Uniqueness and nondegeneracy of sign-changing radial solutions of an almost critical problem

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We study the sign-changing radial solutions for the following semi-linear elliptic equation

\[ \Delta u - u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \]

where \(1 < p < \frac{N+2}{N-2}, \ N \geq 3\). It is well known that this equation has an unique positive radial solution. For the sign-changing radial solutions, the existence is also known. In this paper, we show that such sign-changing radial solution is also unique when \(p\) is close to \(\frac{N+2}{N-2}\). Moreover, those solutions are non-degenerate, i.e., the kernel of the linearized operator is exactly \(N\)-dimensional.

1 Introduction and main results

In this paper we establish the uniqueness and nondegeneracy of sign-changing radially symmetric solutions to the following semi-linear elliptic equation

\[ \Delta u - u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N) \] (1.1)

where \(1 < p < 2^* - 1\) and \(2^* = 2N/(N - 2)\) is the critical Sobolev exponent for the embedding of \(H^1(\mathbb{R}^N)\) into \(L^{2^*}(\mathbb{R}^N)\), \(N \geq 3\). More precisely, for any \(k \in \mathbb{N}\), we will prove that the following ODE problem

\[
\begin{cases}
  u'' + \frac{N-1}{r} u' - u + |u|^{p-1}u = 0, & r \in (0, \infty), \quad N \geq 3, \\
  u'(0) = 0, & \lim_{r \to \infty} u(r) = 0,
\end{cases}
\]

\] (1.2)
has an unique solution \( u \in C^2[0, \infty) \) such that \( u(0) > 0 \) and \( u \) has exactly \( k \) zeros. Moreover, this unique solution is non-degenerate in the space of \( H^1 \)-radial symmetric functions.

The classical work of Gidas, Ni and Nirenberg [10] implies that all the positive solutions of (1.1) are radially symmetric. The uniqueness of positive solutions to (1.1) has been extensively studied during the last thirty years. It was initiated by Coffman [4] with \( p = 3 \) and \( N = 3 \), and then improved by McLeod and Serrin [18] to \( 1 < p \leq \frac{N}{N-2} \), and finally extended by Kwong [16] to all values of exponent \( 1 < p < \frac{N+2}{N-2} \) by shooting method. After these results there have been many extensions and refinements, see for example the works [26], [3], [27] and references therein. In order to obtain their results, they studied the behaviour of solutions \( u(r, \alpha) \) to the initial value problem

\[
\begin{cases}
  u'' + \frac{N-1}{r} u' - u + u^p = 0, & r \in (0, \infty), \ N \geq 3, \\
  u(0) = \alpha, & u'(0) = 0,
\end{cases}
\]

for \( \alpha \in (0, \infty) \) and then obtained series of comparison results between two solutions to (1.3) with different initial values. One feature of their approach is that it can be extended to the \( m \)-Laplacian operator and more general nonlinearities, see [27] for example. However, it seems very hard to apply the approach to sign-changing solutions if one does not understand the complicated intersection between two solutions to 1.1 in the second nodal domain.

For sign-changing radial solutions, the existence results have been established by Coffman [5] and McLeod, Troy and Weissler [19] using shooting techniques and a scaling argument. But for the uniqueness of sign-changing solutions, to our knowledge, there is few work on it. In [6], using Coffman’s approach, Cortazar, Garcia-Huidobro and Yarur study the uniqueness of sign-changing radial solution to

\[
\Delta u + f(u) = 0, \quad \text{in } \mathbb{R}^N,
\]

under some convexity and growth conditions of \( f(u) \). In the canonical case of \( f(u) = |u|^{p-1}u - |u|^{q-1}u \), the conditions on \( p \) and \( q \) is that:

\[
p \geq 1, \ 0 < q < p, \quad \text{and} \quad p + q \leq \frac{2}{N-2}.
\]

Thus it is impossible to take \( q = 1 \) in their result and then the study of uniqueness under weaker assumptions on the function \( f(u) \) remains open. The result we present here is a contribution to this matter which covers the \( q = 1 \) case, and the method we use is the Lyapunov-Schmidt reduction method.

Up to now, the Lyapunov-Schmidt reduction method, which reduces an infinite-dimensional problem to a finite-dimensional one, has been widely used successfully in constructing various solutions, see for example [29], [22], [11], [24]. For the uniqueness problem, Wei [29] applied this method and established the uniqueness and non-degeneracy of boundary spike solutions for the following singularly perturbed Neumann boundary problem:

\[
\begin{cases}
  \epsilon^2 \Delta u - u + u^p = 0 \quad \text{in } \Omega, \\
  u > 0 \quad \text{in } \Omega \text{ and } \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]
where $\epsilon > 0$ is a small parameter, $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, and $p$ is subcritical. The main idea is to reduce the problem in $H^2(\Omega)$ into a finite-dimensional problem on the space of spikes and then compute the number of critical points for a finite-dimensional problem. The same idea has been used successfully by Grossi [13] in computing the number of single-peak solutions of the nonlinear Schrödinger equation

$$
\begin{align*}
-\epsilon^2 \Delta u + V(x)u &= u^p \quad \text{in } \mathbb{R}^N; \\
u &= 0,
\end{align*}
$$

for a suitable class of potentials $V$ and critical point $P$. But for the uniqueness problem, we don’t know whether the Lyapunov-Schmidt reduction method can be used to problems other than the singularly perturbed one. In this direction, Dancer and Wei [7] developed a new idea to deal with the similar problem for the following equation

$$
\Delta u + |u|^{p-1}u = 0 \quad \text{in } \Omega = \mathcal{D} - B_\delta(P)
$$

where $\delta$ is a small constant, $\mathcal{D}$ is a smooth bounded domain and $p > \frac{N+2}{N-2}, N \geq 3$.

The purpose of this paper is to deal with the uniqueness of sign-changing radial solutions to (1.1) by the Liapunov-Schmidt reduction. After setting $p = \frac{N+2}{N-2} - \epsilon$, then problem (1.1) will become a singularly perturbed one and then we can use the idea in [29] and [7] to establish the uniqueness and non-degeneracy of sign-changing solution to (1.1) for sufficient small $\epsilon > 0$.

Our first result concerns the uniqueness of sign-changing radial solution:

**Theorem 1.1.** For any positive integer $k$, there exists a positive constant $\epsilon_0$ such that for $p \in \left(\frac{N+2}{N-2} - \epsilon_0, \frac{N+2}{N-2}\right)$, there exists an unique sign-changing radial solution to (1.1) with $u(0) > 0$ and exactly $k$ zeros.

**Remark.** Using the same idea, we can give a new proof on the uniqueness of positive solution to the equation (1.1) with an almost critical power.

Our second result concerns the eigenvalue estimates associated with the linearized operator at $u_\epsilon$, the solutions obtained in Theorem 1.1:

$$
\overline{\mathcal{L}}_\epsilon \equiv \Delta - 1 + p|u_\epsilon|^{p-1}.
$$

We have the following non-degeneracy result:

**Theorem 1.2.** There exists a number $\epsilon_0 > 0$ such that for $p \in \left(\frac{N+2}{N-2} - \epsilon_0, \frac{N+2}{N-2}\right)$, $u_\epsilon$ is non-degenerate, i.e.,

$$
\ker\{\mathcal{L}_\epsilon\} = \text{span}\left\{\frac{\partial u_\epsilon}{\partial x_1}, \ldots, \frac{\partial u_\epsilon}{\partial x_N}\right\}.
$$

The organization of the paper is as follows. In Section 2 we give some preliminary analysis. In Section 3 a finite dimensional reduction procedure is given. In Section 4 we show the existence and uniqueness. Finally in Section 5 we give the small eigenvalue estimate and complete the proof of Theorem 1.2.

Throughout this paper we denote various generic constants by $C$. We use $O(B), o(B)$ to mean $|O(B)| \leq C|B|, o(B)/|B| \to 0$ as $|B| \to 0$, respectively.

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2 The asymptotic behaviour of the solutions

In this section, we will give some preliminary analysis. First by Pohozaev’s non-existence result, equation (1.1) only has trivial solution \( u = 0 \) when \( p \geq 2^* - 1 \), see [25]. So if \( u_\varepsilon \) is a sign-changing solution to (1.1) with \( p = (N + 2)/(N - 2) - \epsilon, \epsilon > 0 \), then \( u_\varepsilon \) must blow up as \( \epsilon \to 0 \). Moreover, by the result of Felmer, Quaas, Tang and Yu [8],

\[
    u_\varepsilon(0) \to \infty \quad \text{and} \quad R_\varepsilon \to 0
\]

as \( \epsilon \to 0 \), where \( R_\varepsilon \) is the first zero point of \( u_\varepsilon \). Without loss of generality, in this paper, we assume that \( u_\varepsilon(0) > 0 \) and we will consider sign-changing once radial solutions to (1.1). It is difficult to see the relation between \( u_\varepsilon(0) \) and the first radius \( R_\varepsilon \).

To overcome the above mentioned difficulty, we take the so-called Emden-Fowler transformation to \( u_\varepsilon \) as in [7]. Let

\[
    v_\varepsilon(t) = r^\alpha u_\varepsilon(r), \quad r = e^t, \quad \alpha = \frac{2}{p - 1}.
\]

Then \( v_\varepsilon \) satisfies

\[
    v'' - \beta v' - (\gamma + e^{2t})v + |v|^{p-1}v = 0, \quad t \in (-\infty, \infty),
\]

where

\[
    p = \frac{N + 2}{N - 2} - \epsilon, \quad \beta = \frac{(N - 2)^2 \epsilon}{4 - (N - 2)\epsilon} \quad \text{and} \quad \gamma = \frac{(N - 2)^2}{4} - \frac{\beta^2}{4}.
\]

Recall that the corresponding energy functional of equation (1.1) is

\[
    \tilde{E}_\varepsilon(u) = \frac{1}{2} \int_0^\infty \left( |u'|^2 + |u|^2 \right) r^{N-1} \, dr - \frac{1}{p + 1} \int_0^\infty |u|^{p+1} r^{N-1} \, dr,
\]

and by the Emden-Fowler transformation,

\[
    \int_0^\infty |u'|^2 r^{N-1} \, dr = \int_{-\infty}^\infty \left[ |v'|^2 + \gamma |v|^2 \right] e^{-\beta t} \, dt;
\]

\[
    \int_0^\infty |u|^2 r^{N-1} \, dr = \int_{-\infty}^\infty e^{2t} |v|^2 e^{-\beta t} \, dt;
\]

\[
    \int_0^\infty |u|^{p+1} r^{N-1} \, dr = \int_{-\infty}^\infty |v|^{p+1} e^{-\beta t} \, dt.
\]

Thus the corresponding energy functional of equation (2.2) is

\[
    E_\varepsilon(v) = \frac{1}{2} \int_{-\infty}^\infty \left[ |v'|^2 + (\gamma + e^{2t})|v|^2 \right] e^{-\beta t} \, dt - \frac{1}{p + 1} \int_{-\infty}^\infty |v|^{p+1} e^{-\beta t} \, dt,
\]

(2.4)
and \( u(r) \in H^1(\mathbb{R}^N) \) if and only if \( v(t) \in H \), where \( H \) is the Hilbert space defined by

\[
H \equiv \{ v \in H^1(\mathbb{R}) \mid \int_{-\infty}^{\infty} [v'(t)^2 + (\gamma + e^{2t})|v|^2] e^{-\beta t} dt < \infty \}
\]

(2.5)

with the inner product

\[
(v, w) = \int_{-\infty}^{\infty} [v'(t)w'(t) + (\gamma + e^{2t})vw(t)] e^{-\beta t} dt.
\]

(2.6)

Similarly, we define the weighted \( L^2 \)-product as follows:

\[
\langle v, w \rangle = \int_{-\infty}^{\infty} vwe^{-\beta t} dt.
\]

(2.7)

To get the asymptotic behaviour of the solutions, by standard blow-up analysis, we first have the following Lemma:

**Lemma 2.1.** Let \( v_\epsilon \) be a solution of (2.2). Then there exists a positive constant \( C = C(N) \) such that

\[
\|v_\epsilon\|_{\infty} \leq C.
\]

(2.8)

Since the uniqueness of positive solutions is known for \( u \) in ball and annulus, so is it for \( v \) and we have the following a priori estimate of energy of \( v_\epsilon \):

**Lemma 2.2.** Let \( v_\epsilon \) be a solution of (2.2). Then there exists a small positive constant \( \delta \) such that

\[
E_\epsilon(v_\epsilon) < 2E_\epsilon(w_0) + \delta < 3E_\epsilon(w_0),
\]

(2.9)

where \( w_0 \) is the unique positive solution of the following problem

\[
\begin{aligned}
& w'' - \frac{(N-2)^2}{4} w + w^{2^*-1} = 0, w > 0 \quad \text{in } \mathbb{R}; \\
& w(0) = \max_{t \in \mathbb{R}} w(t), \quad w(t) \to 0, \quad \text{as } |t| \to \infty.
\end{aligned}
\]

(2.10)

Using the above a priori estimate of energy, we can follow the argument of [21] to prove the following asymptotic behavior of \( v_\epsilon \):

**Lemma 2.3.** Suppose \( v_\epsilon \) is a sign-changing once solution of (2.2), then \( v_\epsilon \) has exactly one local maximum point \( t_1 \) and one local minimum point \( t_2 \) in \( (-\infty, \infty) \), provided that \( \epsilon \) is sufficiently small. Moreover,

\[
v_\epsilon(t) = w_0(t - t_1) - w_0(t - t_2) + o(1)
\]

(2.11)

and

\[
t_1 < t_2, \quad t_1 \to -\infty, \quad t_2 \to -\infty, \quad |t_2 - t_1| \to \infty, \quad \text{as } \epsilon \to 0
\]

(2.12)

where \( w_0 \) is the unique positive solution to equation (2.10) and \( o(1) \to 0 \) as \( \epsilon \to 0 \).
Proof. First we show that the local maximum point must goes to $-\infty$ as $\epsilon \to 0$. Suppose not, there exists a sequence of local maximum points $t_{\epsilon}$ of $v_{\epsilon}$ such that $t_{\epsilon} \to t_0$. By the estimate of energy of $v_{\epsilon}$ we get $v_{\epsilon}(t + t_{\epsilon}) \to v_0$ in $C_{\text{loc}}^{2}$, where $v_0$ satisfies

$$v'' - (\gamma_0 + e^{2t})v + v^{2^* - 1} = 0, \quad v \geq 0 \text{ in } \mathbb{R},$$

$$v(t) \to 0, \text{ as } |t| \to \infty,$$

where $\gamma_0 = \frac{(N-2)^2}{4}$. But by Pohozaev’s identity, $v_0 \equiv 0$. This contradicts with $v_0(0) \geq \gamma_0^{1/(2^* - 1)} > 0$.

