LOCAL PROFILE OF FULLY BUBBLING SOLUTIONS TO SU(N+1) TODA SYSTEMS

CHANG-SHOU LIN, JUNCHENG WEI, AND LEI ZHANG

ABSTRACT. In this article we prove that for locally defined singular $SU(n+1)$ Toda systems in $\mathbb{R}^2$, the profile of fully bubbling solutions near the singular source can be accurately approximated by global solutions. The main ingredients of our new approach are the classification theorem of Lin-Wei-Ye [20] and the non-degeneracy of the linearized Toda system [20], which make us overcome the difficulties that come from the lack of symmetry and the singular source.

1. INTRODUCTION

Let (M,g) be a compact Riemann surface and Δ the Beltrami-Laplacian operator of the metric g, and K the Gauss curvature. The $SU(n+1)$ Toda system is the following nonlinear PDE

$$\Delta u_i+\sum_{j=1}^n a_{ij}h_je^{u_j}-K(x)=4\pi\sum_j \gamma_{ij}\delta_{q_j},\quad 1\le i\le n, \tag{1.1}$$

where h_i $(i=1,...,n)$ are positive smooth functions on M, δ_q stands for the Dirac measure at $q\in M$, and $A=(a_{ij})$ is the Cartan matrix given by

$$A=\left(\begin{array}{ccccc} 2 & -1 & 0 & ... & 0\\ -1 & 2 & -1 & ... & 0\\ 0 & -1 & 2 & & 0\\ \vdots & \vdots & & \vdots & \\ 0 & \dots & -1 & 2 & -1\\ 0 & \dots & & -1 & 2 \end{array}\right).$$

The Toda system (1.1) is closely related to many different research areas in mathematics. For $n=1$, the equation is reduced to the curvature equation in two dimensional surfaces. Without the singular source and $M=\mathbb{S}^2$, it is the well known Nirenberg problem. In general it is related to the existence of the metric of positive constant curvature with conic singularities ([8, 9, 36, 37]). For the past three decades, equation (1.1) with $n=1$ has been extensively studied. We refer to the readers to [3], [5], [19] and reference therein. For the general n and $h_i\equiv 1$

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($i = 1, .., n$) equation (1.1) is connected with holomorphic curves of M into $\mathbb{C}P^n$ via the classical infinitesimal Plücker formulae, see [13]. This geometric connection is very important to (1.1). From it, we have found that equation (1.1) with $h_i \equiv 1$ is an integrable system, for example, see [11] and [14] and the reference therein in this respect. Recently by using this connection, Lin-Wei-Ye [20] are able to completely classify all the entire solutions of (1.1) in $\mathbb{R}^2$ with one singular source.

In mathematical physics, equation (1.1) has also played an important role in many Chern-Simons gauge theory. For example, in the relativistic $SU(n+1)$ Chern-Simons model proposed by physicists (see [15]) for $n = 1$ and [12] for $n > 1$), in order to explain the physics of high temperature super-conductivity, (1.1) governs the limiting equations as the physical parameters tend to 0. For the past twenty years, the connections of (1.1) with $n = 1$ and the Chern-Simons-Higgs equation have been explored extensively. See [33] and [24]. However, for $n \geq 2$ only very few works are devoted to this direction of research. See [1], [25] and [34].

Recently, equation (1.1) is getting a lot of attention from many mathematicians. For recent development of equation (1.1) and related subjects, we refer the readers to [29, 30, 17, 16, 20, 21, 23, 26, 27, 28, 31, 32, 40] and the reference therein.

One of the fundamental issues concerning (1.1) is to study the bubbling phenomenon, which could lead to establishing the a priori bound of solutions of (1.1). For the case $n = 1$, the bubbling phenomenon has been studied thoroughly for the past twenty years. Basically there are two kinds of bubbling behaviors of solutions near its blowup points. One is the so called "simple blowup", which means the bubbling profile could be well controlled locally by the entire bubbling solutions in $\mathbb{R}^2$. For the case without singular sources, this was proved by Y. Y. Li [19], applying the method of moving planes. With singular sources, this was proved by Bartolucci-Chen-Lin-Tarantello [2] if $\gamma \notin \mathbb{N}$ and recently by Kuo-Lin [18] if $\gamma \in \mathbb{N}$, who use the potential analysis and Pohozaev identity. On the other hand, the non-simple blowup could occur at $\gamma \in \mathbb{N}$ only. The sharp profile of the non-simple blowup has recently been proved in [18]. The study of the bubbling phenomenon is important not only for deriving the a priori bounds, but also for providing a lot of important geometric information concerning (1.1) near blowup points, see [4, 6, 25].

For $n \geq 2$, (1.1) is an elliptic system. It is expected that the behavior of bubbling solutions is more complicated than the case $n = 1$. One major difficulty comes from the partial blown-up phenomenon, that is, after a suitable scaling, the solutions converge to a smaller system. To understand the partial blown-up phenomenon, we have to first study the fully blown-up behavior, and to obtain accurate description of this class of bubbling solutions. When $n = 2$ and (1.1) has no singular souces, the the bubbling behavior of fully bubbling solutions has been studied by Jost-Lin-Wang [17] and Lin-Wei-Zhao [23]. In [17] it is proved that any sequence of fully bubbling solutions is a simple blowup at any blowup point. The proof in [17] has used deep application of holonomy theory, which is a very effective generalization of Pohozaev identity. Unfortunately their holonomy method can not be extended to cover the case with singular sources. The purpose of this article is to extend their results to any $n \geq 2$ and to include (1.1) with singular sources. Before stating our main results, we set up our problem first. Since this is a local problem, for simplicity we consider

$$\Delta u_i^k+\sum_{j=1}^n a_{ij}h_j^k e^{u_j^k}=4\pi\gamma_i^k\delta_0,\quad B_1\subset\mathbb{R}^2. \tag{1.2}$$

For $u^k=(u_1^k,..,u_n^k)$, $h^k=(h_1^k,..,h_n^k)$ and γ_i^k $(i=1,..,n)$ we assume the usual assumptions:

$$\begin{aligned}
(H):\qquad &(i):\quad \frac{1}{C}\le h_i^k\le C,\quad \|h_i^k\|_{C^2(B_1)}\le C,\quad h_i^k(0)=1,\quad i=1,..,n\\
&(ii):\quad \lim_{k\to\infty}\gamma_i^k=\gamma_i>-1,\quad i=1,..,n\\
&(iii):\quad \int_{B_1}h_i^ke^{u_i^k}\le C,\quad i=1,..,n,\ C \text{ is independent of } k.\\
&(iv):\quad |u_i^k(x)-u_i^k(y)|\le C,\quad \text{for all } x,y\in\partial B_1,\quad i=1,..,n.\\
&(v):\quad \max_{K\subset\subset B_1\setminus\{0\}}u_i^k\le C, \text{ and 0 is the only blowup point.}
\end{aligned}$$

If $(u_1^k,...,u_n^k)$ is a global solution of (1.1) in M, it is easy to see that all assumptions of (H) are satisfied. We also note that the assumption (iv) in (H) is necessary for our analysis, without it Chen [10] proved that even for $n=1$ the blowup solutions can be very complicated near their blowup points. The assumption $h_i^k(0)=1$ in (i) is just for convenience.

