h,d,\mu}^{\theta}(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{N-2+\theta}} \, dz \\
\leq C \sum_{j=1}^{k} \frac{1}{(1 + \mu |y - x_j|)^{N-2+\theta}}.
\]

**Proof.** The proof can be found in [20]. We just need to use
\[ W_{h,r,\mu}(y) \leq C \sum_{i=1}^{k} \frac{\mu^{N-2}}{(1 + \mu |y - x_i|)^{N-2}}, \]
and
\[
\int_{\mathbb{R}^N} \frac{1}{|y - z|^{N-2}} W_{h,d,\mu}^{\theta}(z) \sum_{j=1}^{k} \frac{1}{(1 + |z - x_j|)^{N-2+\theta}} \, dz \\
\leq \int_{\mathbb{R}^N} \frac{1}{|\mu y - z|^{N-2}} \left( \sum_{i=1}^{k} \frac{1}{(1 + |z - \mu x_i|)^{N-2}} \right)^{\theta} \sum_{j=1}^{k} \frac{1}{(1 + |z - \mu x_j|)^{N-2+\theta}} \, dz.
\]

\[ \square \]

**Appendix C. Proof of Lemma 2.1**

In this appendix, we prove Lemma 2.1.

First, for any $x \in \Omega$, we need to find a solution $u$, satisfying
\[
-\text{div}(y_N^m Du) = \delta_x, \quad y \in \Omega.
\]

For this purpose, we take a domain $\Omega_1$, satisfying $\Omega \subset \subset \Omega_1 \subset \subset R_+^N$.

Let $u_1 = u - \frac{c_0}{x_N |y-x|^{N-2}}$, where $c_0 > 0$ is chosen in such a way that
\[ -\Delta \frac{c_0}{|y - x|^{N-2}} = \delta_x. \]
Then

$$-\Delta u_1 - \frac{m}{y_N} \frac{\partial u_1}{\partial y_N} = \frac{m}{y_N} \frac{(N - 2)(y_N - x_N)}{|y - x|^{N-1}}.$$  

So, we consider the following problem:

$$
\begin{aligned}
-\Delta u - \frac{m}{y_N} \frac{\partial u}{\partial y_N} &= \frac{a}{y_N} \frac{y_N - x_N}{|y - x|^\alpha}, & \text{in } \Omega_1; \\
\tilde{u}(y) &= \frac{1}{|y - x|^\alpha}, & \text{on } \partial \Omega_1,
\end{aligned}
$$

where $a > 0$ and $\alpha \leq N$ are some constants.

Note that for any $x \in \Omega$, $\frac{1}{|y - x|^\alpha} \leq C$ for any $y \in \partial \Omega_1$.

By the $L^p$ estimate, it is easy to see that if $\alpha < 3$, then $|u(y)| \leq C$.

Suppose now that $\alpha \geq 3$. Then the solution $u$ of (C.21) satisfies $|u| \leq \tilde{u}$, where $\tilde{u}$ is the solution of

$$
\begin{aligned}
-\Delta \tilde{u} - \frac{m}{y_N} \frac{\partial \tilde{u}}{\partial y_N} &= \frac{a}{y_N} \frac{1}{|y - x|^\alpha}, & \text{in } \Omega_1; \\
\tilde{u}(y) &= \frac{1}{|y - x|^\alpha}, & \text{on } \partial \Omega_1,
\end{aligned}
$$

Let $u_1 = \tilde{u} - \frac{a}{(\alpha - 2)(N - \alpha)} \frac{1}{|y - x|^{\alpha - 3}}$. Then

$$
\begin{aligned}
-\Delta u_1 - \frac{m}{y_N} \frac{\partial u_1}{\partial y_N} &= \frac{m(\alpha - 3)}{y_N} \frac{y_N - x_N}{|y - x|^\alpha}, & \text{in } \Omega_1; \\
u_1(y) &= \frac{1}{|y - x|^\alpha} - \frac{a}{(\alpha - 2)(N - \alpha)} \frac{1}{|y - x|^{\alpha - 3}}, & \text{on } \partial \Omega_1,
\end{aligned}
$$

By the $L^p$ estimate, it is easy to see that if $\alpha < 4$, then $|u_1(y)| \leq C$. So, we have proved that if $\alpha < 4$, the solution of (C.21) satisfies $|u(y)| \leq \frac{C}{|y - x|^{\alpha - \beta}}$. Now we can continue this procedure to prove that for any $\alpha \leq N$, the solution of (C.21) satisfies $|u(y)| \leq \frac{C}{|y - x|^{\alpha - \beta}}$.

From the above discussion, we know that (C.20) has a solution $S(y, x)$, satisfying

$$S(y, x) = \frac{a}{|y - x|^{N-2}} + O\left(\frac{1}{|y - x|^{N-3}}\right)$$

as $y \to x$.

By adding a constant to $S$, we can always assume that $S(y, x) > 0$. So, the Green function $G(y, x)$ for (C.20) satisfies

$$0 < G(y, x) < S(y, x) \leq \frac{C}{|y - x|^{N-2}},$$

and the result follows.
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