Next we show that the distance of local maximum point and zero point of $v_{\epsilon}$ goes to $\infty$. Suppose not, using the same notation above, there exists $d \in \mathbb{R}$ such that, $v_{\epsilon}(t + t_{\epsilon}) \to v_0$ in $C_{\text{loc}}^{2}((-\infty, d))$, where $v_0$ satisfies

$$v'' - (\gamma_0 + e^{2t})v + v^{2^* - 1} = 0, \quad v \geq 0 \text{ in } (-\infty, d),$$

$$v(d) = 0, \quad v(t) \to 0, \text{ as } |t| \to \infty.$$

This is also a contradiction to the Pohozaev’s identity.

Now we show that there only exists one local maximum point. Suppose not, there are at least are two local maximum points $t_1$ and $t_2$. We first show that $|t_1 - t_2| \to \infty$. Suppose not, $|t_1 - t_2|$ is bounded, then using the same notations, $v_{\epsilon}(t + t_{\epsilon}) \to v_0$ in $C_{\text{loc}}^{2}(\mathbb{R})$, where $v_0$ satisfies (2.13). Moreover since $v'_{\epsilon}(0) = 0$, $v_0'(0) = 0$ and then applying Lemma 4.2 in [21] and the argument right after the proof of Lemma 4.2, we get a contradiction. Thus $|t_1 - t_2| \to \infty$. Then we have a lower bound of the energy functional $E_{\epsilon}(v_{\epsilon}) > 2E_{\epsilon}(w_0) + C_1 > 2E_{\epsilon}(w_0) + \delta$ for some $C_1 > 0$ independent of $\epsilon$ small, which contradicts with Lemma 2.2.

For the negative part, we can get the similar result and complete the proof. \hfill \Box

Now we set

$$S_{\epsilon}[v] = v'' - \beta v' - (\gamma + e^{2t})v + |v|^{p-1}v.$$  \hspace{1cm} (2.15)

To get more accurate information on asymptotic behaviour, we introduce the function $w$ be the unique positive solution of

$$\begin{cases}
    w'' - \frac{(N-2)^2}{4}w + w^p = 0 & \text{in } \mathbb{R}; \\
    w(0) = \max_{t \in \mathbb{R}} w(t), \quad w(t) \to 0, \text{ as } |t| \to \infty.
\end{cases}$$  \hspace{1cm} (2.16)

It is standard that

$$\begin{cases}
    w(t) = A_{\epsilon,N}e^{-(N-2)t/2} + O(e^{-p(N-2)t/2}), \quad t \geq 0; \\
    w'(t) = -\frac{N-2}{2}A_{\epsilon,N}e^{-(N-2)t/2} + O(e^{-p(N-2)t/2}), \quad t \geq 0,
\end{cases}$$  \hspace{1cm} (2.17)

where $A_{\epsilon,N} > 0$ is a constant depending only on $\epsilon$ and $N$. Actually the function $w(t)$ can be written explicitly and has the following form

$$w(t) = \gamma_0^{\frac{1}{p-1}} \left( \frac{p+1}{2} \right)^{\frac{1}{p-1}} \left[ \cosh \left( \frac{p-1}{2} \gamma_0^{1/2} t \right) \right]^{-\frac{2}{p-1}}.$$
Testing (2.17) with $w$ and $w'$ and integrating by parts, one arrives at the following identity:

$$\int_{\mathbb{R}} |w'|^2 \, dt = \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_{\mathbb{R}} w^{p+1} \, dt = \gamma_0 \frac{p-1}{p+3} \int_{\mathbb{R}} w^2 \, dt. \quad (2.18)$$

Note that $w \notin H$ when $N = 3, 4$. For $t_1, t_2$ obtained in Lemma 2.3, we set $w_{j,t_j}$ to be the unique solution of

$$v'' - (\gamma_0 + e^{2s})v + w_{p} = 0, \quad \text{where } w_{t_j}(s) = w(s - t_j), \ j = 1, 2 \quad (2.19)$$

in the Hilbert space $H$. The existence and uniqueness of $w_{j,t_j}$ are derived from the Riesz’s representation theorem.

Using the ODE analysis, we can obtain the following asymptotic expansion of $w_{j,t_j}$ for whose proof we postpone to Appendix A:

**Lemma 2.4.** For $\epsilon$ sufficiently small,

$$w_{j,t_j} = w_{t_j} + \phi_{j,t_j} + O(e^{2t_j}), \quad (2.20)$$

where for $N = 3$,

$$\phi_{j,t_j}(s) = -e^{t_j^2A_{\epsilon,3}}e^{-s^2/2}(1 - e^{-e^s}); \quad (2.21)$$

for $N = 4$,

$$\phi_{j,t_j}(s) = -e^{t_j}A_{\epsilon,4}e^{-s}[1 - \rho_0(1 - e^{2s})], \quad (2.22)$$

and

$$\rho_0(r) = 2\sqrt{r}K_1(2\sqrt{r}),$$

where $K_k(z)$ is the modified Bessel function of second kind and satisfies

$$z^2K_k''(z) + zK_k'(z) - (z^2 + 1)K_k(z) = 0;$$

for $N = 5$,

$$\phi_{j,t_j}(s) = -e^{3t_j/2}A_{\epsilon,5}e^{-3s/2}[1 - (1 + e^s)e^{-e^s}]; \quad (2.23)$$

for $N = 6$,

$$\phi_{j,t_j} = -e^{2t_j}A_{\epsilon,6}e^{-2s}[1 - u_0(1 - e^{16s})],$$

where

$$u_0(r) = 8\sqrt{r}K_2(4r^{1/4}),$$

where $K_k(z)$ is the modified Bessel function of second kind and satisfies

$$z^2K_k''(z) + zK_k'(z) - (z^2 + 4)K_k(z) = 0;$$

for $N > 6, \phi_{j,t_j} = 0.$
Remark. By the maximum principle, we have the following useful estimates:

\[ 0 < w_{j,t} < w_t, \quad -w_t < \phi_{j,t} < 0, \quad (2.24) \]

and

\[ \left| w'_{j,t} \right| \leq c_1 w_t \leq c_2, \quad \left| w'_{j,t} \right| \leq c_1 w_{j,t} \leq c_2, \quad (2.25) \]

where \( c_1, c_2 \) are two positive constants independent of \( \epsilon \) small.

From the above Lemma 2.4 and (2.12), we see that

\[ w_{j,t} = w_t + o(1) = w_{0,t} + o(1) \]

in all the cases for \( j = 1, 2 \). Thus by (2.11),

\[ v_{\epsilon}(t) = w_{\epsilon,t} + o(1), \quad (2.26) \]

where

\[ w_{\epsilon,t}(t) = w_{1,t_1}(t) - w_{2,t_2}(t). \quad (2.27) \]

Before studying the properties of \( w_{\epsilon,t} \), we need some preliminary Lemmas. The first one is a useful inequality:

**Lemma 2.5.** For \( x \geq 0, y \geq 0, \)

\[ |x^p - y^p| \leq \begin{cases} |x - y|^p, & \text{if } 0 < p < 1, \\ p|x - y|(x^{p-1} + y^{p-1}), & \text{if } 1 \leq p < \infty. \end{cases} \quad (2.28) \]

The following Lemma is proved in Proposition 1.2 of [2].

**Lemma 2.6.** Let \( f \in C(R) \cap L^\infty(R), g \in C(R) \) be even and satisfy for some \( \alpha \geq 0, \beta \geq 0, \gamma_0 \in R, \)

\[ f(x)\exp(\alpha |x|)|x|^\beta \rightarrow \gamma_0 \quad \text{as } |x| \rightarrow \infty, \]

\[ \int_R |g(x)|\exp(\alpha |x|)(1 + |x|^\beta)dx < \infty. \]

Then

\[ \exp(\alpha |y|)|y|^\beta \int_R g(x + y)f(x)dx \rightarrow \gamma_0 \int_R g(x)\exp(-\alpha x_1)dx \quad \text{as } |y| \rightarrow \infty. \]

Next we state a useful Lemma about the interactions of two \( w \)'s:

**Lemma 2.7.** For \( |r - s| \gg 1 \) and \( \eta > \theta > 0 \), there hold

\[ w^n(t - r)w^\theta(t - s) = O(w^\theta(|r - s|)); \quad (2.29) \]

\[ \int_{-\infty}^{\infty} w^n(t - r)w^\theta(t - s) \, dt = (1 + o(1))w^\theta(|r - s|) \int_{-\infty}^{\infty} w^n(t)e^{\theta \sqrt{\gamma_0}t} \, dt, \quad (2.30) \]

where \( o(1) \rightarrow 0 \) as \( |t - s| \rightarrow \infty. \)
Proof. The conclusion follows from (2.17) and the Lebesgue’s Dominated Convergence Theorem, see for example [17].

Now we have the following error estimates:

**Lemma 2.8.** For $\epsilon$ sufficiently small and $t_1, t_2$ satisfy (2.12), there is a constant $C$ independent of $\epsilon, t_1$ and $t_2$ such that

\[
\|S_\epsilon[w_{\epsilon,t}]\|_\infty + \int_{-\infty}^{\infty} |S_\epsilon[w_{\epsilon,t}]| e^{-\beta t} dt \leq C[\beta + e^{\tau t_2} + e^{-\tau|t_1-t_2|/2}] \quad \text{for } N = 3;
\]

\[
\|S_\epsilon[w_{\epsilon,t}]\|_\infty + \int_{-\infty}^{\infty} |S_\epsilon[w_{\epsilon,t}]| e^{-\beta t} dt \leq C[\beta + t_2^2 e^{2t_2} + e^{-\tau|t_1-t_2|}] \quad \text{for } N = 4;
\]

\[
\|S_\epsilon[w_{\epsilon,t}]\|_\infty + \int_{-\infty}^{\infty} |S_\epsilon[w_{\epsilon,t}]| e^{-\beta t} dt \leq C[\beta + e^{2t_2} + e^{-\gamma(N-2)|t_1-t_2|/2}] \quad \text{for } N \geq 5,
\]

where $\tau$ is a constant satisfying $\frac{1}{2} < \tau < \frac{\min(p,2)}{2}$.

**Proof.** By the equation satisfied by $w_{j,t}$, we have

\[
S_\epsilon[w_{\epsilon,t}] = -\beta w_{\epsilon,t}' - (\gamma - \gamma_0)w_{\epsilon,t} + |w_{\epsilon,t}|^{p-1}w_{\epsilon,t} - w_{t_1}^p + w_{t_2}^p. \tag{2.31}
\]

From the exponential decay of $w_j$, (2.24), (2.25) and $\gamma - \gamma_0 = -\frac{\beta^2}{\tau}$, we deduce that

\[
|\beta w_{\epsilon,t}'| \leq C\beta(w_{t_1} + w_{t_2}),
\]

and

\[
|\gamma - \gamma_0|w_{\epsilon,t}| \leq C\beta^2(w_{t_1} + w_{t_2}).
\]

Next, we divide $(-\infty, \infty)$ into 2 intervals $I_1, I_2$ defined by

\[
I_1 = (-\infty, \frac{t_1 + t_2}{2}) \quad \text{and} \quad I_2 = \left[\frac{t_1 + t_2}{2}, \infty\right).
\]

Then on $I_i$, $i = 1, 2$, we have $w_{t_j} \leq w_{t_i}$ and then $w_{j,t_j} \leq w_{i,t_i}$ by the maximum principle for $i \neq j$. So on $I_1$ we use inequality (2.28) to get

\[
|w_{\epsilon,t}|^{p-1}w_{\epsilon,t}^p - w_{t_1}^p + w_{t_2}^p \leq C w_{t_1}^{p-1}w_{t_1} + C w_{t_2}^{p-1}w_{t_1},
\]

for any $\tau \in (0, 1)$.

Similarly on $I_2$ the following inequality holds,

\[
|w_{\epsilon,t}|^{p-1}w_{\epsilon,t}^p - w_{t_1}^p + w_{t_2}^p \leq C w_{t_2}^{p-1}w_{t_2} + C w_{t_2}^{p-1}w_{t_2},
\]

for any $\tau \in (0, 1)$.

By the above inequalities and using Lemma 2.6, the desired result follows.
In order to obtain the a priori estimate of $t_1, t_2$ and compute the energy expansion $E_{\epsilon}[w_{\epsilon,t}]$, we need to estimate

$$\|v_\epsilon - w_{\epsilon,t}\|_\infty \text{ and } \|v_\epsilon - w_{\epsilon,t}\|_H.$$ 

**Lemma 2.9.** For $\epsilon$ sufficiently small, there is a constant $C$ independent of $\epsilon$ such that

$$v_\epsilon = w_{\epsilon,t} + \phi_\epsilon,$$  

(2.32)

where

$$\left\{ \begin{array}{l l}
\|\phi_\epsilon\|_\infty + \|\phi_\epsilon\|_H \leq C[\beta + e^{\tau t_2} + e^{-\tau|t_1-t_2|/2}] & \text{ for } N = 3; \\
\|\phi_\epsilon\|_\infty + \|\phi_\epsilon\|_H \leq C[\beta + t_2^2 e^{\tau t_2} + e^{-\tau|t_1-t_2|}] & \text{ for } N = 4; \\
\|\phi_\epsilon\|_\infty + \|\phi_\epsilon\|_H \leq C[\beta + e^{2\tau t_2} + e^{-\tau(N-2)|t_1-t_2|/2}] & \text{ for } N \geq 5,
\end{array} \right.$$  

(2.33)

where $\tau$ satisfies $\frac{1}{2} < \tau < \frac{\min\{p,2\}}{2}$.