Let

$$-2\log\varepsilon_k=\max_{x\in B_1,i=1,..,n}\Big(\frac{\tilde{u}_i^k(x)}{1+\gamma_i^k}\Big),\qquad \text{where } \tilde{u}_i^k(x)=u_i^k(x)-2\gamma_i^k\log|x|, \tag{1.3}$$

and

$$v_i^k(y)=\tilde{u}_i^k(\varepsilon_k y)+2(1+\gamma_i^k)\log\varepsilon_k,\quad i=1,..,n \tag{1.4}$$

Then clearly v_i^k satisfies

$$\Delta v_i^k(y)+\sum_{j=1}^n a_{ij}|y|^{2\gamma_j^k}h_j^k(\varepsilon_k y)e^{v_j^k}=0,\quad |y|\le\varepsilon_k^{-1}. \tag{1.5}$$

Our major assumption is $v^k=(v_1^k,...,v_n^k)$ converges to a $SU(n+1)$ Toda system uniformly over all compact subsets of $\mathbb{R}^2$:

Definition 1.1. *We say u^k of (1.2) is a fully bubbling sequence if v^k converges in $C^{1,\alpha}_{loc}(\mathbb{R}^2)$ to $v=(v_1,..,v_n)$ that solves the following $SU(n+1)$ Toda system in $\mathbb{R}^2$:*

$$\begin{aligned}
&\Delta v_i+\sum_{j=1}^n a_{ij}|y|^{2\gamma_j}e^{v_j}=0,\quad \mathbb{R}^2,\quad i=1,..,n\\
&\int_{\mathbb{R}^2}e^{v_i}<\infty,\quad i=1,..,n.
\end{aligned} \tag{1.6}$$

The main purpose of this paper is to show that a fully bubbling sequence u^k can be sharply approximated by a sequence of global solutions $U^k = (U_1^k, .., U_n^k)$ of

$$\Delta U_i^k + \sum_{j=1}^n a_{ij} e^{U_j^k} = 4\pi\gamma_i^k \delta_0, \quad \text{in} \quad \mathbb{R}^2, \quad i = 1, ..., n. \tag{1.7}$$

Theorem 1.1. *Let (H) hold and u^k be a fully bubbling sequence described in Definition 1.1, then there exists a sequence of global solutions $U^k = (U_1^k, .., U_n^k)$ of (1.7) such that*

$$|u_i^k(\varepsilon_k y) - U_i^k(\varepsilon_k y)| \le C(\sigma)\varepsilon_k^\sigma (1 + |y|)^\sigma, \quad |y| \le \varepsilon_k^{-1}, \quad i = 1, ..n$$

for $\sigma \in (0, \min\{1, 2+2\gamma_1, .., 2+2\gamma_n\})$ and $C(\sigma) > 0$ independent of k. Moreover, there exists $C > 0$ independent of k, such that

$$|\tilde{U}_i^k(\varepsilon_k y) + 2(2 + \gamma_i^k + 2\gamma_{n+1-i}^k)\log(1 + |y|)| \le C, \quad \text{for } |y| \le \varepsilon_k^{-1}, i = 1, .., n,$$

where $\tilde{U}_i^k(x) = U_i^k(x) + 2\gamma_i^k \log|x|$ is the regular part of U_i^k.

The following corollary is immediately implied by Theorem 1.1:

Corollary 1.1. *Let u^k, ε_k be the same as in Theorem 1.1, v^k be defined by (1.4). Then for $i = 1, .., n$,*

$$|v_i^k(y) + 2(2 + \gamma_i^k + \gamma_{n+1-i}^k)\log(1 + |y|)| \le C, \quad \text{for } |y| \le \varepsilon_k^{-1}.$$

If $n = 2$ and $\gamma_i = 0 (i = 1, 2)$, Corollary 1.1 was proved by Jost-Lin-Wang [17] by a deep application of holonomy theory. It is easy to see that Theorem 1.1 is stronger than Corollary 1.1 even for this special case. The approach of Jost-Lin-Wang can not be employed to deal with the singular case. In this article our proof of Theorem 1.1 is purely based on PDE methods and the essential part relies on the important classification theorem of Lin-Wei-Ye [20] on all global solutions of the $SU(n+1)$ Toda system and the non-degeneracy of the linearized system.

When $n = 2$ and $\gamma_i = 0 (i = 1, 2)$, Theorem 1.1 was proved by Lin-Wei-Zhao [23]. The idea there was to fix the initial Cauchy data $(v_1^k(0), v_2^k(0), \partial_z v_1^k(0), \partial_z v_2^k(0), \partial_{zz} v_1^k(0))$ of v_i^k (Theorem 2.2 of [23]). When $n > 2$, it seems very difficult to fix the initial data because the number of parameters involved for solving the algebraic equation is huge (which is $n^2 + 2n$). To overcome this difficulty, the new idea we introduced in this paper is that instead of fixing Cauchy data at one single point we only need to match v_1^k with U_1^k at some specific chosen $n^2 + 2n$ points in $\mathbb{R}^2$.

Estimates similar to those in Theorem 1.1 for single Liouville equations or non-singular $SU(3)$ Toda systems can be found in [4, 41, 42] and [23], respectively. However all the previous approaches either require differentiation of blowup solutions at the origin ([4, 41, 23]) or the symmetry of global solutions ([42]). None of them can be employed to prove Theorem 1.1, since the singularity at the origin does not allow differentiations of blowup solutions, and the lack of symmetry of global solutions makes it impossible to use ODE theories. In this article we employ new ideas to determine precise information of blowup solutions on $n^2 + 2n$ carefully chosen points.

For some applications such as constructing blowup solutions, more refined estimates than those in Theorem 1.1 are needed. For $SU(3)$ Toda systems with no singularity, Lin-Wei-Zhao [23] obtained more delicate estimates for this case based on Corollary 1.1.

The organization of the article is as follows. In section two we list some facts on the $SU(n+1)$ Toda system and the non-degeneracy of the linearized system. The proof of Theorem 1.1 is in section three. One key point in the proof of Theorem 1.1 is to determine n^2+2n points in $\mathbb{R}^2$ in a specific way. Since this part is somewhat elaborate and elementary, we put it separately in section four.

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2. SOME FACTS ON THE LINEARIZED $SU(n+1)$ SYSTEM

First we list some facts on the entire solutions of $SU(n+1)$ Toda systems with singularities. For more details see [20]. Let $u=(u_1,..,u_n)$ solve

$$\begin{cases} \Delta u_i+\sum_{j=1}^n a_{ij}e^{u_j}=4\pi\gamma_i\delta_0, \quad \mathbb{R}^2, \quad i=1,..,n \\ \int_{\mathbb{R}^2} e^{u_i}<\infty \end{cases}$$

where $A=(a_{ij})_{n\times n}$ is the Cartan matrix and $\gamma_i>-1$. Then let

$$U_i=\sum_{j=1}^n a^{ij}u_j, \quad i=1,...,n$$

where $(a^{ij})_{n\times n}=A^{-1}$. Clearly $(U_1,...,U_n)$ satisfies

$$\Delta U_i+e^{\sum_{j=1}^n a_{ij}U_j}=4\pi\alpha_i\delta_0, \quad \alpha_i=\sum_{j=1}^n a^{ij}\gamma_j, \quad i=1,...,n.$$

The classification theorem of Lin-Wei-Ye ([20]) asserts

$$e^{-U_1}=|z|^{-2\alpha_1}(\lambda_0+\sum_{i=1}^n\lambda_i|P_i(z)|^2) \tag{2.1}$$

where

$$P_i(z)=z^{\mu_1+...+\mu_i}+\sum_{j=0}^{i-1}c_{ij}z^{\mu_1+...+\mu_j}, \quad i=1,...,n \tag{2.2}$$

$$\mu_i=1+\gamma_i, \tag{2.3}$$

c_{ij} $(j<i)$ are complex numbers and $\lambda_i>0$ $(0\le i\le n)$ satisfies

$$\lambda_0...\lambda_n=2^{-n(n+1)}\Pi_{1\le i\le j\le n}(\sum_{k=i}^j\mu_k)^{-2}. \tag{2.4}$$

Furthermore if $\mu_{j+1}+...+\mu_i \notin \mathbb{N}$ for some $j<i$, then $c_{ij}=0$. Let

$$\tilde{U}_1 = U_1 - 2\alpha_1 \log|z|,$$

then

$$\tilde{U}_1 = -\log(\lambda_0 + \sum_{i=1}^{n} \lambda_i |P_i(z)|^2). \tag{2.5}$$