**Proof.** We may follow the arguments given in the proof of Lemma 2.4 in [14]. First by the properties of $w_{j,t}$'s we can choose proper $t_j$'s such that the maximum points $r_j$ and minimum points $s_j$ of $v_\epsilon$ are also the ones of $w_{\epsilon,t}$, respectively. Let $v_\epsilon = w_{\epsilon,t} + \phi_\epsilon$, then $\phi_\epsilon \to 0$ and satisfies

$$\phi'' - \beta \phi' - (\gamma + e^{2t})\phi + p|w_{\epsilon,t}|^{p-1}\phi + S_\epsilon[w_{\epsilon,t}] + N_\epsilon[\phi] = 0,$$

where

$$N_\epsilon[\phi] = |w_{\epsilon,t} + \phi|^{p-1}(w_{\epsilon,t} + \phi) - |w_{\epsilon,t}|^{p-1}w_{\epsilon,t} - p|w_{\epsilon,t}|^{p-1}\phi.$$

Now we prove the estimates for $\phi_\epsilon$ by contradiction. Denote the right hand side order term of (2.33) by $K_\epsilon$ and suppose that

$$\|\phi_\epsilon\|_\infty/K_\epsilon \to \infty.$$

Let $\tilde{\phi}_\epsilon = \phi_\epsilon/\|\phi_\epsilon\|_\infty$, then $\tilde{\phi}_\epsilon$ satisfies

$$\tilde{\phi}'' - \beta \tilde{\phi}' - (\gamma + e^{2t})\tilde{\phi} + p|w_{\epsilon,t}|^{p-1}\tilde{\phi} + \frac{S_\epsilon[w_{\epsilon,t}]}{\|\phi_\epsilon\|_\infty} + \frac{N_\epsilon[\phi_\epsilon]}{\|\phi_\epsilon\|_\infty} = 0.$$  

(2.34)

Note that

$$\left| \frac{S_\epsilon[w_{\epsilon,t}]}{\|\phi_\epsilon\|_\infty} \right| \leq CK_\epsilon/\|\phi_\epsilon\|_\infty, \quad \left| \frac{N_\epsilon[\phi_\epsilon]}{\|\phi_\epsilon\|_\infty} \right| \leq C\|\phi_\epsilon\|_\infty^{\min\{p-1,1\}}.$$  

(2.35)

Let $t_\epsilon$ be such that $\tilde{\phi}_\epsilon(t_\epsilon) = \|\tilde{\phi}_\epsilon\|_\infty = 1$ (the same proof applies if $\tilde{\phi}_\epsilon(t_\epsilon) = -1$). Then by (2.34), (2.35) and the Maximum Principle, we have $|t_\epsilon - t_1| \leq C$ or $|t_\epsilon - t_2| \leq C$. Thus $|t_\epsilon - r_\epsilon| \leq C$ or $|t_\epsilon - s_\epsilon| \leq C$. Without loss of generality, we assume that $|t_\epsilon - r_\epsilon| \leq C$. Then
by the usual elliptic regular theory, we may take a subsequence \( \tilde{\phi}_\epsilon(t + r_\epsilon) \to \tilde{\phi}_0(t) \) as \( \epsilon \to 0 \) in \( C^1_{\text{loc}}(\mathbb{R}) \) since \( |r_\epsilon - t_1| \to 0 \), where \( \phi_0 \) satisfies
\[
\tilde{\phi}_0'' - \frac{(N - 2)^2}{4} \tilde{\phi}_0 + \frac{N + 2}{N - 2} \tilde{\phi}_0^4/(N - 2) = 0, \quad \text{and} \quad \tilde{\phi}_0'(0) = 0,
\]
which implies \( \tilde{\phi}_0 \equiv 0 \). This contradicts to the fact that \( 1 = \tilde{\phi}_\epsilon(t_\epsilon) \to \tilde{\phi}_0(t_0) \) for some \( t_0 \). Therefore we complete the proof.

The following is the basic technical estimate in this paper which gives the a priori estimates for \( t_1 \) and \( t_2 \):

**Lemma 2.10.** For \( \epsilon \) sufficient small we have for \( N = 3 \),
\[
\begin{align*}
 t_1 &= \log a + 2 \log b + 3 \log \beta; \\
 t_2 &= \log a + \log \beta,
\end{align*}
\]
where \( a, b \) are constants and
\[
a \to a_{0,3}, \quad b \to b_{0,3}.
\]
Here \( a_{0,3}, b_{0,3} \) are positive constants.

For \( N = 4 \),
\[
\begin{align*}
 t_1 - t_2 &= \log b + \log \beta; \\
 -2t_2 e^{2t_2} &= a\beta,
\end{align*}
\]
where \( a, b \) are constants and
\[
a \to a_{0,4}, \quad b \to b_{0,4}.
\]
Here \( a_{0,4}, b_{0,4} \) are positive constants.

For \( N \geq 5 \),
\[
\begin{align*}
 t_1 &= \frac{1}{2} \log a + \frac{2}{N-2} \log b + \frac{N+2}{2(N-2)} \log \beta; \\
 t_2 &= \frac{1}{2} \log a + \frac{1}{2} \log \beta,
\end{align*}
\]
where \( a, b \) are constants and
\[
a \to a_{0,N}, \quad b \to b_{0,N}.
\]
Here \( a_{0,N}, b_{0,N} \) are positive constants.

**Proof.** Here we only give the proof for \( N = 3 \), for \( N > 3 \), the proof is similar. From \( S_\epsilon[v_\epsilon] = 0 \) and \( v_\epsilon = w_{\epsilon,t} + \phi_\epsilon \) we deduce
\[
L_{\epsilon,t}[-\phi] + S_\epsilon[w_{\epsilon,t}] + N_\epsilon[\phi] = 0,
\]
(2.39)
Similarly we can obtain
\[ L_{\epsilon,t}[\phi] = \phi'' - \beta \phi' - (\gamma + e^{2t})\phi + p |w_{\epsilon,t}|^{p-1}\phi, \]
and
\[ N_{\epsilon}[\phi] = |w_{\epsilon,t} + \phi|^{p-1}(w_{\epsilon,t} + \phi) - |w_{\epsilon,t}|^{p-1}w_{\epsilon,t} - p |w_{\epsilon,t}|^{p-1}\phi. \]

Multiplying (2.39) by \( w'_{1,t} \) and integrating over \( \mathbb{R} \), we obtain
\[ \int_{-\infty}^{\infty} L_{\epsilon,t}[\phi] w'_{1,t} dt + \int_{-\infty}^{\infty} S_{\epsilon}[w_{\epsilon,t}] w'_{1,t} dt + \int_{-\infty}^{\infty} N_{\epsilon}[\phi] w'_{1,t} dt = 0. \]

Integrating by parts and using Lemma 2.9 we have
\[
\int_{-\infty}^{\infty} L_{\epsilon,t}[\phi] w'_{1,t} dt = \int_{-\infty}^{\infty} \left[ w''_{1,t} + \beta w''_{1,t} - (\gamma + e^{2t}) w'_{1,t} + p |w_{\epsilon,t}|^{p-1} w'_{1,t} \right] \phi dt \\
= \int_{-\infty}^{\infty} \beta \left[ (\gamma_0 + e^{2t}) w'_{1,t} - w''_{1,t} \right] \phi dt - (\gamma - \gamma_0) \int_{-\infty}^{\infty} w'_{1,t} \phi dt \\
+ 2 \int_{-\infty}^{\infty} e^{2t} w_{1,t} \phi dt + p \int_{-\infty}^{\infty} \left[ |w_{\epsilon,t}|^{p-1} w'_{1,t} - w''_{1,t} \right] \phi dt \\
= o(\left[ \beta + e^{t_1} + e^{t_2} + e^{-|t_1-t_2|/2} \right]).
\]

Similarly we can obtain
\[ \int_{-\infty}^{\infty} L_{\epsilon,t}[\phi] w'_{2,t} dt = o(\left[ \beta + e^{t_1} + e^{t_2} + e^{-|t_1-t_2|/2} \right]). \]

For the nonlinearity term using (2.28) we get
\[
\left| N_{\epsilon}[\phi] \right| = \left| w_{\epsilon,t} + \phi |^{p-1}(w_{\epsilon,t} + \phi) - |w_{\epsilon,t}|^{p-1}w_{\epsilon,t} - p |w_{\epsilon,t}|^{p-1}\phi \right| \leq C|\phi|^{\min\{p,2\}}.
\]

So using the exponential decay of \( w \) and taking \( \tau > \max\{\frac{1}{2}, \frac{1}{p}\} \) we deduce
\[ \int_{-\infty}^{\infty} N_{\epsilon}[\phi] w'_{1,t} dt = O(\|\phi\|_{L_{\infty}^2}^2) = o(\left[ \beta + e^{t_1} + e^{t_2} + e^{-|t_1-t_2|/2} \right]). \]

Similarly we can obtain
\[ \int_{-\infty}^{\infty} N_{\epsilon}[\phi] w'_{2,t} dt = o(\left[ \beta + e^{t_1} + e^{t_2} + e^{-|t_1-t_2|/2} \right]). \]

To estimate \( \int_{-\infty}^{\infty} S_{\epsilon}[w_{\epsilon,t}] w'_{1,t} dt \), we write
\[
\int_{-\infty}^{\infty} S_{\epsilon}[w_{\epsilon,t}] w'_{1,t} dt = \int_{-\infty}^{\infty} \left[ - \beta w'_{\epsilon,t} - (\gamma - \gamma_0) w_{\epsilon,t} + |w_{\epsilon,t}|^{p-1} w_{\epsilon,t} - w''_{1,t} + w''_{2,t} \right] w'_{1,t} dt \\
= E_1 + E_2 + E_3,
\]
where
\begin{align*}
E_1 &= -\beta \int_{-\infty}^{\infty} w_{e,t}' w_{1,t_1} dt; \\
E_2 &= -(\gamma - \gamma_0) \int_{-\infty}^{\infty} w_{e,t} w_{1,t_1}' dt; \\
E_3 &= \int_{-\infty}^{\infty} \left(|w_{e,t}|^{p-1} w_{e,t} - w_{t_1}^p + w_{t_2}^p\right) w_{1,t_1}' dt.
\end{align*}

Using (2.25) and Lemma 2.7 we obtain
\begin{align*}
E_1 &= -\beta \int_{-\infty}^{\infty} |w_{1,t_1}'|^2 dt + \beta \int_{-\infty}^{\infty} w_{1,t_1} w_{2,t_2}' dt = -\beta \int_{-\infty}^{\infty} |w'|^2 dt + o(\beta).
\end{align*}

Note that \( \gamma - \gamma_0 = -\beta^2/4 \) and using (2.25) we get
\begin{align*}
E_2 &= \frac{\beta^2}{4} \int_{-\infty}^{\infty} w_{e,t} w_{1,t_1}' dt = O(\beta^2).
\end{align*}

To estimate \( E_3 \), following the argument in the proof of Lemma 2.8. We divide \((-\infty, \infty)\) into two intervals \( I_1, I_2 \) defined by
\begin{align*}
I_1 &= (-\infty, \frac{t_1 + t_2}{2}), \quad I_2 = [\frac{t_1 + t_2}{2}, \infty).
\end{align*}

On \( I_1 \) the following equality holds:
\begin{align*}
|w_{e,t}|^{p-1} w_{e,t} - w_{t_1}^p + w_{t_2}^p &= \left[(w_{1,t_1} - w_{2,t_2})^p - w_{t_1}^p + pw_{1,t_1}^{p-1} w_{2,t_2}ight] - pw_{1,t_1}^{p-1} w_{2,t_2} \\
&\quad + \left[(w_{t_1} + \phi_{1,t_1})^p - w_{t_1}^p - pw_{t_1}^{p-1} \phi_{1,t_1}\right] + pw_{t_1}^{p-1} \phi_{1,t_1}.
\end{align*}

We use inequality (2.28) to get
\begin{align*}
|w_{1,t_1} - w_{2,t_2})^p - w_{t_1}^p + pw_{1,t_1}^{p-1} w_{2,t_2}| &\leq C w_{t_1}^{p-\delta} w_{t_2}^\delta, \\
|w_{t_1} + \phi_{1,t_1})^p - w_{t_1}^p - pw_{t_1}^{p-1} \phi_{1,t_1}| &\leq C w_{t_1}^{p-\delta} \phi_{1,t_1}^\delta,
\end{align*}
for any \( 1 < \delta < 2 \).

Then using Lemma 2.7 and integrating by parts, we get
\begin{align*}
\int_{I_1} \left| |w_{e,t}|^{p-1} w_{e,t} - w_{t_1}^p + w_{t_2}^p\right| w_{1,t_1}' dt \\
= \int_{-\infty}^{\infty} w_{t_1}^p w_{t_2}' dt - \int_{-\infty}^{\infty} w_{t_1}^p \phi_{1,t_1}' dt + o(e^{-|t_1-t_2|/2}) + o(e^{t_1}).
\end{align*}

On the other hand, on \( I_2 \), using \( w_{1,t_1} \leq w_{2,t_2} \), (2.25) and inequality (2.28) we get
\begin{align*}
\left| |w_{e,t}|^{p-1} w_{e,t} - w_{t_1}^p + w_{t_2}^p\right| w_{1,t_1}' &\leq C w_{t_1}^\delta w_{t_2}^{p+1-\delta} + C \phi_{2,t_2}^\delta w_{t_2}^{p+1-\delta},
\end{align*}
for any $1 < \delta < 2$.

Using Lemma 2.7 we get

\[
\int_{I_2} \left[ |w_{e,t}|^{p-1} w_{e,t} - w_{p,t_1}^p + w_{p,t_2}^p \right] w_{t_1}' dt = o(e^{-|t_1-t_2|/2}) + o(e^t).
\]

Thus

\[
E_3 = \frac{1}{2} e^{-|t_1-t_2|/2} A_{e,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt + \frac{1}{2} e^{t_1} A_{e,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt + o(e^{-|t_1-t_2|/2}) + o(e^t),
\]

and then

\[
\int_{-\infty}^{\infty} S_e [w_{e,t}] w_{1,t_1}' dt = -\beta \int_{-\infty}^{\infty} |w'|^2 dt + \frac{1}{2} e^{-|t_1-t_2|/2} A_{e,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt + o(\beta) + o(e^{-|t_1-t_2|/2}) + o(e^t) \quad (2.40)
\]

Similarly,

\[
\int_{-\infty}^{\infty} S_e [w_{e,t}] w_{2,t_2}' dt = \beta \int_{-\infty}^{\infty} |w'|^2 dt + \frac{1}{2} \left[ e^{-|t_1-t_2|/2} - e^{t_2} \right] A_{e,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt + o(\beta) + o(e^{-|t_1-t_2|/2}) + o(e^t) \quad (2.41)
\]

Combining all the estimates above, one can see that $\beta, e^{t_2}$ and $e^{-|t_1-t_2|/2}$ must have the same order.

Therefore,

\[
\begin{cases}
-\beta \int_{-\infty}^{\infty} |w'|^2 dt + \frac{1}{2} e^{-|t_1-t_2|/2} A_{e,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt = o(\beta); \\
\beta \int_{-\infty}^{\infty} |w'|^2 dt + \frac{1}{2} \left[ e^{-|t_1-t_2|/2} - e^{t_2} \right] A_{e,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt = o(\beta).
\end{cases}
\]

Let

\[
e^{t_2} = a\beta, \quad e^{-|t_1-t_2|/2} = b\beta,
\]

then

\[
a = \frac{4 \int_{-\infty}^{\infty} |w'|^2 dt}{A_{e,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt} + o(1) \to a_{0,3},
\]

and

\[
b = \frac{2 \int_{-\infty}^{\infty} |w'|^2 dt}{A_{e,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt} + o(1) \to b_{0,3},
\]

where $a_{0,3}, b_{0,3}$ are positive constants.
Thus
\[
\begin{aligned}
  t_1 &= \log a + 2 \log b + 3 \log \beta; \\
  t_2 &= \log a + \log \beta,
\end{aligned}
\]
where
\[
a \to a_{0,3}, \quad b \to b_{0,3}.
\]

We now introduce the following configuration space:
\[
\Lambda \equiv \begin{cases} 
  \{ t = (t_1, t_2) | 1/2 a_{0,3} < t_2 < 3/2 a_{0,3}, \frac{1}{2} b_{0,3} < e^{(t_1-t_2)/2} < \frac{3}{2} b_{0,3} \} & \text{for } N = 3; \\
  \{ t = (t_1, t_2) | 1/2 a_{0,4} < -2t_2e^{2t_2} < 3/2 a_{0,4}, \frac{1}{2} b_{0,4} < e^{t_1-t_2} < \frac{3}{2} b_{0,4} \} & \text{for } N = 4; \\
  \{ t = (t_1, t_2) | 1/2 a_{0,N} < e^{2t_2} < 3/2 a_{0,N}, \frac{1}{2} b_{0,N} < e^{(N-2)(t_1-t_2)/2} < \frac{3}{2} b_{0,N} \} & \text{for } N \geq 5.
\end{cases}
\]

Then by Lemma 2.10, for $\epsilon$ sufficient small, $t = (t_1, t_2) \in \Lambda$ if $v_\epsilon$ is a sign-changing solution to equation (2.2). In the next section, we will show an one-to-one correspondence between the sign-changing solution of (1.1) and the critical points of some functional in $\Lambda$.

3 The existence result: Liapunov-Schmidt reduction

In this section we outline the main steps of the so called Liapunov-Schmidt reduction method or localized energy method, which reduces the infinite problem to finding a critical point for a functional on a finite dimensional space. A very important observation is the reduction Lemma 3.6. To achieve this, we first study the solvability of a linear problem and then apply some standard fixed point theorem for contraction mapping to solve the nonlinear problem. Since the procedure has been by now standard (see for example [17] and the references therein), we will omit most of the details.

3.1 An auxiliary linear problem

In this subsection we study a linear theory which allows us to perform the finite-dimensional reduction procedure.

First observing that orthogonality to $\frac{\partial w_\epsilon}{\partial t_j}$ in $H$, $j = 1, 2$, is equivalent to orthogonality to the following functions
\[
Z_{\epsilon,t_j} := -\left(\partial_{t_j} w_\epsilon\right)^\prime\prime + \beta (\partial_{t_j} w_\epsilon)^\prime + (\gamma + e^{2t}) \partial_{t_j} w_\epsilon, \quad j = 1, 2,
\]
in the weighted $L^2$-product $\langle \cdot , \cdot \rangle_\epsilon$. 

\[
\text{(3.1)}
\]
By (2.27) and elementary computations, we obtain for \( j = 1, 2, \)
\[
\partial_t w_{c,t} = (-1)^{j+1} \partial_t w_{j,t} = (-1)^{j+1} (\partial_t w_t + \partial_t \phi_{t,t}) + O(e^{2t}), \tag{3.2}
\]
and
\[
Z_{c,t} = (-1)^j \left[ pw'_t w_j - \beta \left( \partial_t w_{j,t} \right)' - (\gamma - \gamma_0) \partial_t wc_{jt} \right]. \tag{3.3}
\]

In this section, we consider the following linear problem:
\[
\begin{aligned}
\left\{ 
L_{c,t}[\phi] &:= \phi'' - \beta \phi' - (\gamma + e^{2t}) \phi + p|w_{c,t}|^{p-1} \phi = h + \sum_{j=1}^{2} c_j Z_{c,t}; \\
\langle \phi, Z_{c,t} \rangle_{e} & = 0, \quad j = 1, 2,
\end{aligned}
\tag{3.4}
\]

where \( t \in \Lambda. \)

For the above linear problem, we have the following result:

**Proposition 3.1.** Let \( \phi \) satisfy (3.4) with \( \|h\|_{\infty} < \infty \). Then for \( \epsilon \) sufficiently small, we have
\[
\|\phi\|_{\infty} \leq C \|h\|_{\infty}, \tag{3.5}
\]
where \( C \) is a positive constant independent of \( \epsilon \) and \( t \in \Lambda. \)

**Proof.** The proof is now standard, see for example [17].

Using Fredholm’s alternative we can show the following existence result:

**Proposition 3.2.** There exists \( \epsilon_0 > 0 \) such that for any \( \epsilon < \epsilon_0 \) the following property holds true. Given \( h \in L^\infty(\mathbb{R}) \), there exists a unique pair \((\phi, c_1, c_2)\) such that
\[
\begin{aligned}
L_{c,t}[\phi] &= h + \sum_{j=1}^{2} c_j Z_{c,t}; \\
\langle \phi, Z_{c,t} \rangle_{e} &= 0, \quad j = 1, 2.
\end{aligned}
\tag{3.6}
\]

Moreover, we have
\[
\|\phi\|_{\infty} + \sum_{j=1}^{2} |c_j| \leq C \|h\|_{\infty} \tag{3.7}
\]
for some positive constant \( C. \)

**Proof.** The result follows from Proposition 3.1 and the Fredholm’s alternative theorem, see for example [17].

In the following, if \( \phi \) is the unique solution given in Proposition 3.2, we set
\[
\phi = A_\epsilon(h). \tag{3.8}
\]

Note that (3.7) implies
\[
\|A_\epsilon(h)\|_{\infty} \leq C \|h\|_{\infty}. \tag{3.9}
\]
3.2 The nonlinear projected problem

This subsection is devoted to the solvability of the following non-linear projected problem:

\[
\begin{align*}
\begin{cases}
(w_{\epsilon,t} + \phi)'' - \beta(w_{\epsilon,t} + \phi)' - (\gamma + e^{2t})(w_{\epsilon,t} + \phi) + |w_{\epsilon,t} + \phi|^{p-1}(w_{\epsilon,t} + \phi) = & \sum_{j=1}^{2} c_j Z_{\epsilon,t,j}; \\
\langle \phi, Z_{\epsilon,t,j} \rangle_{\epsilon} = 0, \ j = 1, 2.
\end{cases}
\end{align*}
\]

The first equation in (3.10) can be written as

\[
\phi'' - \beta\phi' - (\gamma + e^{2t})\phi + p|w_{\epsilon,t}|^{p-1}\phi = -S_\epsilon[w_{\epsilon,t}] - N_\epsilon[\phi] + \sum_{j=1}^{2} c_j Z_{\epsilon,t,j},
\]

where

\[
N_\epsilon[\phi] = |w_{\epsilon,t} + \phi|^{p-1}(w_{\epsilon,t} + \phi) - |w_{\epsilon,t}|^{p-1}w_{\epsilon,t} - p|w_{\epsilon,t}|^{p-1}\phi_{\epsilon,t}.
\]

First, we have the following estimates:

**Lemma 3.3.** For \( t \in \Lambda \) and \( \epsilon \) sufficiently small, we have for \( \|\phi\|_{\infty} + \|\phi_1\|_{\infty} + \|\phi_2\|_{\infty} \leq 1; \)

\[
\|N_\epsilon[\phi]\|_{\infty} \leq C\|\phi\|_{\infty}^{\min\{p, 2\}};
\]

\[
\|N_\epsilon[\phi_1] - N_\epsilon[\phi_2]\|_{\infty} \leq C(\|\phi_1\|_{\infty}^{\min\{p-1, 1\}} + \|\phi_2\|_{\infty}^{\min\{p-1, 1\}})\|\phi_1 - \phi_2\|_{\infty}.
\]

**Proof.** These inequalities follows from the mean-value theorem and inequality (2.28).

By the standard fixed point theorem for contraction mapping and Implicit Function Theorem, Lemma 2.8 and 3.3, we have the following Proposition:

**Proposition 3.4.** For \( t \in \Lambda \) and \( \epsilon \) sufficiently small, there exists a unique \( \phi = \phi_{\epsilon,t} \) such that (3.10) holds. Moreover, \( t \to \phi_{\epsilon,t} \) is of class \( C^1 \) as a map into \( H \), and we have

\[
\|\phi_{\epsilon,t}\|_{\infty} + \sum_{j=1}^{2} |c_j| \leq \begin{cases}
C[\beta + e^{\tau t_2} + e^{-\tau|t_1-t_2|/2}] & \text{for } N = 3; \\
C[\beta + t_2 e^{2\tau t_2} + e^{-\tau|t_1-t_2|}] & \text{for } N = 4; \\
C[\beta + e^{2\tau t_2} + e^{-\tau(N-2)|t_1-t_2|/2}] & \text{for } N \geq 5,
\end{cases}
\]

where \( \tau \) satisfies \( \frac{1}{2} < \tau < \frac{\min\{p, 2\}}{2} \).
3.3 Energy expansion for reduced energy functional

In this subsection we expand the quantity

\[ K_\epsilon(t) = E_\epsilon[w_{\epsilon,t} + \phi_{\epsilon,t}] : \Lambda \to \mathbb{R} \]  

(3.15)

in terms of \( \epsilon \) and \( t \), where \( \phi_{\epsilon,t} \) is obtained in Proposition 3.4.

**Lemma 3.5.** For \( t \in \Lambda \) and \( \epsilon \) sufficiently small, we have for \( N = 3 \),

\[ K_\epsilon(t) = \left( \frac{1}{2} - \frac{1}{p+1} \right) (e^{-\beta t_1} + e^{-\beta t_2}) \int_{-\infty}^{\infty} w^{p+1} dt + \frac{1}{2} e^{2t_2} A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt \]

\[ + e^{-|t_1-t_2|/2} A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt + o(\beta) + o(e^{t_2}) + o(e^{-|t_1-t_2|/2}) \]

\[ = \tilde{K}_\epsilon(t) + o(\beta) + o(e^{t_2}) + o(e^{-|t_1-t_2|/2}). \]

For \( N = 4 \),

\[ K_\epsilon(t) = \left( \frac{1}{2} - \frac{1}{p+1} \right) (e^{-\beta t_1} + e^{-\beta t_2}) \int_{-\infty}^{\infty} w^{p+1} dt - \frac{1}{4} t_2 e^{2t_2} A_{\epsilon,4} \int_{-\infty}^{\infty} w^p e^t dt \]

\[ + e^{-|t_1-t_2|} A_{\epsilon,4} \int_{-\infty}^{\infty} w^p e^t dt + o(\beta) + o(t_2 e^{2t_2}) + o(e^{-|t_1-t_2|}) \]

\[ = \tilde{K}_\epsilon(t) + o(\beta) + o(t_2 e^{2t_2}) + o(e^{-|t_1-t_2|}). \]

For \( N \geq 5 \),

\[ K_\epsilon(t) = \left( \frac{1}{2} - \frac{1}{p+1} \right) (e^{-\beta t_1} + e^{-\beta t_2}) \int_{-\infty}^{\infty} w^{p+1} dt + \frac{1}{2} e^{2t_2} \int_{-\infty}^{\infty} w^2 e^{2t} dt \]

\[ + e^{-(N-2)|t_1-t_2|/2} A_{\epsilon,N} \int_{-\infty}^{\infty} w^p e^{(N-2)t/2} dt + o(\beta) + o(e^{2t_2}) + o(e^{-(N-2)|t_1-t_2|/2}) \]

\[ = \tilde{K}_\epsilon(t) + o(\beta) + o(e^{2t_2}) + o(e^{-(N-2)|t_1-t_2|/2}). \]