The following lemma classifies the solutions of the linearized system under a mild growth condition at infinity:

Lemma 2.1. *Let $\Phi_1,...,\Phi_n$ solve the linearized $SU(n+1)$ Toda system:*

$$\Delta\Phi_i + e^{u_i}(\sum_{j=1}^{n} a_{ij}\Phi_j) = 0, \quad in \quad \mathbb{R}^2, \quad i=1,...,n.$$

If

$$|\Phi_i(x)| \le C(1+|x|)^{\sigma}, \quad x \in \mathbb{R}^2 \tag{2.6}$$

for $\sigma \in (0, \min\{1, 2\mu_1, ..., 2\mu_n\})$, then

$$e^{-U_1}\Phi_1(z) = \sum_{k=0}^{n} m_{kk}|z|^{2\beta_k} + 2\sum_{k=1}^{n-1}|z|^{\beta_k}\sum_{l=k+1}^{n}|z|^{\beta_l} Re(m_{kl}e^{-i(\mu_{k+1}+..+\mu_l)\theta}) \tag{2.7}$$

where $\theta = arg(z)$,

$$\beta_0 = -\alpha_0, \quad \beta_i = \alpha_i - \alpha_{i+1} + i, \quad \beta_n = \alpha_n + n, \tag{2.8}$$

$m_{kk} \in \mathbb{R}$ for $k=0,..,n$, $m_{kl} \in \mathbb{C}$ for $k<l$. Obviously $m_{kl}=0$ if $\mu_{k+1}+..+\mu_l \notin \mathbb{N}$.

Proof of Lemma 2.1: This lemma is proved in [20] when all Φ_i are bounded functions. Here we mention the minor modifications when a mild growth condition in (2.6) is assumed. Let

$$w_i(y) = \frac{1}{2\pi}\int_{\mathbb{R}62} \log|y-\eta| e^{u_i(\eta)}(\sum_{j=1}^{n} a_{ij}\Phi_j(\eta))d\eta.$$

By (2.7) and $e^{u_i} = O(|z|^{-4-2\nu_{n+1-i}})$ we see that $e^{u_i(z)}(\sum_{j=1}^{n} a_{ij}\Phi_j(z)) = O(|z|^{-2-\delta})$ for some $\delta > 0$ when $|z|$ is large. Thus $w_i(y) = O(\log|y|)$ for $|y|$ large. From $\Delta(\Phi_i - w_i) = 0$ in $\mathbb{R}^2$ and $|\Phi_i(z) - w_i(z)| \le O(|z|^{1-\delta})$ for some $\delta > 0$ we have

$$\Phi_i = w_i + C.$$

Then using the integral representation of Φ_i we can further obtain $\nabla^k\Phi_i = O(|z|^{-k})$ as $|z| \to \infty$. Then the remaining part of the proof is the same as Lemma 6.1 of [20]. $\square$

From (2.8) it is easy to verify that

$$\beta_i - \beta_{i-1} = \mu_i, \quad 1 \le i \le n. \tag{2.9}$$

Thus β_i is increasing in i. Using (2.1) and (2.9) in (2.7), we have

$$\Phi_1 = \frac{1}{\lambda_0 + \sum_i \lambda_i |P_i(z)|^2} \Big\{ \sum_{k=0}^{n} m_{kk} |z|^{2\beta_k + 2\alpha_0} + 2\sum_{k=0}^{n-1} |z|^{\beta_k+\alpha_0} \sum_{l=k+1}^{n} |z|^{\beta_l+\alpha_0} Re(m_{kl} e^{-i(\mu_{k+1}+..+\mu_l)\theta}) \Big\}$$
$$= \frac{1}{\lambda_0 + \sum_i \lambda_i |P_i(z)|^2} \Big\{ \sum_{k=0}^{n} m_{kk} |z|^{2\mu_1+...+2\mu_k} + 2\sum_{k=0}^{n-1} |z|^{\mu_1+...+\mu_k} \Big(\sum_{l=k+1}^{n} |z|^{\mu_1+...+\mu_l} Re(m_{kl} e^{-i(\mu_{k+1}+...+\mu_l)\theta}) \Big) \Big\}. \tag{2.10}$$

Lemma 2.2. $m_{nn} = 0$.

Proof of Lemma 2.2: Consider the equation for Φ_1:

$$\Delta\Phi_1 + e^{u_1}(2\Phi_1 - \Phi_2) = 0, \quad \mathbb{R}^2. \tag{2.11}$$

From the classification theorem of Lin-Wei-Ye, we have

$$e^{u_1} = O(|z|^{-2-2\mu_n}), \quad |z| > 1.$$

On the other hand from (2.10) we have

$$\Delta\Phi_1 = c(\lambda_0, .., \lambda_{n-1}, \mu_1, ..., \mu_n) m_{nn} |z|^{-2} + O(|z|^{-2-\delta}), \quad |z| > 1$$

where $c(\lambda_0, .., \lambda_{n-1}, \mu_1, ..., \mu_n)$ and δ are two positive numbers. Then it is easy to see from (2.11) that $m_{nn} = 0$ because by (2.6)

$$e^{u_1}(2\Phi_1 - \Phi_2) \le C(1+|z|)^{-2-\varepsilon}$$

for some $\varepsilon > 0$. Lemma 2.2 is established. □

From Lemma 2.2 we see that there are n^2+2n unknowns in Φ_1. We write Φ_1 as

$$\Phi_1 = \frac{1}{\lambda_0 + \sum_{i=1}^n \lambda_i |P_i(z)|^2} \Big\{ \sum_{k=0}^{n-1} m_{kk} |z|^{2\mu_1+..+2\mu_k} + 2\sum_{k=0}^{n-1} |z|^{2\mu_1+..+2\mu_k} \sum_{l=k+1}^{n} Re(\bar{m}_{kl} z^{\mu_{k+1}+..+\mu_l}) \Big\}. \tag{2.12}$$

3. THE PROOF OF THEOREM 1.1

Recall that $v^k = (v_1^k, .., v_n^k)$ satisfies (1.5) and v^k converges in $C^{1,\alpha}_{loc}(\mathbb{R}^2)$ to $v = (v_1, ..., v_n)$, where v satisfies (1.6). By the classification theorem of Lin-Wei-Ye [20], there exists $\Lambda = (\lambda_i, c_{ij})(i = 0, .., n, j < i)$ such that $\tilde{U}_1(z)$ is defined in (2.5) where λ_i and P_i satisfy (2.4) and (2.2), respectively. To emphasize the dependence of Λ, we denote v_i and U_i as $v_i(z, \Lambda)$ and $U_i(z, \Lambda)$, respectively.