**Proof.** Here again, we only give the proof for \( N = 3 \). By the definition of \( K_\epsilon(t) \), we can re-write it as

\[ K_\epsilon(t) = E_\epsilon[w_{\epsilon,t}] + K_1 + K_2 - K_3, \]

(3.16)

where

\[ K_1 = \int_{-\infty}^{\infty} \left[ w_{\epsilon,t}' \phi_{\epsilon,t}' + (\gamma + e^{2t}) w_{\epsilon,t} \phi_{\epsilon,t} \right] e^{-\beta t} dt - \int_{-\infty}^{\infty} |w_{\epsilon,t}|^{p-1} w_{\epsilon,t} \phi_{\epsilon,t} e^{-\beta t} dt; \]

\[ K_2 = \frac{1}{2} \int_{-\infty}^{\infty} \left[ |\phi_{\epsilon,t}'|^2 + (\gamma + e^{2t}) |\phi_{\epsilon,t}|^2 - p |w_{\epsilon,t}|^{p-1} |\phi_{\epsilon,t}|^2 \right] e^{-\beta t} dt; \]

\[ K_3 = \frac{1}{p+1} \int_{-\infty}^{\infty} \left[ |w_{\epsilon,t} + \phi_{\epsilon,t}|^{p+1} - |w_{\epsilon,t}|^{p+1} - (p+1) |w_{\epsilon,t}|^{p-1} w_{\epsilon,t} \phi_{\epsilon,t} \right. \]

\[ \left. - \frac{1}{2} p(p+1) |w_{\epsilon,t}|^{p-1} |\phi_{\epsilon,t}|^2 \right] e^{-\beta t} dt. \]
Integrating by parts and using Lemma 2.8, 2.9, we have
\[ |K_1| = \left| - \int_{-\infty}^{\infty} S_\varepsilon[w_{\varepsilon,t}] \phi_{\varepsilon,t} e^{-\beta t} \, dt \right| = o\left( [\beta + \varepsilon^{2} + e^{-|t_1 - t_2|/2}] \right). \tag{3.17} \]

To estimate \( K_2 \), note that \( \phi_{\varepsilon,t} \) satisfies
\[
\begin{align*}
\phi''_{\varepsilon,t} - \beta \phi'_{\varepsilon,t} - (\gamma + e^{2t}) \phi_{\varepsilon,t} &= -|w_{\varepsilon,t} + \phi_{\varepsilon,t}|^{p-1}(w_{\varepsilon,t} + \phi_{\varepsilon,t}) + |w_{\varepsilon,t}|^{p-1}w_{\varepsilon,t} - S_\varepsilon[w_{\varepsilon,t}] + \sum_{j=1}^{2} c_j Z_{\varepsilon,t_j}.
\end{align*}
\tag{3.18}
\]

Integrating by parts and using the orthogonality condition (3.10), we have
\[
\begin{align*}
2K_2 &= \int_{-\infty}^{\infty} \left[ |w_{\varepsilon,t} + \phi_{\varepsilon,t}|^{p-1}(w_{\varepsilon,t} + \phi_{\varepsilon,t}) - |w_{\varepsilon,t}|^{p-1}w_{\varepsilon,t} - p|w_{\varepsilon,t}|^{p-1}\phi_{\varepsilon,t} + S_\varepsilon[w_{\varepsilon,t}] \right] \phi_{\varepsilon,t} e^{-\beta t} \, dt.
\end{align*}
\]

By the mean value theorem and inequality (2.28) we get
\[
\left| |w_{\varepsilon,t} + \phi_{\varepsilon,t}|^{p-1}(w_{\varepsilon,t} + \phi_{\varepsilon,t}) - |w_{\varepsilon,t}|^{p-1}w_{\varepsilon,t} - p|w_{\varepsilon,t}|^{p-1}\phi_{\varepsilon,t} \right| \leq C|\phi_{\varepsilon,t}|^{\min\{p,2\}}.
\]

So using Lemma 2.8 and 2.9 we deduce
\[
K_2 = o\left( [\beta + \varepsilon^{2} + e^{-|t_1 - t_2|/2}] \right). \tag{3.19}
\]

For \( K_3 \), using the mean value theorem and inequality (2.28),
\[
\begin{align*}
&\left| |w_{\varepsilon,t} + \phi_{\varepsilon,t}|^{p+1} - |w_{\varepsilon,t}|^{p+1} - (p + 1)|w_{\varepsilon,t}|^{p-1}w_{\varepsilon,t}\phi_{\varepsilon,t} \right| - \frac{1}{2} p(p + 1)|w_{\varepsilon,t}|^{p-1} |\phi_{\varepsilon,t}|^2 \\
&\leq C|\phi_{\varepsilon,t}|^{\min\{p+1,3\}}.
\end{align*}
\]

So, again, using Lemma 2.8 and 2.9 it follows that
\[
K_3 = o\left( [\beta + \varepsilon^{2} + e^{-|t_1 - t_2|/2}] \right). \tag{3.20}
\]

Combing with (3.16), (3.17), (3.19), (3.20), and the estimates in Appendix B, we obtain the desired estimates.

We will end this section with a reduction lemma which is important for both the existence and uniqueness:

**Lemma 3.6.** \( v_{\varepsilon,t} = w_{\varepsilon,t} + \phi_{\varepsilon,t} \) is a critical point of \( E_\varepsilon \) if and only if \( t \) is a critical point of \( K_\varepsilon \) in \( \Lambda \).
Proof. The proof follows from the proofs in [12], [29]. For the sake of completeness, we include a proof here.

By Proposition 3.4, there exists an \( \epsilon_0 \) such that, for \( 0 < \epsilon < \epsilon_0 \), we have a \( C^1 \) map \( t \to \phi_{\epsilon,t} \) from \( \Lambda \) into \( H \) such that

\[
S_\epsilon[v_{\epsilon,t}] = \sum_{j=1}^{2} c_j(t)Z_{\epsilon,t_j}, \quad v_{\epsilon,t} = w_{\epsilon,t} + \phi_{\epsilon,t},
\]

(3.21)

for some constants \( c_j \), which also are of class \( C^1 \) in \( t \).

First integrating by parts we get

\[
\partial_j K_\epsilon(t) = \int_{-\infty}^{\infty} \left[ v'_{\epsilon,t}(\partial_j w_{\epsilon,t} + \partial_j \phi_{\epsilon,t})' + (\gamma + e^{\beta t})v_{\epsilon,t}(\partial_j w_{\epsilon,t} + \partial_j \phi_{\epsilon,t}) \right] e^{-\beta t} dt
\]

\[= -\int_{-\infty}^{\infty} |v_{\epsilon,t}|^{p-1} v_{\epsilon,t}(\partial_j w_{\epsilon,t} + \partial_j \phi_{\epsilon,t}) e^{-\beta t} dt\]  

(3.22)

which means that \( t \) is a critical point of \( K_\epsilon \).

On the other hand, let \( t_* \in \Lambda \) be a critical point of \( K_\epsilon \), that is \( \partial_j K_\epsilon(t_*) = 0, j = 1, 2, \) by (3.22) we get

\[
0 = \partial_j K_\epsilon(t_*) = -\int_{-\infty}^{\infty} S_\epsilon[v_{\epsilon,t_*}](\partial_j w_{\epsilon,t_*} + \partial_j \phi_{\epsilon,t_*}) e^{-\beta t} dt = 0,
\]

for \( j = 1, 2 \). Hence by (3.21) we have

\[
\sum_{j=1}^{2} c_j(t_*) \int_{-\infty}^{\infty} Z_{\epsilon,t_j,*}(\partial_j w_{\epsilon,t_*} + \partial_j \phi_{\epsilon,t_*}) e^{-\beta t} dt = 0.
\]

By Proposition 3.4 and the fact \( \langle \phi_{\epsilon,t_*}, Z_{\epsilon,t_j,*}\rangle = 0 \),

\[
\langle Z_{\epsilon,t_j,*}, \partial_j \phi_{\epsilon,t_*}\rangle = -\langle \phi_{\epsilon,t_*}, \partial_j Z_{\epsilon,t_j,*}\rangle = o(1).
\]

(3.23)

On the other hand,

\[
\int_{-\infty}^{\infty} Z_{\epsilon,t_j,*} \partial_j w_{\epsilon,t_*} e^{-\beta t} dt = \langle Z_{\epsilon,t_j,*}, \partial_j w_{\epsilon,t_*}\rangle = \delta_{ij} p \int_{-\infty}^{\infty} w^{p-1}|w'|^2 dt + o(1).
\]

(3.24)

By (3.23) and (3.24), the matrix

\[
\int_{-\infty}^{\infty} Z_{\epsilon,t_j,*}(\partial_j w_{\epsilon,t_*} + \partial_j \phi_{\epsilon,t_*}) e^{-\beta t} dt
\]

is diagonally dominant and thus is non-singular, which implies \( c_j(t_*) = 0 \) for \( i = 1, 2 \). Hence \( v_{\epsilon,t_*} = w_{\epsilon,t_*} + \phi_{\epsilon,t_*} \) is a critical point of \( E_\epsilon \). This finishes the proof.

\[\square\]

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Remark. Note that in the proof the theorem, we assume that the solution $v_\epsilon$ of equation (2.2) can be written as $v_\epsilon = w_{\epsilon,t} + \phi_\epsilon$ with $\phi_\epsilon$ satisfying
\[ \langle \phi_\epsilon, Z_{\epsilon,t_j} \rangle_\epsilon = 0, \quad j = 1, 2. \quad (3.25) \]

In general, using (3.24) we can decompose
\[ \phi_\epsilon = \overline{\phi_\epsilon} + \sum_{j=1}^{2} d_j \partial_t w_{\epsilon,t}, \]
where $\overline{\phi_\epsilon}$ satisfies (3.25) and $d_j = O(\|\phi_\epsilon\|_\infty)$. Thus we can write
\[ v_\epsilon = w_{\epsilon,t} + \sum_{j=1}^{2} d_j \partial_t w_{\epsilon,t} + \overline{\phi_\epsilon} \]
and get the desired result using the same argument for $w_{\epsilon,t} + \sum_{j=1}^{2} d_j \partial_t w_{\epsilon,t}$.

4 The uniqueness result

By Lemma 3.6, the number of sign-changing once solutions of (2.2) equals to the number of critical points of $K_\epsilon(t)$. To count the number of critical points of $K_\epsilon(t)$, we need to compute $\partial K_\epsilon(t)$ and $\partial^2 K_\epsilon(t)$.

Recall that $K_\epsilon(t)$ and $\tilde{K}_\epsilon(t)$ are defined in (3.15) and Lemma 3.5. The crucial estimate to prove uniqueness of $v_\epsilon$ and $u_\epsilon$ is the following Proposition:

**Proposition 4.1.** $K_\epsilon(t)$ is of $C^2$ in $\Lambda$ and for $\epsilon$ sufficiently small, we have

1. $K_\epsilon(t) - \tilde{K}_\epsilon(t) = o(\beta)$;
2. $\partial K_\epsilon(t) - \partial \tilde{K}_\epsilon(t) = o(\beta)$ uniformly for $t \in \Lambda$;
3. if $t_\epsilon \in \Lambda$ is a critical point of $K_\epsilon$, then
   \[ \partial^2 K_\epsilon(t_\epsilon) - \partial^2 \tilde{K}_\epsilon(t_\epsilon) = o(\beta). \]

The proof of Proposition 4.1 will be delayed until the end of this section. Let us now use it to prove the uniqueness of $v_\epsilon$.

**Proof of theorem 1.1** By lemma 3.6, we just need to prove that $K_\epsilon(t)$ has only one critical point in $\Lambda$. We prove it in the following steps as in [29].

**Step 1.** By (2) of proposition 4.1, both $K_\epsilon(t)$ and $\tilde{K}_\epsilon(t)$ have no critical points on $\partial \Lambda$ and a continuous deformation argument shows that $\partial \tilde{K}_\epsilon(t)$ has the same degree as $\partial K_\epsilon(t)$ on $\Lambda$. By the definition of $\tilde{K}_\epsilon(t)$, we have $\deg(\tilde{K}_\epsilon(t), \Lambda, 0) = 1$ and thus $\deg(\partial K_\epsilon(t), \Lambda, 0) = 1$. 

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Step 2. At each critical point \( t_\epsilon \) of \( K_\epsilon(t) \), we have
\[
\text{deg}(\partial K_\epsilon(t), \Lambda \cap B_\delta(t_\epsilon), 0) = 1,
\]
for \( \delta_\epsilon \) sufficiently small.

This follows from (3) of Proposition 4.1 and the fact that the eigenvalues of the matrix \( \beta^{-1}(\partial t_\epsilon \partial j_\epsilon \widetilde{K}_\epsilon(t_\epsilon)) \) are positive and away from 0.

Step 3. From step 2, we deduce that \( K_\epsilon(t) \) has only a finite number of critical points in \( \Lambda \), say, \( k_\epsilon \). By the properties of degree, we have
\[
\text{deg}(\partial K_\epsilon(t), \Lambda, 0) = k_\epsilon.
\]

By step 1, \( k_\epsilon = 1 \) and then Theorem 1.1 is thus proved.

In the rest of this section, we shall prove Proposition 4.1.

Proof of Proposition 4.1 The proof of part (1) follows from Lemma 3.5. We now prove part (2) of Proposition 4.1 as follows:
\[
\partial t_\epsilon K_\epsilon(t) = \int_{-\infty}^{\infty} \left[ v'_{\epsilon,t}(\partial t_\epsilon v_{\epsilon,t})' + (\gamma + e^{2t})v_{\epsilon,t} \partial t_\epsilon v_{\epsilon,t} \right] e^{-\beta t} dt - \int_{-\infty}^{\infty} |v_{\epsilon,t}|^{p-1} v_{\epsilon,t} \partial t_\epsilon v_{\epsilon,t} e^{-\beta t} dt
\]
\[
= -\int_{-\infty}^{\infty} S_\epsilon[v_{\epsilon,t}] \partial t_\epsilon v_{\epsilon,t} e^{-\beta t} dt
\]
\[
= J_1 + J_2,
\]
where
\[
J_1 \equiv -\int_{-\infty}^{\infty} S_\epsilon[w_{\epsilon,t} + \phi_{\epsilon,t}] \partial t_\epsilon w_{\epsilon,t} e^{-\beta t} dt.
\]
and
\[
J_2 \equiv -\int_{-\infty}^{\infty} S_\epsilon[w_{\epsilon,t} + \phi_{\epsilon,t}] \partial t_\epsilon \phi_{\epsilon,t} e^{-\beta t} dt.
\]

Using similar argument as in Lemma 2.10 for \( N = 3 \), we can obtain
\[
J_1 = \begin{cases} 
-\beta \int_{-\infty}^{\infty} |w'|^2 dt + \frac{1}{2} e^{(t_1 - t_2)/2} A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt + o(\beta), & j = 1; \\
-\beta \int_{-\infty}^{\infty} |w'|^2 dt - \frac{1}{2} \left[ e^{(t_1 - t_2)/2} - e^{t_2} \right] A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt + o(\beta), & j = 2.
\end{cases}
\]
By (3.10) and Proposition 3.4,
\[
J_2 = - \sum_{i=1}^{2} c_i(t) \int_{-\infty}^{\infty} Z_{\epsilon,t_i} \partial_{t_j} \phi_{\epsilon,t} e^{-\beta t} dt
\]
\[
= \sum_{i=1}^{2} c_i(t) \int_{-\infty}^{\infty} \phi_{\epsilon,t} \partial_{t_j} Z_{\epsilon,t} e^{-\beta t} dt = o(\beta).
\]

(4.4)

Combining the above two estimates (4.3) and (4.4), part (2) of Proposition 4.1 is thus proved.

In the rest we shall prove part (3) of Proposition 4.1. Using (4.2),
\[
\partial_{t_i} \partial_{t_j} K_{\epsilon}(t) = \partial_{t_i} \left[ - \int_{-\infty}^{\infty} S_{\epsilon}[v_{\epsilon,t}] \partial_{t_j} v_{\epsilon,t} e^{-\beta t} dt \right]
\]
\[
= - \int_{-\infty}^{\infty} S_{\epsilon}[v_{\epsilon,t}] \partial_{t_j} v_{\epsilon,t} e^{-\beta t} dt - \int_{-\infty}^{\infty} \partial_{t_i} S_{\epsilon}[v_{\epsilon,t}] \partial_{t_j} v_{\epsilon,t} e^{-\beta t} dt.
\]

By (3.21) we get
\[
\partial_{t_i} S_{\epsilon}[v_{\epsilon,t}] = 2 \sum_{k=1}^{2} c_k(t) \partial_{t_i} Z_{\epsilon,t_k} + 2 \sum_{k=1}^{2} \partial_{t_i} c_k(t) Z_{\epsilon,t_k}
\]

Let \( t_\epsilon \) be a critical point of \( K_{\epsilon}(t) \) in \( \Lambda \), then
\[
S_{\epsilon}[v_{\epsilon,t_\epsilon}] = 0 \quad \text{and} \quad c_k(t_\epsilon) = 0,
\]
which implies
\[
\partial_{t_i} S_{\epsilon}[v_{\epsilon,t}] \bigg|_{t=t_\epsilon} = 2 \sum_{k=1}^{2} \partial_{t_i} c_k(t_\epsilon) Z_{\epsilon,t_k,\epsilon}.
\]

Note that
\[
\partial_{t_i} S_{\epsilon}[v_{\epsilon,t}] = L_{\epsilon} \partial_{t_i} v_{\epsilon,t} + p [ |v_{\epsilon,t}|^{p-1} - |w_{\epsilon,t}|^{p-1} ] \partial_{t_i} v_{\epsilon,t} =: T_{\epsilon} \partial_{t_i} v_{\epsilon,t}.
\]

(4.5)

As in Lemma 3.2, multiplying (4.5) by \( \partial_{t_j} w_{\epsilon,t_j} \) and integrating by parts, we get \( \partial_{t_i} c_k(t_\epsilon) = O(\beta) \). Hence
\[
\int_{-\infty}^{\infty} \partial_{t_i} S_{\epsilon}[v_{\epsilon,t}] \partial_{t_j} \phi_{\epsilon,t} e^{-\beta t} dt \bigg|_{t=t_\epsilon} = \sum_{k=1}^{2} \partial_{t_i} c_k(t_\epsilon) \int_{-\infty}^{\infty} Z_{\epsilon,t_k,\epsilon} (\partial_{t_j} \phi_{\epsilon,t}) e^{-\beta t} dt
\]
\[
= - \sum_{k=1}^{2} \partial_{t_i} c_k(t_\epsilon) \int_{-\infty}^{\infty} (\partial_{t_j} Z_{\epsilon,t_k,\epsilon}) \phi_{\epsilon,t} e^{-\beta t} dt = o(\beta),
\]

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and then
\[
\partial_t \partial_j K(t) = - \int_{-\infty}^{\infty} \partial_t S(e) \partial_j v e^{-\beta t} dt \bigg|_{t=t_*} = - \int_{-\infty}^{\infty} L_e \partial_t w e^{-\beta t} dt \bigg|_{t=t_*} + o(\beta).
\]

Note that
\[
\int_{-\infty}^{\infty} L_e \partial_t \phi w e^{-\beta t} dt = \int_{-\infty}^{\infty} \partial_t \phi \partial_j w e^{-\beta t} dt = o(\beta),
\]
since
\[
L_e \partial_t w = -Z_{t_j} + p|v e|^{p-1} \partial_t w = O(\beta^r).
\]

Therefore,
\[
\partial_t \partial_j K(t) = - \int_{-\infty}^{\infty} L_e \partial_t w e^{-\beta t} dt \bigg|_{t=t_*} + o(\beta).
\]

Using the following important estimate:
\[
\int_{-\infty}^{\infty} L_e \partial_t \phi w e^{-\beta t} dt = \begin{cases} 
\frac{1}{4} e^{(t_1-t_2)/2} A_c, e^{\int_{-\infty}^{\infty} w^p e^{t/2} dt + o(\beta)}, & \text{for } i = j = 1; \\
\frac{1}{4} e^{(t_1-t_2)/2} A_c, e^{\int_{-\infty}^{\infty} w^p e^{t/2} dt + o(\beta)}, & \text{for } i \neq j; \\
-\left[\frac{1}{4} e^{(t_1-t_2)/2} + \frac{1}{2} e^{t^2}\right] A_c, e^{\int_{-\infty}^{\infty} w^p e^{t/2} dt + o(\beta)}, & \text{for } i = j = 2,
\end{cases}
\]

which will be proved in Appendix C, we get the desired result.

\[\square\]

5 The non-degeneracy result and eigenvalue estimates

In this section we shall study the eigenvalue estimates for
\[
L_e(\phi) := \Delta \phi - \phi + p|u_e|^{p-1} \phi
\]
and prove Theorem 1.2.

**Proof of Theorem 1.2.** Let \( \lambda_k, c_k(\theta) \) with \( \theta \in S^{N-1} \) be the eigenvalues and eigenfunctions of the Laplace-Beltrami operator on \( S^{N-1} \). Then
\[
\lambda_0 = 0 < \lambda_1 = \cdots = \lambda_N = N - 1 < \lambda_{N+1} \leq \cdots,
\]
and $e_k$ are normalized so that they form a complete orthonormal basis of $L^2(S^{N-1})$. In fact the set of eigenvalues is given by $\{j(N-2+j) \mid j \geq 0\}$.

Suppose $\phi$ satisfies

$$L_\epsilon(\phi) = 0 \text{ in } R^N, \phi(x) \to 0 \text{ as } |x| \to \infty.$$ 

Put

$$\phi_k(r) = \int_{S^{N-1}} \phi(r, \theta)e_k(\theta)d\theta,$$

then $\phi_k(r) \to 0$ as $r \to \infty$, and it satisfies

$$\phi''_k + \frac{N-1}{r}\phi'_k - \phi_k + p|u_\epsilon|^{p-1}\phi_k + \frac{(-\lambda_k)}{r^2}\phi_k = 0 \text{ in } (0, \infty) \text{ and } \lim_{r \to \infty} \phi_k(r) = 0, \quad (5.2)$$

for $k = 0, 1, \ldots$. We claim that $\phi_k = 0$ for $k \geq N + 1$.

To this end, let us consider the eigenvalues of the problem

$$\phi''_k + \frac{N-1}{r}\phi'_k - \phi_k + p|u_\epsilon|^{p-1}\phi_k + \frac{\nu}{r^2}\phi_k = 0 \text{ in } (0, \infty) \text{ and } \lim_{r \to \infty} \phi_k(r) = 0. \quad (5.3)$$

The $l$-th eigenvalue of (5.3) can be characterized variationally as

$$\nu_l(p) = \max_{\dim(V) < l} \inf_{\phi \in V^\perp} \frac{\int_0^\infty \left[|\phi'|^2 + |\phi|^2\right] r^{N-1} dr - \int_0^\infty |u_\epsilon|^{p-1}|\phi|^2 r^{N-1} dr}{\int_0^\infty |\phi|^2 r^{N-3} dr}, \quad (5.4)$$

where $V$ runs through subspaces of $H^1_0(\mathbb{R}^N)$ and $V^\perp$ is the set of $\phi \in H^1_{0,r}(\mathbb{R}^N)$ satisfying $\int_0^\infty \phi ur^{N-3} = 0$ for all $u \in V$, and $H^1_{0,r}(\mathbb{R}^N)$ be the space of radial functions in $H^1_0(\mathbb{R}^N)$. Thanks to Hardy’s inequality:

$$\frac{(N-2)^2}{4} \int_0^\infty |\phi|^2 r^{N-3} dr \leq \int_0^\infty |\phi'|^2 r^{N-1} dr,$$

the eigenvalues $\nu_1(p) \leq \nu_2(p) \leq \cdots$ are well defined. Using Hardy’s embedding and a simple compactness argument involving the fast decay of $|u_\epsilon|^{p-1}$, there is an extremal for $\nu_l(p)$ which represents a solution to problem (5.3) for $\nu = \nu_l(p)$.

To prove Theorem 1.2 we need to know whether and when $\nu_l(p)$ equals $-\lambda_k$. To show this more information about solutions is required. So we consider the corresponding problems for $u_\epsilon$ using the Emden-Fowler transformation. Then the eigenvalue problem (5.3) becomes

$$\hat{L}_\epsilon[\psi] := \psi'' - \beta \psi' - (\gamma + e^{2t}) \psi + p|u_\epsilon|^{p-1}\psi = -\nu \psi \text{ in } (-\infty, \infty) \text{ and } \lim_{|t| \to \infty} \psi(t) = 0. \quad (5.5)$$

For the proof of Theorem 1.2, let us consider first the radial mode $k = 0$, namely $\lambda_k = 0$. The following result, which contains elements of independent interest, gives the small eigenvalue estimates of $L_\epsilon$ and shows that $\psi_k = 0$ for the mode $k = 0$. 

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Proposition 5.1. For $\epsilon$ small enough, the eigenvalue problem

$$L_\epsilon \phi_\epsilon = \mu_\epsilon \phi_\epsilon$$

has exactly two small eigenvalues $\mu^j_\epsilon$, $j = 1, 2$, which satisfy

$$\frac{\mu^j_\epsilon}{\epsilon} \rightarrow -c_0 \xi_j, \quad \text{up to a subsequence as } \epsilon \rightarrow 0, \text{ for } j = 1, 2,$$

where $\xi_j$'s are the eigenvalues of the Hessian matrix $\nabla^2 \tilde{K}_\epsilon$ and $c_0$ is a positive constant. Furthermore, the corresponding eigenfunctions $\phi^j_\epsilon$'s satisfy

$$\phi^j_\epsilon = \sum_{i=1}^2 \left[ a_{ij} + o(1) \right] \partial_i w_{\epsilon,t} + O(\epsilon), \quad j = 1, 2,$$

where $a_j = (a_{1,j}, \ldots, a_{2,j})^T$ is the eigenvector associated with $\xi_j$, namely,

$$\nabla^2 \tilde{K}_\epsilon a_j = \xi_j a_j.$$

Remark. By (5.6) we know that $\mu_\epsilon \neq 0$ and then obtain the non-degeneracy of $v_\epsilon$ in the space of $H^1$-radial symmetric functions.

Proof of proposition 5.1. To prove this Proposition, one may follow the arguments given in Section 5 of [29] or Section 2 of [14] and the following estimates

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L_\epsilon[\partial_i w_{\epsilon,t}] \partial_j w_{\epsilon,t} e^{-\beta t} dt = \begin{cases} -\frac{1}{4} e^{(t_1-t_2)/2} A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt + o(\beta), & \text{for } i = j = 1; \\ \frac{1}{4} e^{(t_1-t_2)/2} A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt + o(\beta), & \text{for } i \neq j; \\ -\left[ \frac{1}{4} e^{(t_1-t_2)/2} + \frac{1}{2} e^{t_2} \right] A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt + o(\beta), & \text{for } i = j = 2, \end{cases}$$

given in Appendix C.

Let us consider now mode 1 for (5.2), namely $k = 1, \ldots, N$, for which $\lambda_k = N - 1$. In this case we have an explicit solution $u'_\epsilon(r)$. Now we show that $\phi_k = C_k u'_\epsilon$ for some constants $C_k$ for $k = 1, \ldots, N$. This is not trivial since $u'_\epsilon(r)$ changes sign once. Suppose that $\phi_k$ solve (5.2).