Let $\tilde{U}_i(z, \Lambda) = \sum_{j=1}^n a^{ij} v_j(z, \Lambda)$ and $W_i^k = \sum_j a^{ij} v_j^k$. Then $\tilde{U}_1$ is of the form in (2.5). The main idea of the proof of Theorem 1.1 is that we look for a sequence of global solutions $v(z, \Lambda_k)$ such that v_1^k is sufficiently close to $v_1(z, \Lambda_k)$ at n^2+2n

points. Since Λ_k has at most n^2+2n components, we shall prove that by choosing the n^2+2n points carefully we have $\Lambda_k \to \Lambda$. Set

$$\Theta(p) = (\frac{\partial \tilde{U}_1}{\partial \lambda_0}(p),..,\frac{\partial \tilde{U}_1}{\partial \lambda_{n-1}}(p),\frac{\partial \tilde{U}_1}{\partial c^1_{10}}(p),...,\frac{\partial \tilde{U}_1}{\partial c^2_{n,n-1}}(p))'$$

where $()'$ stands for transpose. For $p_1,...,p_{n^2+2n} \in \mathbb{C}$, consider the following $(n^2+2n)\times(n^2+2n)$ matrix

$$\mathbf{M} = (\Theta(p_1),...,\Theta(p_{n^2+2n})). \tag{3.1}$$

By choosing $p_1,...,p_{n^2+2n}$ carefully we make $\mathbf{M}$ invertible. The choice of $p_1,...,p_{n^2+2n}$ is described in section four. On the other hand we write (2.12) as

$$\begin{aligned} &\Phi_1(z)(\lambda_0+\sum_i \lambda_i |P_i(z)|^2) \\ = &\sum_{k=0}^{n-1} m_{kk}|z|^{2\mu_1+..+2\mu_k} + 2\sum_{k=0}^{n-1}\sum_{l=k+1}^{n} |z|^{2\mu_1+..+2\mu_k+\mu_{k+1}+..+\mu_l} \\ &(\cos((\mu_{k+1}+..+\mu_l)\theta)m^1_{kl} + \sin((\mu_{k+1}+..+\mu_l)\theta)m^2_{kl}). \\ = &\mathbf{X}\hat{\Theta}(z). \end{aligned} \tag{3.2}$$

where

$$\mathbf{X} = (m_{00},...,m_{n-1,n-1},m^1_{01},...,m^2_{n-1,n}), \quad m_{kl} = m^1_{kl}+\sqrt{-1}m^2_{kl}.$$

So $\hat{\Theta}(z)$ is a column vector (so is $\Theta(p)$). Our choice of $p_1,...,p_{n^2+2n}$ (explained in section four) also makes

$$\mathbf{M_1} = (\hat{\Theta}(p_1),...,\hat{\Theta}(p_{n^2+2n}))$$

invertible.

We have known that $W_i^k \to \tilde{U}_i(z,\Lambda)$ in $C^{1,\alpha}$ norm over any fixed compact subset of $\mathbb{R}^2$. Now we find a sequence of global solutions $U(z,\Lambda_k)$ as follows: Let

$$\tilde{U}_i(z,\Lambda_k)(z) = U_i(z,\Lambda_k) - 2\alpha_i^k \log|z|, \quad i=1,..,n$$

where $\alpha_i^k = \sum_j a_{ij}\gamma_j^k$.

$$\tilde{U}_1(z,\Lambda_k) = -\log(\lambda_0^k + \sum_{i=1}^n \lambda_i^k |P_i^k(z)|^2)$$

where

$$P_i^k(z) = z^{\mu_1^k+...+\mu_i^k} + \sum_{j=0}^{i-1} c_{ij}^k z^{\mu_1^k+...+\mu_j^k}.$$

We claim that from $\tilde{U}_1(p_l,\Lambda_k) = W_1^k(p_l)$ $(l=1,..,n^2+2n)$ we have $\Lambda_k \to \Lambda$, which is

$$\lambda_i^k \to \lambda_i,\ c_{ij}^k \to c_{ij}. \tag{3.3}$$

Indeed, from the definition of p_l, the matrix

$$(\Theta(p_1),...,\Theta(p_{n^2+2n}))$$

is invertible, $W_1^k(p_l)=\tilde{U}_1(p_l,\Lambda)+o(1)$ for $l=1,...,n^2+2n$, (3.3) clearly follows. Other components of $(U_1(z,\Lambda_k),...,U_n(z,\Lambda_k))$ are determined by $U_1(z,\Lambda_k)$. Clearly $(\tilde{U}_1(z,\Lambda_k,...,\tilde{U}_n(z,\Lambda_k))$ satisfies

$$\Delta\tilde{U}_i(y,\Lambda_k)+|y|^{2\gamma_i^k}e^{\sum_j a_{ij}\tilde{U}_j(y,\Lambda_k)}=0,\quad \text{in}\quad \mathbb{R}^2,\quad i=1,..,n.$$

Let $\Phi_i^k=W_i^k-\tilde{U}_i(\cdot,\Lambda_k)$. By (1.5) and the definition of W_i^k we have

$$\Delta W_i^k+|y|^{2\gamma_i^k}h_i^k(\varepsilon_k y)e^{\sum_j a_{ij}W_j^k(y)}=0,\quad |y|\le\varepsilon_k^{-1}.$$

Hence the equation for $(\Phi_1^k,...,\Phi_n^k)$ can be written as

$$\Delta\Phi_i^k(y)+|y|^{2\gamma_i^k}e^{\xi_i^k(y)}(\sum_j a_{ij}\Phi_j^k(y))=O(\varepsilon_k|y|)|y|^{2\gamma_i^k}e^{\sum_j a_{ij}W_j^k}\tag{3.4}$$

where, by the mean value theorem,

$$e^{\xi_i^k}=\frac{e^{\sum_j a_{ij}W_j^k}-e^{\sum_j a_{ij}\tilde{U}_j(\cdot,\Lambda_k)}}{\sum_j a_{ij}(W_j^k-\tilde{U}_j(\cdot,\Lambda_k))}=\int_0^a e^{\sum_j a_{ij}(tW_j^k+(1-t)\tilde{U}_j(\cdot,\Lambda_k))}dt.$$

By Theorem 4.1 and Theorem 4.2 of [21], $e^{\xi_i^k}$ converges uniformly to $e^{v_i(\cdot,\Lambda)}$ over all compact subsets of $\mathbb{R}^2$, moreover,

$$|y|^{2\gamma_i^k}e^{\xi_i^k(y)}=O(1+|y|)^{-4-2\gamma_{n+1-i}+o(1)},\quad |y|\le\varepsilon_k^{-1}.\tag{3.5}$$

Also by Theorem 4.1 and Theorem 4.2 of [21] we can estimate the right hand side of (3.4). Thus (3.4) can be written as

$$\Delta\Phi_i^k+|y|^{2\gamma_i^k}e^{\xi_i^k(y)}(\sum_{j=1}^n a_{ij}\Phi_j^k)=\frac{O(\varepsilon_k)}{(1+|y|)^{3+2\gamma_{n+1-i}+o(1)}},\ \text{in}\ |y|\le\varepsilon_k^{-1}.\tag{3.6}$$

It is immediate to observe that the oscillation of Φ_i^k on $\partial B_{\varepsilon_k^{-1}}$ is finite. Thus for convenience we use the following functions to eliminate the oscillation of Φ_i^k on $\partial B_{\varepsilon_k^{-1}}$:

$$\begin{cases}\Delta\psi_i^k=0, & \text{in}\quad B_{\varepsilon_k^{-1}},\\ \psi_i^k=\Phi_i^k-\frac{1}{2\pi\varepsilon_k^{-1}}\int_{\partial B_{\varepsilon_k^{-1}}}\Phi_i^k, & \text{on}\quad \partial B_{\varepsilon_k^{-1}}.\end{cases}$$

Clearly,

$$|\psi_i^k(y)|\le C\varepsilon_k|y|,\quad |y|\le\varepsilon_k^{-1}.\tag{3.7}$$

Let $\tilde{\Phi}_i^k=\Phi_i^k-\psi_i^k$, then by (3.6) and (3.7) we have

$$\Delta\tilde{\Phi}_i^k+|y|^{2\gamma_i^k}e^{\xi_i^k(y)}(\sum_{j=1}^n a_{ij}\tilde{\Phi}_j^k)=\frac{O(\varepsilon_k)}{(1+|y|)^{3+2\gamma_{n+1-i}+o(1)}},\ \text{in}\ |y|\le\varepsilon_k^{-1}\tag{3.8}$$

and

$$\tilde{\Phi}_1^k(p_l)=O(\varepsilon_k),\quad l=1,...,n^2+2n.\tag{3.9}$$

Set

$$H_k=\max_i\max_{|y|\le\varepsilon_k^{-1}}\frac{|\tilde{\Phi}_i^k(y)|}{(1+|y|)^\sigma\varepsilon_k^\sigma}$$

for any fixed $\sigma \in (0, \min\{1, 2\mu_1, .., 2\mu_n\})$. Our goal is to show that H_k is bounded. We prove this by contradiction. Suppose $H_k \to \infty$ and let y_k be where the maximum is attained. Let

$$\hat{\Phi}_i^k(y) = \frac{\tilde{\Phi}_i^k(y)}{H_k(1+|y_k|)^\sigma \varepsilon_k^\sigma}.$$