We first multiply equation of $\phi_k$ by $u'_\epsilon$ and the equation of $u'_\epsilon$ by $\phi_k$, and integrate over the ball $B_\epsilon$ centered at the origin with radius $r$. Since they satisfy the same equation, we get

$$\phi'_k(r) u'_\epsilon(r) - \phi_k(r) u''_\epsilon(r) = 0,$$

from which we get $\phi_k = C_k u'_\epsilon$ for some constants $C_k$.

Finally let us consider modes 2 and higher. Assume now that $k \geq N + 1$ for which $\lambda_k \geq 2N$. Since $u'_\epsilon(r)$ has exactly one zero in $(0, \infty)$ and $\lambda_k > \lambda_1$, by the standard Sturm-Liouville
comparison theorem, \( \phi_k \) does not change sign in \((0, \infty)\). On the other hand, by Sturm-Liouville theory, it is well known that the eigenfunctions corresponding to \( \nu_1 \) much change sign in \((0, \infty)\) at least \( l - 1 \) times. Thus the only possibility for equation (5.2) to have a nontrivial solution for a given \( k \geq N + 1 \) is that \( \lambda_k = -\nu_1(p) \). In the next proposition we shall show that \(-\nu_1(p) \to \lambda_1 = N - 1\) as \( p \to \frac{N+2}{N-2} \). Therefore we get \( \lambda_k \neq -\nu_1(p) \) for \( k \geq N + 1 \) when \( p \) is closed to \( \frac{N+2}{N-2} \) and then complete the proof of Theorem 1.2.

\[ \square \]

**Proposition 5.2.** As \( p \uparrow \frac{N+2}{N-2} \), we have that \(-\nu_1(p) \to \lambda_1 = N - 1\) for \( l \leq 2 \).

**Proof of proposition 5.2.** One may follow the arguments in Section 3 of [7]. Note that by the Emden-Fowler transformation, the eigenvalues have a variational characterization

\[
\nu_1(p) = \max_{\dim(W) < l} \inf_{\psi \in W^l} \frac{\int_{-\infty}^{\infty} [\psi'^2 + (\gamma + e^{2t})|\psi|^2] e^{-\beta t} \, dt - p \int_{-\infty}^{\infty} |v_c|^2 |\psi|^2 e^{-\beta t} \, dt}{\int_{-\infty}^{\infty} |\psi|^2 e^{-\beta t} \, dt},
\]

(5.8)

where \( W \) runs through the subspaces of \( H \) and \( W^l \) is the set of \( \psi \in W \) satisfying \( \int_{-\infty}^{\infty} \psi v e^{-\beta t} \, dt = 0 \) for all \( v \in W \). Note that the term involving the weight is relatively compact and it follows from a previous argument that the eigenvalues exist.

Observe that the limiting eigenvalue problem

\[
\psi'' - \frac{(N - 2)^2}{4} \psi + \frac{N + 2}{N - 2} \frac{\psi^2}{w_0^{\frac{1}{2}}} \psi = \mu \psi, \quad \psi(\pm \infty) = 0,
\]

admits eigenvalues

\[
\mu_1 = N - 1, \quad \mu_2 = 0, \quad \mu_3 < 0, \quad \cdots,
\]

(5.9)

where the corresponding eigenfunction for the principal eigenvalue \( \mu_1 \) is positive and denoted by \( \Psi_1 \). A simple computation shows that we can take \( \Psi_1 = w_0^{\frac{N}{N-2}} \). Now we take \( \psi_j = w_j^{\frac{N+1}{N-2}}, \quad j = 1, 2 \). Let \( W \) be a given one-dimensional subspace. Then there exists \( c_1, c_2 \) (not all equal to 0) such that \( \int_{-\infty}^{\infty} \left( \sum_{j=1}^{2} c_j \psi_j \right) v e^{-\beta t} \, dt = 0 \) for all \( v \in W \). We then compute that

\[
\int_{-\infty}^{\infty} [\psi'^2 + (\gamma + e^{2t})|\psi|^2] e^{-\beta t} \, dt - p \int_{-\infty}^{\infty} |v|^2 |\psi|^2 e^{-\beta t} \, dt
\]

\[
\leq \sum_{j=1}^{2} c_j^2 (\gamma + e^{2t}) \int_{-\infty}^{\infty} |\psi|^2 e^{-\beta t} \, dt,
\]

and hence by variational characterization of \( \nu_2 \) we deduce that

\[
\nu_1(p) \leq \nu_2(p) \leq -(N - 1) + o(1), \quad l = 1, 2.
\]

(5.10)

On the other hand, according to (5.9), \( \nu_1(p) \to \mu_k \geq -(N - 1) \) for some \( k \). Thus we have \( \nu_1(p) \to -(N - 1) \) as \( p \to \frac{N+2}{N-2} \) for \( l \leq 2 \).

\[ \square \]
6 Appendices

6.1 Appendix A

In this subsection we shall give the estimates of \( w_{j,t_j} \) for \( j = 1, 2 \). Recall that \( w_{j,t} \) is the unique solution to the following equation

\[
v'' - (\gamma_0 + e^{2s})v + w_{t_j}^p = 0, \quad v \in H
\]

whose existence is given by the Riesz’s representation theorem. Here \( w \) is the unique positive even solution of

\[
w'' - \gamma_0 w + w^p = 0.
\]

In fact, the function \( w(t) \) can be written explicitly and has the following form

\[
w(t) = \gamma_0^{\frac{1}{2}} \left( \frac{p + 1}{2} \right)^{\frac{1}{p-1}} \left[ \cosh \left( \frac{p - 1}{2} \gamma_0^{1/2} t \right) \right]^{-\frac{2}{p-1}} = A_{\epsilon,N} \left[ e^{\frac{p-1}{2} \gamma_0^{1/2} t} + e^{-\frac{p-1}{2} \gamma_0^{1/2} t} \right]^{-\frac{2}{p-1}}.
\]

Note that now \( w \) has the following expansion

\[
\left\{ \begin{array}{l}
w(t) = A_{\epsilon,N} e^{-\sqrt{\gamma_0} t} + O(e^{-\sqrt{\gamma_0} t}), \quad t \geq 0; \\
w'(t) = -\sqrt{\gamma_0} A_{\epsilon,N} e^{-\sqrt{\gamma_0} t} + O(e^{-\sqrt{\gamma_0} t}), \quad t \geq 0,
\end{array} \right.
\]

where \( A_{\epsilon,N} > 0 \) is a constant depending on \( \epsilon \) and \( N \).

To get the estimates of \( w_{j,t_j} \), we write \( w_{j,t_j} = w_{t_j} + \phi \), then by (6.1) and (6.2), \( \phi \) satisfies

\[
\phi'' - (\gamma_0 + e^{2s})\phi - e^{2s} w_{t_j} = 0.
\]

Note that as \( s \to \infty \), \( e^{2s} w_{t_j}(s) \to e^{\frac{N-2}{2} t_j} A_{\epsilon,N} e^{-\frac{N-2}{2} \gamma_0 s} \). Hence when \( N > 6 \), \( \phi \in H \) and \( \phi = O(e^{2t_j}) \). Therefore,

\[
w_{j,t_j} = w_{t_j} + O(e^{2t_j}), \quad \text{when } N > 6.
\]

Next we consider \( N \leq 6 \), let \( \phi_N \) be the unique solution of

\[
\phi'' - (\gamma_0 + e^{2s})\phi - e^{-\frac{N-2}{2} s} = 0, \quad |\phi(s)| \to 0, \quad \text{as } |s| \to \infty,
\]

then

\[
w_{j,t_j} = w_{t_j} + e^{\frac{N-2}{2} t_j} A_{\epsilon,N} \phi_N + O(e^{2t_j}) =: w_{t_j} + \phi_{j,t_j} + O(e^{2t_j}), \quad \text{when } N \leq 6.
\]

The rest of this subsection will be devoted to the solvability of \( \phi_N \). A key observation is that

\[
\phi_0 = -e^{-\frac{N-2}{2} s}
\]
is a special solution of (6.5). Thus if we write
\[ \phi_N = \phi_0 + \phi, \]
in order to find a solution of (6.5) which satisfies the decay condition at \( \infty \), let
\[ \phi(s) = e^{-\frac{N-2}{x^2}} \bar{\phi}(\lambda_N e^{(N-2)s}), \quad \text{where} \quad \lambda_N = (N-2)^{-(N-2)}. \] (6.8)
Then \( \bar{\phi} \) satisfies
\[ \bar{\phi}''(s) = s^{-\frac{N-2}{x^2}} \bar{\phi}(s), \quad \bar{\phi}(0) = 1, \quad \bar{\phi}(\infty) = 0 \] (6.9)
and thus
\[ \phi_N = -e^{-\frac{N-2}{x^2}} \left[ 1 - \bar{\phi}(\lambda_N e^{(N-2)s}) \right]. \]

In the case of \( N = 3 \), \( \lambda_3 = 1 \) and \( \bar{\phi} = e^{-s} \). Then
\[ \phi_3 = -e^{-s/2} \left( 1 - e^{s} \right). \]
In the case of \( N = 4 \), \( \lambda_4 = 1/4 \) and
\[ \bar{\phi}(r) = 2\sqrt{r} K_1(2\sqrt{r}) =: \rho_0, \]
where \( K_1(z) \) is the modified Bessel function of second kind and satisfies
\[ z^2 K_1''(z) + z K_1'(z) - (z^2 + 1) K_1(z) = 0, \]
see for example [20]. Then
\[ \phi_4 = -e^{-s} \left[ 1 - \rho_0 \left( \frac{1}{4} e^{2s} \right) \right]. \]

For \( N = 5 \),
\[ \phi_5 = -e^{-3s/2} \left[ 1 - (1 + e^s)e^{-e^s} \right]. \]
In the case of \( N = 6 \),
\[ \phi_6 = -e^{-2s} \left[ 1 - u_0 \left( \frac{1}{16} e^{4s} \right) \right], \]
where \( u_0 \) satisfies
\[ u''(r) = \frac{u(r)}{r^{3/2}}, \quad u(0) = 1, \quad u(\infty) = 0. \]
Actually, we have
\[ u_0(r) = 8\sqrt{r} K_2(4r^{1/4}), \]
where \( K_2(z) \) is the modified Bessel function of second kind and satisfies
\[ z^2 K_2''(z) + z K_2'(z) - (z^2 + 4) K_2(z) = 0. \]
6.2 Appendix B

In this appendix we expand the quality $E_\varepsilon[w_{t,\varepsilon}]$ in terms of $\varepsilon$ and $t$.

**Lemma 6.1.** For $t \in \Lambda$ and $\varepsilon$ sufficiently small, we have for $N = 3$,

$$E_\varepsilon[w_{t,\varepsilon}] = \left( \frac{1}{2} - \frac{1}{p + 1} \right) (e^{-\beta t_1} + e^{-\beta t_2}) \int_{-\infty}^{\infty} w^{p+1} dt + \frac{1}{2} e^{t_2} A_{t,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt$$

$$+ e^{-|t_1-t_2|/2} A_{t,3} \int_{-\infty}^{\infty} w^p e^{t/2} dt + o(\beta) + o(e^x) + o(e^{-|t_1-t_2|/2}).$$

For $N = 4$,

$$E_\varepsilon[w_{t,\varepsilon}] = \left( \frac{1}{2} - \frac{1}{p + 1} \right) (e^{-\beta t_1} + e^{-\beta t_2}) \int_{-\infty}^{\infty} w^{p+1} dt - \frac{1}{4} t_2 e^{2t_2} A_{t,4} \int_{-\infty}^{\infty} w^p e^t dt$$

$$+ e^{-|t_1-t_2|} A_{t,4} \int_{-\infty}^{\infty} w^p e^t dt + o(\beta) + o(t_2 e^{2t_2}) + o(e^{-|t_1-t_2|}).$$

For $N \geq 5$,

$$E_\varepsilon[w_{t,\varepsilon}] = \left( \frac{1}{2} - \frac{1}{p + 1} \right) (e^{-\beta t_1} + e^{-\beta t_2}) \int_{-\infty}^{\infty} w^{p+1} dt + \frac{1}{2} e^{2t_2} \int_{-\infty}^{\infty} w^2 e^{2t} dt$$

$$+ e^{-(N-2)|t_1-t_2|/2} A_{t,N} \int_{-\infty}^{\infty} w^p e^{(N-2)t/2} dt + o(\beta) + o(e^{2t_2}) + o(e^{-(N-2)|t_1-t_2|/2}).$$

**Proof.** Since the proofs are similar for different cases, we give the details for $N = 3$ here. Integrating by parts we get

$$E_\varepsilon[w_{t,\varepsilon}] = \frac{1}{2} \int_{-\infty}^{\infty} \left[ - S_\varepsilon[w_{t,\varepsilon}] + |w_{t,\varepsilon}|^{p-1} w_{t,\varepsilon} \right] w_{t,\varepsilon} e^{-\beta t} dt - \frac{1}{p+1} \int_{-\infty}^{\infty} |w_{t,\varepsilon}|^{p+1} e^{-\beta t} dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left[ \beta w_{t,\varepsilon} + (\gamma - \gamma_0) w_{t,\varepsilon} + w_{t_1} - w_{t_2} \right] w_{t,\varepsilon} e^{-\beta t} dt - \frac{1}{p+1} \int_{-\infty}^{\infty} |w_{t,\varepsilon}|^{p+1} e^{-\beta t} dt$$

$$= E_1 + E_2 + E_3 - E_4 + E_5,$$

where

$$E_1 = \frac{\beta}{2} \int_{-\infty}^{\infty} w_{t,\varepsilon} w_{t,\varepsilon} e^{-\beta t} dt = \frac{\beta^2}{4} \int_{-\infty}^{\infty} w_{t,\varepsilon}^2 e^{-\beta t} dt = O(\beta^2);$$

$$E_2 = \frac{(\gamma - \gamma_0)}{2} \int_{-\infty}^{\infty} w_{t,\varepsilon}^2 e^{-\beta t} dt = -\frac{\beta^2}{8} \int_{-\infty}^{\infty} w_{t,\varepsilon}^2 e^{-\beta t} dt = O(\beta^2);$$

$$E_3 = -\frac{1}{2} \int_{-\infty}^{\infty} w_{t_1}^p w_{t_2} e^{-\beta t} dt - \frac{1}{2} \int_{-\infty}^{\infty} w_{t_1} w_{t_2}^p e^{-\beta t} dt;$$

$$E_4 = \frac{1}{p+1} \int_{-\infty}^{\infty} \left[ |w_{t_1} - w_{t_2}|^{p+1} - w_{t_1}^p w_{t_1} - w_{t_2}^p w_{t_2} \right] e^{-\beta t} dt;$$

$$E_5 = 30$$
\[ E_5 = \left( \frac{1}{2} - \frac{1}{p+1} \right) \left[ \int_{-\infty}^{\infty} w^p_{t_1} w_{1,t_1} e^{-\beta t} \, dt + \int_{-\infty}^{\infty} w^p_{t_2} w_{2,t_2} e^{-\beta t} \, dt \right]. \]