This definition immediately implies

$$|\hat{\Phi}_i^k(y)| \le \frac{(1+|y|)^\sigma}{(1+|y_k|)^\sigma}. \tag{3.10}$$

Next we write the equation for $(\hat{\Phi}_1^k, .., \hat{\Phi}_n^k)$ as

$$\Delta \hat{\Phi}_i^k + |y|^{2\gamma_i^k} e^{\xi_i^k} (\sum_j a_{ij} \hat{\Phi}_j^k) = \frac{O(\varepsilon_k^{1-\sigma})(1+|y|)^{-3-2\gamma_{n+1-i}+o(1)}}{H_k(1+|y_k|)^\sigma},$$

and we observe that $\hat{\Phi}_i^k$ has no oscillation on $\partial B_{\varepsilon_k^{-1}}$.

We first consider when, along a subsequence, $y_k \to y^*$. In this case, $(\hat{\Phi}_1^k, .., \hat{\Phi}_n^k)$ converges to $(\Phi_1, ..., \Phi_n)$ that satisfies

$$\begin{cases} \Delta \Phi_i + e^{u_i} \sum_j a_{ij} \Phi_j = 0, \quad \text{in} \quad \mathbb{R}^2, \quad i = 1, ..., n \\ |\Phi_i(y)| \le C(1+|y|)^\sigma, \quad i =, 1.., n, \quad \sigma \in (0, \min\{1, 2\mu_1, ..., 2\mu_n\}), \\ \Phi_1(p_l) = 0, \quad l = 1, ..., n^2+2n. \end{cases} \tag{3.11}$$

where $u_i(y) = v_i(y) + 2\gamma_i \log |y|$. Note that the last equation in (3.11) holds because of (3.9). First by the first two equations of (3.11) and Lemma 2.1 we have (2.7). Then by (3.2) we have

$$\mathbf{M}\hat{\Theta}(p_l) = 0, \quad l = 1, ..., n^2+2n.$$

Since $\mathbf{M}$ is invertible, we have

$$m_{00} = ... = m_{n-1,n-1} = m_{1,0}^1 = ... = m_{n,n-1}^2 = 0.$$

Thus $\Phi_1 \equiv 0$, which means $\Phi_i \equiv 0$ for all i. This is a contradiction to $|\Phi_i(y^*)| = 1$ for some i.

The only remaining case we need to consider is when $y_k \to \infty$. To get a contradiction we evaluate

$$\begin{aligned} & \hat{\Phi}_i^k(y_k) - \hat{\Phi}_i^k(0) \\ = & \int_{B_{\varepsilon_k^{-1}}} (G_k(y_k, \eta) - G_k(0, \eta)) \Big(|\eta|^{2\gamma_i^k} e^{\xi_i^k(\eta)} (\sum_j a_{ij} \tilde{\Phi}_j^k(\eta)) \\ & + \frac{O(\varepsilon_k^{1-\sigma})(1+|\eta|)^{-3-2\gamma_{n+1-i}+o(1)}}{H_k(1+|y_k|)^\sigma} \Big) d\eta \end{aligned} \tag{3.12}$$

where G_k is the Green's function on $B_{\varepsilon_k^{-1}}$ with Dirichlet boundary condition. To evaluate the right hand side of the term above we use (3.10),(3.5) and the following estimate of the Green's function (see [28] for the proof) :

For $y\in\Omega_k$, let

$$\begin{aligned}\Sigma_1 &= \{\eta\in\Omega_k; \quad |\eta|<|y|/2 \quad\}\\ \Sigma_2 &= \{\eta\in\Omega_k; \quad |y-\eta|<|y|/2 \quad\}\\ \Sigma_3 &= \Omega_k\setminus(\Sigma_1\cup\Sigma_2).\end{aligned}$$

Then for $|y|>2$,

$$|G_k(y,\eta)-G_k(0,\eta)|\le\begin{cases}C(\log|y|+|\log|\eta||), & \eta\in\Sigma_1,\\ C(\log|y|+|\log|y-\eta||), & \eta\in\Sigma_2,\\ C|y|/|\eta|, & \eta\in\Sigma_3.\end{cases}\tag{3.13}$$

Then it is easy to see that the right hand side of (3.12) is $o(1)$. Thus we obtain a contradiction to the definition of y_k since there exists an i such that $|\Phi_i^k(y_k)|=1$. Theorem 1.1 is established. □

4. THE DETERMINATION OF $p_1,..,p_{n^2+2n}$

In this section we explain how p_1, ... p_{n^2+2n} are chosen to make the matrices $\mathbf{M}$ and $\mathbf{M_1}$ both invertible.

First we list some facts that can be verified easily by direct computation: Let

$$c_{ij}^1=Re(c_{ij}),\quad c_{ij}^2=Im(c_{ij}).$$

Using (2.4) (recalling that $\tilde{U}_1=-\log(\lambda_0+\sum_{i=1}^n\lambda_i|P_i(z)|^2)$) we have

$$\begin{aligned}\frac{\partial\tilde{U}_1}{\partial\lambda_0} &= \frac{\frac{\lambda_n}{\lambda_0}|P_n(z)|^2-1}{\lambda_0+\sum_i\lambda_i|P_i(z)|^2},\\ \frac{\partial\tilde{U}_1}{\partial\lambda_i} &= \frac{\frac{\lambda_n}{\lambda_i}|P_n(z)|^2-|P_i(z)|^2}{\lambda_0+\sum_i\lambda_i|P_i(z)|^2}, \quad i=1,...,n-1,\\ \frac{\partial\tilde{U}_1}{\partial c_{ij}^1} &= -\frac{2\lambda_iRe(z^{\mu_1+...+\mu_j}\bar{P}_i)}{\lambda_0+\sum_i\lambda_i|P_i(z)|^2} \quad j<i,\quad i=1,..,n\\ \frac{\partial\tilde{U}_1}{\partial c_{ij}^2} &= \frac{2\lambda_iIm(z^{\mu_1+...+\mu_j}\bar{P}_i)}{\lambda_0+\sum_i\lambda_i|P_i(z)|^2} \quad j<i,\quad i=1,..,n\end{aligned}\tag{4.1}$$

It is easy to verify that for $|z|$ large

$$\begin{aligned}&z^{\mu_1+..+\mu_j}\bar{P}_i\\ =\;& |z|^{2\mu_1+..+2\mu_j+\mu_{j+1}+..+\mu_i}\left(e^{-\sqrt{-1}(\mu_{j+1}+..+\mu_i)\theta}+O(|z|^{-\delta})\right)\end{aligned}$$

for some $\delta>0$ that depends only on $\mu_1,...,\mu_n$. Thus for $|z|$ large

$$\begin{aligned}&\frac{\partial\tilde{U}_1}{\partial c_{ij}^1}(z)(\lambda_0+\sum_k\lambda_k|P_k(z)|^2)\\ =\;& -2\lambda_i|z|^{2\mu_1+..+2\mu_j+\mu_{j+1}+..+\mu_i}\left(\cos((\mu_{j+1}+..+\mu_i)\theta)+O(|z|^{-\delta})\right)\end{aligned}\tag{4.2}$$

$$
\begin{aligned}
(4.3) \qquad & \frac{\partial \tilde{U}_1}{\partial c_{ij}^2}(z)(\lambda_0+\sum_k \lambda_k |P_k(z)|^2) \\
= \quad & -2\lambda_i |z|^{2\mu_1+..+2\mu_j+\mu_{j+1}+..+\mu_i}\Big(\sin((\mu_{j+1}+..+\mu_i)\theta)+O(|z|^{-\delta})\Big).
\end{aligned}
$$