First for \( E_3 \), by Lemma 2.6 we have
\[ E_3 = -e^{-|t_1-t_2|/2} A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2} \, dt + o(\beta) + o(\epsilon^2) + o(e^{-|t_1-t_2|/2}). \]

To estimate \( E_4 \), we divide \( \mathbb{R} \) into two intervals \( I_1, I_2 \) defined by
\[ I_1 = (-\infty, \frac{t_1 + t_2}{2}), \quad I_2 = [\frac{t_1 + t_2}{2}, \infty). \]

So on \( I_1 \) the following equality holds:
\[
\begin{align*}
\frac{1}{p+1} \left[ |w_{1,t_1} - w_{2,t_2}|^{p+1} &- w^p_{t_1} w_{1,t_1} - w^p_{t_2} w_{2,t_2} \right] \\
= \frac{1}{p+1} \left[ (w_{1,t_1} - w_{2,t_2})^{p+1} - w^p_{t_1} w_{1,t_1} + (p+1)w^p_{t_1} w_{2,t_2} \right. \\
&+ \frac{1}{p+1} \left[ (w_{1,t_1} + \phi_{1,t_1})^p - w^p_{t_1} - pw_{t_1}^{p-1} \phi_{1,t_1} \right] w_{1,t_1} + \frac{p}{p+1} w^p_{t_1} \phi_{1,t_1} \\
&+ \frac{p}{p+1} w_{t_1}^{p-1} \phi_{1,t_1} - \frac{1}{p+1} w^p_{t_2} w_{2,t_2}. \\
\end{align*}
\]

As in the proof of Lemma 2.10, by the mean value theorem and inequality (2.28) we have
\[
\left| \frac{1}{p+1} \left[ |w_{1,t_1} - w_{2,t_2}|^{p+1} - w^p_{t_1} w_{1,t_1} - w^p_{t_2} w_{2,t_2} \right] \right| + w^p_{t_1} w_{2,t_2} - \frac{p}{p+1} w^p_{t_1} \phi_{1,t_1} \leq C w_{t_1}^{p+1-\delta} w_{t_2}^{\delta},
\]

for any \( 1 < \delta < 2 \).

Using Lemma 2.7 and integrating by parts, we get
\[
\begin{align*}
\frac{1}{p+1} \int_{I_1} \left[ |w_{t_1} - w_{t_2}|^{p+1} - w^p_{t_1} w_{1,t_1} - w^p_{t_2} w_{2,t_2} \right] e^{-\beta t} \, dt \\
= -\frac{p}{p+1} e^{t_1} A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2} \, dt - e^{-|t_1-t_2|/2} A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2} \, dt + o(e^{-|t_1-t_2|/2}).
\end{align*}
\]

Similarly,
\[
\begin{align*}
\frac{1}{p+1} \int_{I_2} \left[ |w_{t_1} - w_{t_2}|^{p+1} - w^p_{t_1} w_{1,t_1} - w^p_{t_2} w_{2,t_2} \right] e^{-\beta t} \, dt \\
= -\frac{p}{p+1} e^{t_2} A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2} \, dt - e^{-|t_1-t_2|/2} A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2} \, dt + o(e^{-|t_1-t_2|/2}).
\end{align*}
\]

Hence
\[
E_4 = -\frac{p}{p+1} e^{t_2} A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2} \, dt - 2e^{-|t_1-t_2|/2} A_{\epsilon,3} \int_{-\infty}^{\infty} w^p e^{t/2} \, dt + o(e^{-|t_1-t_2|/2}).
\]
Regarding the term $E_5$, by the Lemma 2.6 we have

$$E_5 = \left(\frac{1}{2} - \frac{1}{p+1}\right) \left[ \int_{-\infty}^{\infty} w_t^{p+1} e^{-\beta t} dt + \int_{-\infty}^{\infty} w_t^p \phi_{1,t} e^{-\beta t} dt ight] + \int_{-\infty}^{\infty} w_t^{p+1} e^{-\beta t} dt + \int_{-\infty}^{\infty} w_t^p \phi_{2,t} e^{-\beta t} dt$$

$$= \left(\frac{1}{2} - \frac{1}{p+1}\right) (e^{-\beta t_1} + e^{-\beta t_2}) \int_{-\infty}^{\infty} w^{p+1} dt$$

$$- \left(\frac{1}{2} - \frac{1}{p+1}\right) e^{t_1} A_{t,3} \int_{-\infty}^{\infty} w^{p} e^{t/2} dt + o(\beta).$$

Combining the above estimates for $E_1, E_2, E_3, E_4$ and $E_5$, we obtain

$$E_{e}[w_{t,t}] = \left(\frac{1}{2} - \frac{1}{p+1}\right) (e^{-\beta t_1} + e^{-\beta t_2}) \int_{-\infty}^{\infty} w^{p+1} dt + \frac{1}{2} e^{t_2} A_{t,3} \int_{-\infty}^{\infty} w^{p} e^{t/2} dt + e^{-|t_1-t_2|/2} A_{t,3} \int_{-\infty}^{\infty} w^{p} e^{t/2} dt + o(\beta) + o(e^{\beta t}) + o(e^{-|t_1-t_2|/2}).$$

$\square$

### 6.3 Appendix C

In this section we give the technical proof of (4.6) for $N = 3$, that is,

$$\int_{-\infty}^{\infty} L_e[\partial_{i,t} w_{t,t}] \partial_{j,t} w_{t,t} e^{-\beta t} dt = \begin{cases} -\frac{1}{4} e^{(t_1-t_2)/2} A_{t,3} \int_{-\infty}^{\infty} w_{t} e^{t/2} dt + o(\beta), & \text{for } i = j = 1; \\ \frac{1}{4} e^{(t_1-t_2)/2} A_{t,3} \int_{-\infty}^{\infty} w_{t} e^{t/2} dt + o(\beta), & \text{for } i \neq j; \\ \left[-\frac{1}{4} e^{(t_1-t_2)/2} + \frac{1}{2} e^{t_2}\right] A_{t,3} \int_{-\infty}^{\infty} w_{t} e^{t/2} dt + o(\beta), & \text{for } i = j = 2. \end{cases}$$

(6.10)

**Proof.** Note that by (3.1) and (3.3), we obtain

$$L_e[\partial_{i,t} w_{t,t}] = -Z_{e,t_i} + p|v_{e,t}|^{p-1} \partial_{j,t} w_{t,t}$$

$$= (-1)^j \left[ - p w_{e,t_i}^{p-1} w_{t_i}^p + \beta (\partial_{j,t} w_{j,t_j})' + (\gamma - \gamma_0) \partial_{j,t} w_{j,t_j} - p|v_{e,t}|^{p-1} \partial_{j,t} w_{j,t_j} \right],$$

and by the definition of $w_{e,t},$

$$\partial_{j,t} w_{e,t} = (-1)^j \partial_{j,t} w_{j,t_j} = (-1)^{j+1} \partial_{j,t} w_{j,t_j} + O(e^{2\beta}).$$

(6.11)

In order to calculate the integration, we divide $(-\infty, \infty)$ into two intervals $I_1, I_2$ defined by

$$I_1 = (-\infty, \frac{t_1 + t_2}{2}), \quad I_2 = \left[ \frac{t_1 + t_2}{2}, \infty \right).$$

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First we compute the case of $i \neq j$. By (6.11) and (6.12) we get

$$\int_{-\infty}^{\infty} L_{c}[\partial_{t_i} w_{c,t}] \partial_{t_j} w_{c,t} e^{-\beta t} \, dt = \int_{-\infty}^{\infty} p|v_{c,t}|^{p-1} w_{t_i} w_{t_j} - \int_{t_1}^{t_2} p|v_{c,t}|^{p-1} w_{t_i} w_{t_j} + o(\beta)$$

$$= - \int_{t_2}^{\infty} p|v_{c,t}|^{p-1} w_{t_i} w_{t_j} + o(\beta)$$

$$= \int_{-\infty}^{\infty} w_{t_i}^p w_{t_j}^{\prime\prime} + o(\beta)$$

$$= \frac{1}{4} e^{(t_1-t_2)/2} A_{c,3} \int_{-\infty}^{\infty} w^p e^{t/2} \, dt + o(\beta).$$

For the case of $i = j = 1$, recall that $v_{c,t} = w_{c,t} + \phi$, where $\phi = \phi_{c,t}$ is given by Proposition 3.4. Then on $I_1$:

$$p|v_{c,t}|^{p-1}(w_{t_1}^l)^2 - p|w_{t_1}|^{p-1}(w_{t_1}^l)^2$$

$$= -p(p-1)w_{t_1}^{p-2}(w_{t_1}^l)^2 w_{t_2} + p(p-1)w_{t_1}^{p-2}(w_{t_1}^l)^2 \phi + o(\beta).$$

So by (6.11) and (6.12) we obtain

$$\int_{-\infty}^{\infty} L_{c}[\partial_{t_1} w_{c,t}] \partial_{t_1} w_{c,t} e^{-\beta t} \, dt$$

$$= - \int_{I_1} \int \frac{p|v_{c,t}|^{p-1}(w_{t_1}^l)^2 e^{-\beta t}}{2} \, dt + \int_{I_1} \frac{p|v_{c,t}|^{p-1}(w_{t_1}^l)^2 e^{-\beta t}}{2} \, dt + o(\beta)$$

$$= - \int_{I_1} p(p-1)w_{t_1}^{p-2}(w_{t_1}^l)^2 w_{t_2} \, dt + \int_{I_1} p(p-1)w_{t_1}^{p-2}(w_{t_1}^l)^2 \phi + o(\beta)$$

$$= T_1 + T_2 + o(\beta).$$

Recall that $w_{j,t_j}$ satisfies

$$w_{j,t_j}^{\prime\prime} - (\gamma_0 + e^{2t})w_{j,t_j} + w_{t_j}^p = 0.$$

So $\partial_{t_j} w_{j,t_j}$ and $\partial_{t_j}^2 w_{j,t_j}$ satisfy

$$(\partial_{t_j} w_{j,t_j})^{\prime\prime} - (\gamma_0 + e^{2t})\partial_{t_j} w_{j,t_j} + p w_{t_j}^{p-1}(\partial_{t_j} w_{t_j}) = 0,$$

and

$$(\partial_{t_j}^2 w_{j,t_j})^{\prime\prime} - (\gamma_0 + e^{2t})(\partial_{t_j}^2 w_{j,t_j}) + p w_{t_j}^{p-1}(\partial_{t_j}^2 w_{t_j}) + p(p-1)w_{t_j}^{p-2}(\partial_{t_j} w_{t_j})^2 = 0,$$

which implies

$$p(p-1)w_{t_j}^{p-2}(w_{t_j}^l)^2 = -L_{c}[\partial_{t_j}^2 w_{1,t_1}] + o(\beta).$$
Hence

\[ T_2 = - \int_{I_1} \phi L_e[D_{t_1}^2 w_{t_1}] + o(\beta). \]

By (2.31) and Proposition 3.4, on \( I_1 \) we have on \( I_1 \)

\[ L_e[\phi] = \beta w_{t_1} + pw_{t_1}^{p-1}w_{t_2} + o(\beta). \]

Thus

\[
T_2 = - \int_{\mathbb{R}} w_{t_1}^{p} [\beta w_{t_1} + pw_{t_1}^{p-1}w_{t_2}] dt + o(\beta) \\
= - \int_{\mathbb{R}} w_{t_1}^{p}pw_{t_1}^{p-1}w_{t_2} dt + o(\beta). \tag{6.14}
\]

On the other hand,

\[
T_1 = - \int_{\mathbb{R}} p(p - 1)w_{t_1}^{p-2}(w_{t_1}')^2 w_{t_2} dt + o(\beta) \\
= \int_{\mathbb{R}} L_0[w_{t_1}]w_{t_2} dt + o(\beta) \\
= \int_{\mathbb{R}} w_{t_1}^{p}L_0[w_{t_2}] dt + o(\beta) \\
= \int_{\mathbb{R}} w_{t_1}^{p}w_{t_2}^{p} + pw_{t_1}^{p-1}w_{t_2} dt + o(\beta) \\
= - \int_{\mathbb{R}} w_{t_1}^{p}w_{t_2}^{p} + \int_{\mathbb{R}} w_{t_1}^{p}pw_{t_1}^{p-1}w_{t_2} dt + o(\beta), \tag{6.15}
\]

where

\[ L_0[\phi] := \phi'' - \gamma_0 \phi + pw_{t_1}^{p-1} \phi. \]

Combining (6.13), (6.14) and (6.15), we get the desired result for \( i = j = 1 \). The proof for \( i = j = 2 \) is similar, we omit the details here.

\[ \square \]

References