By the definition of $P_i(z)$ in (2.2),

$$
|P_i(z)|^2 = |z|^{2\mu_1+..+2\mu_i}(1+O(|z|^{-\delta})). \tag{4.4}
$$

We also note that

$$
\frac{\partial \tilde{U}_1}{\partial \lambda_i} = \frac{\lambda_0}{\lambda_i}\frac{\partial \tilde{U}_1}{\partial \lambda_0} + \frac{\frac{\lambda_0}{\lambda_i} - |P_i(z)|^2}{\lambda_0+\sum_i \lambda_i |P_i(z)|^2}, \quad i=1,...,n-1.
$$

First we explain how to choose the n^2+2n points to make $\mathbf{M}$ invertible ($\mathbf{M}$ is defined in (3.1)). Clearly the factor $\lambda_0+\sum_k \lambda_k |P_k(p_l)|^2$ can be taken out from the $l-th$ column, thus for $|p_l|>>1$, $\mathbf{M}$ is invertible if and only if

$$
\mathbf{M_2} := (\Theta_1(p_1),...,\Theta_1(p_{n^2+2n}))
$$

is invertible, where, according to (4.2), (4.3) and (4.4)

$$
\begin{aligned}
& \Theta_1(p_l) \\
= \quad & \Big(|p_l|^{2a_n}(1+O(\frac{1}{|p_l|^\delta}), |p_l|^{2a_{n-1}+a_{n,n-1}}\cos(a_{n,n-1}\theta_l)(1+O(\frac{1}{|p_l|^\delta})), \\
& |p_l|^{2a_{n-1}+a_{n,n-1}}\sin(a_{n,n-1}\theta_l)(1+O(\frac{1}{|p_l|^\delta})), ...\qquad \Big)'
\end{aligned}
$$

where

$$
a_0=0, \quad a_i=\mu_1+..+\mu_i \ (i=1,...,n), \quad a_{ij}=\mu_{j+1}+..+\mu_i \ (i=1,..,n, j<i),
$$

$\theta_l = arg(p_l)$, $\delta>0$ only depends on μ_1,..,μ_n. Note that $a_{ij}=a_i-a_j$ and $2a_j+a_{ij}=a_i+a_j$. The powers of $|p_l|$ are arranged in a non-increasing manner (so the largest power is $2a_n$, the second largest power is $2a_{n-1}+a_{n,n-1}$, etc). The powers of $|p_l|$ are either $2a_i$ or a_i+a_j. Here we note that some powers appear only once (for example $2a_n$). Some powers appear only twice (for example $2a_{n-1}+a_{n,n-1}$), and it is possible that some powers appear more than twice.

Let

$$
p_l = s^{1+\varepsilon l} N e^{\sqrt{-1}\theta_l}, \quad l=1,..,n^2+2n
$$

where $N>>s>>1>>\varepsilon>0$ are constants only depending on $\mu_1,...,\mu_n$, n. The angles θ_l also only depend on these parameters. We shall determine these constants and angles. On each row a power of N can be taken out, therefore $\mathbf{M_2}$ is invertible iff

$$
(\Theta_2(p_1),..,\Theta_2(p_{n^2+2n}))
$$

is invertible, where

$$\Theta_2(p_l)=\Big(s^{2a_n(1+\varepsilon l)}(1+O(\frac{1}{|p_l|^\delta})),$$
$$s^{(2a_{n-1}+a_{n,n-1})(1+\varepsilon l)}\cos(a_{n,n-1}\theta_l)(1+O(\frac{1}{|p_l|^\delta})),$$
$$s^{(2a_{n-1}+a_{n,n-1})(1+\varepsilon l)}\sin(a_{n,n-1}\theta_l)(1+O(\frac{1}{|p_l|^\delta})),...,\Big)'$$

Hence for fixed s, if N is sufficiently large, $\mathbf{M_2}$ is invertible iff the following matrix is invertible:

$$\mathbf{M_3}=(\Theta_3(p_1),...,\Theta_3(p_{n^2+2n}))$$

where

$$\begin{aligned}&\Theta_3(p_l)\\ =\;&(s^{2a_n(1+\varepsilon l)},s^{(2a_n+a_{n,n-1})(1+\varepsilon l)}\cos(a_{n,n-1}\theta_l),s^{(2a_n+a_{n,n-1})(1+\varepsilon l)}\sin(a_{n,n-1}\theta_l),..)'.\end{aligned}$$

We start with the largest entry in $\mathbf{M_3}$: $s^{2a_n(1+\varepsilon(n^2+2n))}$, which is in row one and column n^2+2n. We divide row 1 by $s^{2a_n(1+\varepsilon(n^2+2n))}$ (we call this **operation one**), then the entries in row one become

$$s^{2a_n\varepsilon(l-n^2-2n)},\ \text{for } l=1,...,n^2+2n.$$

Next we subtract a multiple of row one from other rows to eliminate the last entry in each row (we call this **operation two**). For any entry in the cofactor matrix of 1, if before **operation two** it is of the form s^aA, it becomes $s^a(A+O(s^{-\delta}))$ after **operation two**. Indeed, for example, let $s^{2a_{i_0}(1+\varepsilon l)}$ be an entry before **operation two**. The last entry of the same row is $s^{2a_{i_0}(1+\varepsilon(n^2+2n))}$. In **operation two** we subtract the $s^{2a_{i_0}(1+\varepsilon(n^2+2n))}$ multiple of the first row. The entry in row 1 and the same column of $s^{2a_{i_0}(1+\varepsilon l)}$ is $s^{2a_n\varepsilon(l-n^2-2n)}$. Thus after **operation two** $s^{2a_{i_0}(1+\varepsilon l)}$ becomes

$$\begin{aligned}&s^{2a_{i_0}(1+\varepsilon l)}-s^{2a_0(1+\varepsilon(n^2+2n))}s^{2a_n\varepsilon(l-n^2-2n)}\\ =\;&s^{2a_{i_0}(1+\varepsilon l)}(1-s^{(2a_0-2a_n)\varepsilon(n^2+2n-l)})\\ =\;&s^{2a_{i_0}(1+\varepsilon l)}(1+O(s^{-\delta}))\end{aligned}$$

where we have used $a_{i_0}<a_n$.

Similarly if an entry before **operation two** is

$$s^{(2a_j+a_{ij})(1+\varepsilon l)}\cos(a_{ij}\theta_l),$$

after **operation two** it becomes

$$s^{(2a_j+a_{ij})(1+\varepsilon l)}(\cos(a_{ij}\theta_l)+O(s^{-\delta})),$$

for some $\delta>0$. Eventually s will be chosen large to eliminate the influence of all the perturbations.

Our strategy is to use high powers of s to simplify the matrix. After the aforementioned row operations it is clear that we only need to consider the cofactor matrix of 1, which we use A_1 to denote. The highest power of s in A_1 is shared by two entries:

$$s^{(2a_{n-1}+a_{n,n-1})(1+\varepsilon(n^2+2n-1))}(\cos(a_{n,n-1}\theta_{n^2+2n-1})+O(s^{-\delta}))$$

and

$$s^{(2a_{n-1}+a_{n,n-1})(1+\varepsilon(n^2+2n-1))}(\sin(a_{n,n-1}\theta_{n^2+2n-1})+O(s^{-\delta})).$$

We recall that the previous one is in row one of A_1. We choose $\theta_{n^2+2n-1}=0$. In A_1 we divide the first row by $s^{(2a_{n-1}+a_{n,n-1})(1+\varepsilon(n^2+2n-1))}$, then the largest entry in row 1 of A_1 becomes $1+O(s^{-\delta})$. We then subtract from other rows a multiple of the first row to eliminate the last entry of each row. By the same reason as before, after these row operations the invertibility of A_1 is equivalent to the invertibility of the cofactor matrix A_2 of $1+O(s^{-\delta})$, a $(n^2+2n-2)\times(n^2+2n-2)$ matrix which is barely changed after these transformations. In fact, each entry in A_2 is only multiplied a factor $1+O(s^{-\delta})$ after these transformations.

As we continue this process we face three situations. If the highest power of s without the ε part is not repeated, we just apply the same type of row operations as in **operation one** and **operation two**. If the highest power of s without the ε part is shared by only two entries (one is a cosine term, one is a sine term), we just take the corresponding angle to be 0, so the cosine term will dominate all other terms and this case is similar to the previous case. Finally we may run into the following situation: A power of s without the ε part is shared by more than two indices:

$$\exists i_0, j_0, i_1, j_1, \text{ such that } 2a_{j_0}+a_{i_0,j_0}=2a_{j_1}+a_{i_1,j_1}, \quad j_0\neq j_1.$$
$$\exists i_0, j_0, i_1, \text{ such that } 2a_{j_0}+a_{i_0,j_0}=2a_{j_1}.$$

In this case we first prove the following simple but important lemma.

Lemma 4.1. *There exist $\varepsilon_0>0$ that depends only on $\mu_1,..,\mu_n$ and n such that for $\varepsilon\in(0,\varepsilon_0)$,*

$$\frac{|p_a|^{l_1}}{|p_b|^{l_2}}\to\infty \text{ as } s\to\infty, \forall a,b\in\{1,...,n^2+2n\}. \tag{4.5}$$

where l_1, l_2 are two numbers in the set $\{2a_1,..,2a_n,...,2a_j+a_{ij},...,\}$ that satisfy $l_1>l_2$.

Proof of Lemma 4.1: Suppose $|p_a|^{l_1}=s^{(1+\varepsilon a)l_1}$, $|p_b|^{l_2}=s^{(1+\varepsilon b)l_2}$, it is easy to see that for all $a,b\in\{1,..,n^2+2n\}$, $(1+\varepsilon a)l_1>(1+\varepsilon b)l_2$ if $l_1>l_2$ and ε is sufficiently small. The smallness of ε is clearly determined by the set

$$\{2a_1,..,2a_n,...,2a_j+a_{ij},...,\}.$$

Lemma 4.1 is established. □

Next we prove two more Calculus lemmas.

Lemma 4.2. *Let $N_1 < N_2 < ... < N_k$ be positive numbers. Then there exist $\theta_1, \theta_2, ..., \theta_k$ such that the following matrix*

$$M_{Nk} = \begin{pmatrix} 1 & ... & ... & ... & ... & 1 \\ \sin(N_1\theta_1) & ... & ... & ... & ... & \sin(N_1\theta_k) \\ \cos(N_1\theta_1) & ... & ... & ... & ... & \cos(N_1\theta_k) \\ ... & ... & ... & ... & ... & ... \\ \sin(N_k\theta_1) & ... & ... & ... & ... & \sin(N_k\theta_k) \\ \cos(N_k\theta_1) & ... & ... & ... & ... & \cos(N_k\theta_k) \end{pmatrix}$$

satisfies

$$0 < c_1(N_1, ..., N_k) < |det(M_{Nk})| < c_2(N_1, ..., N_k).$$

for positive constants c_1 and c_2 that only depend on $N_1, ..., N_k$.

Proof of Lemma 4.2: We use the Taylor expansion of $\sin(N\theta)$ and $\cos(N\theta)$:

$$\sin(N_i\theta_j) = \sum_{l=1}^{k}(-1)^{l+1}\frac{(N_i\theta_j)^{2l-1}}{(2l-1)!} + O((N_i\theta_j)^{2k+1}).$$

$$\cos(N_i\theta_j) = \sum_{l=0}^{k}(-1)^{l}\frac{(N_i\theta_j)^{2l}}{(2l)!} + O((N_i\theta_j)^{2k+2}).$$

We apply the following elementary operations on M_{Nk}: First we subtract a multiple of the first row from other odd number rows to eliminate the first order terms of θ_i ($i = 1, .., k$). After the cancelation it is easy to see that, the entry of row $2j-1$ ($j > 1$) and column r ($r > 1$) is of the form

$$\sum_{l=2}^{k}(-1)^{l+1}(a_{l,j}\theta_r)^{2l-1} + O(\theta_r)^{2k+1}$$

for some positive constant $a_{l,j}$, which satisfies $a_{l,j} < a_{l,j+1}$. In the second step we use row three to eliminate all the $O(\theta^3)$ terms of other odd number rows starting from row 5. After the second step, the entry of row $2j-1$ ($j > 2$) and column r ($r > 2$) is of the form

$$\sum_{l=3}^{k}(-1)^{l+1}(\tilde{a}_{l,j}\theta_r)^{2l-1} + O(\theta_r)^{2k+1},$$

with $\tilde{a}_{l,j} > 0$ satisfying $\tilde{a}_{l,j} < \tilde{a}_{l,j+1}$.

After $k-1$ such operations we see that the entry of row $2j-1$ and column r is a multiple of θ_r^{2j-1} plus lower order terms. Clearly we can use the terms on row $2k-1$ to eliminated all the $O(\theta^{2k-1})$ terms in other odd number rows. Then we can use row $2k-3$ to remove the $O(\theta^{2k-3})$ terms in other odd number rows. After such operations the entry of row $2j-1$ and column r is $C\theta_r^{2j-1} + O(\theta_r^{2k+1})$. Similar operations can be applied to even number rows. Thus after a finite number of elementary row operations (including multiplying a constant on each row) the matrix M_{Nk} is transformed to

$$\tilde{M}_{Nk} = \begin{pmatrix} 1 & 1 & \dots & \dots & \dots & 1 \\ \theta_1 & \theta_2 & \dots & \dots & \dots & \theta_k \\ \theta_1^2 & \theta_2^2 & \dots & \dots & \dots & \theta_k^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \theta_1^{2k-1} & \theta_2^{2k-1} & \dots & \dots & \dots & \theta_k^{2k-1} \\ \theta_1^{2k} & \theta_2^{2k} & \dots & \dots & \dots & \theta_k^{2k} \end{pmatrix} + \text{ a minor matrix }.$$

The (i,j) entry of the second matrix is $O(\theta_i^{2k+1})$. Now we choose $\theta_i = i\varepsilon$ for some $\varepsilon > 0$ that depends only on $N_1, ..., N_k$. For ε sufficiently small, $\tilde{M}_{Nk}$ is invertible if and only if the first matrix is invertible. Finally we observe that the first matrix of $\tilde{M}_{Nk}$ is a Vandermonde matrix. Lemma 4.2 is established. □

The proof of the following lemma is very similar and is omitted.

Lemma 4.3. *Let $N_1 < N_2 < ... < N_k$ be positive numbers. Then there exist $\theta_1, \theta_2, ..., \theta_k$ such that the following matrix*

$$M_{2Nk} = \begin{pmatrix} \sin(N_1\theta_1) & \dots & \dots & \dots & \dots & \sin(N_1\theta_k) \\ \cos(N_1\theta_1) & \dots & \dots & \dots & \dots & \cos(N_1\theta_k) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \sin(N_k\theta_1) & \dots & \dots & \dots & \dots & \sin(N_k\theta_k) \\ \cos(N_k\theta_1) & \dots & \dots & \dots & \dots & \cos(N_k\theta_k) \end{pmatrix}$$

satisfies

$$0 < c_1(N_1, ..., N_k) < |det(M_{2Nk})| < c_2(N_1, ..., N_k).$$

for positive constants c_1 and c_2 that only depend on $N_1, ..., N_k$.

Now we go back to the case that after finite steps of reduction, the highest power of s without the ε part is M and is shared by more than 2 indices. Our goal is to make the following matrix invertible:

$$\mathbf{A_2} = \begin{pmatrix} B & C \\ D & F \end{pmatrix} \cdot (1 + O(s^{-d}))$$

where the last term $(1+O(s^{-d}))$ means each entry in $\begin{pmatrix} B & C \\ D & F \end{pmatrix}$ is multiplied by a quantity of the magnitude $1+O(s^{-d})$, even though these quantities are different from one another. C is either of the form

$$\begin{pmatrix} s^{M(1+\varepsilon(l+1))}\cos(N_1\theta_{l+1}) & \dots & s^{M(1+\varepsilon(l+k))}\cos(N_1\theta_{l+k}) \\ s^{M(1+\varepsilon(l+1))}\sin(N_1\theta_{l+1}) & \dots & s^{M(1+\varepsilon(l+k))}\sin(N_1\theta_{l+k}) \\ \dots & \dots & \dots \\ s^{M(1+\varepsilon(l+1))}\cos(N_T\theta_{l+1}) & \dots & s^{M(1+\varepsilon(l+k))}\cos(N_T\theta_{l+k}) \\ s^{M(1+\varepsilon(l+1))}\sin(N_T\theta_{l+1}) & \dots & s^{M(1+\varepsilon(l+k))}\sin(N_T\theta_{l+k}) \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & \dots & 1 \\ s^{M(1+\varepsilon(l+1))}\cos(N_1\theta_{l+1}) & \dots & s^{M(1+\varepsilon(l+k))}\cos(N_1\theta_{l+k}) \\ s^{M(1+\varepsilon(l+1))}\sin(N_1\theta_{l+1}) & \dots & s^{M(1+\varepsilon(l+k))}\sin(N_1\theta_{l+k}) \\ \dots & \dots & \dots \\ s^{M(1+\varepsilon(l+1))}\cos(N_T\theta_{l+1}) & \dots & s^{M(1+\varepsilon(l+k))}\cos(N_T\theta_{l+k}) \\ s^{M(1+\varepsilon(l+1))}\sin(N_T\theta_{l+1}) & \dots & s^{M(1+\varepsilon(l+k))}\sin(N_T\theta_{l+k}) \end{pmatrix}$$

We take the first case as an example. Clearly C gives the leading term of $det(\mathbf{A_2})$. Correspondingly B is of the form

$$B = \begin{pmatrix} s^{M(1+\varepsilon)}\cos(N_1\theta_1) & \dots & s^{M(1+\varepsilon l)}\cos(N_1\theta_l) \\ s^{M(1+\varepsilon)}\sin(N_1\theta_l) & \dots & s^{M(1+\varepsilon l)}\sin(N_1\theta_l) \\ \dots & \dots & \dots \\ s^{M(1+\varepsilon)}\cos(N_T\theta_1) & \dots & s^{M(1+\varepsilon l)}\cos(N_T\theta_l) \\ s^{M(1+\varepsilon)}\sin(N_T\theta_1) & \dots & s^{M(1+\varepsilon l)}\sin(N_T\theta_l) \end{pmatrix}$$

The importance of Lemma 4.1 is that it makes F minor. For matrices D and F, we just write one row vector of (D,F) as a representative:

$$\left(s^{H(1+\varepsilon)},\dots,s^{H(1+\varepsilon l)},s^{H(1+\varepsilon(l+1))},\dots,s^{H(1+\varepsilon(l+k))}\right)$$

where

$$\left(s^{H(1+\varepsilon)},\dots,s^{H(1+\varepsilon l)}\right)$$

is a row vector of D,

$$\left(s^{H(1+\varepsilon(l+1))},\dots,s^{H(1+\varepsilon(l+k))}\right)$$

is a row vector of F. Here we note that $H < M$, other rows of $\mathbf{A_2}$ may have sine or cosine terms.

Now we take $s^{M(1+\varepsilon(l+1))}$ out of the $2k$ rows of (B,C), after this operation B and C become $\tilde{B}$ and $\tilde{C}$:

$$\tilde{B} = \begin{pmatrix} s^{-M\varepsilon l}\cos(N_1\theta_1) & s^{-M\varepsilon(l-1)}\cos(N_1\theta_2) & \dots & s^{-M\varepsilon}\cos(N_1\theta_l) \\ s^{-M\varepsilon l}\sin(N_1\theta_l) & s^{-M\varepsilon(l-1)}\sin(N_1\theta_2) & \dots & s^{-M\varepsilon}\sin(N_1\theta_l) \\ \dots & \dots & \dots & \dots \\ s^{-M\varepsilon l}\cos(N_T\theta_1) & s^{-M\varepsilon(l-1)}\cos(N_T\theta_2) & \dots & s^{-M\varepsilon}\cos(N_T\theta_l) \\ s^{-M\varepsilon l}\sin(N_T\theta_l) & s^{-M\varepsilon(l-1)}\sin(N_T\theta_2) & \dots & s^{-M\varepsilon}\sin(N_T\theta_l) \end{pmatrix}$$

$$\tilde{C} = \begin{pmatrix} \cos(N_1\theta_{l+1}) & s^{M\varepsilon}\cos(N_1\theta_{l+2}) & \dots & s^{M(k-1)\varepsilon}\cos(N_1\theta_{l+k}) \\ \sin(N_1\theta_{l+1}) & s^{M\varepsilon}\sin(N_1\theta_{l+2}) & \dots & s^{M(k-1)\varepsilon}\sin(N_1\theta_{l+k}) \\ \dots & \dots & \dots & \dots \\ \cos(N_T\theta_{l+1}) & s^{M\varepsilon}\cos(N_T\theta_{l+2}) & \dots & s^{M(k-1)\varepsilon}\cos(N_T\theta_{l+k}) \\ \sin(N_T\theta_{l+1}) & s^{M\varepsilon}\sin(N_T\theta_{l+2}) & \dots & s^{M(k-1)\varepsilon}\sin(N_T\theta_{l+k}) \end{pmatrix}$$

After these row operations the major part of $\mathbf{A_2}$ becomes

$$\mathbf{A_3} = (A_{31}, A_{32}) = \begin{pmatrix} \tilde{B} & \tilde{C} \\ D & F \end{pmatrix}$$

Starting from the second column of A_{32} we take away the power of s. For example we divide the second column of A_{32} by $s^{M\varepsilon}$, the third column by $s^{2M\varepsilon}$ and the $k-th$ column by $s^{M(k-1)\varepsilon}$. Now we see the influence of the representative row vector in F. Before this set of column operations it is

$$\left(s^{H(1+\varepsilon(l+1))},...,s^{H(1+\varepsilon(l+k))}\right)$$

After these column operations it becomes (using $H<M$)

$$s^{H(1+\varepsilon(l+1))}\left(1,O(s^{-d}),...,O(s^{-d})\right).$$

Note that this computation is very similar to those in the proof of Lemma 4.1. We use $\tilde{F}$ to represent the new matrix after the column operations on F.

After these column operations, $\tilde{C}$ becomes

$$\tilde{C}_1=\begin{pmatrix} \cos(N_1\theta_{l+1}) & \cos(N_1\theta_{l+2}) & ... & \cos(N_1\theta_{l+k}) \\ \sin(N_1\theta_{l+1}) & \sin(N_1\theta_{l+2}) & ... & \sin(N_1\theta_{l+k}) \\ ... & ... & ... & ... \\ \cos(N_T\theta_{l+1}) & \cos(N_T\theta_{l+2}) & ... & \cos(N_T\theta_{l+k}) \\ \sin(N_T\theta_{l+1}) & \sin(N_T\theta_{l+2}) & ... & \sin(N_T\theta_{l+k}) \end{pmatrix}$$

By Lemma 4.3, $\tilde{C}_1$ is invertible, which means its row vectors are linearly independent. Thus there is a combination of its row vectors to cancel the representative vector in $\tilde{F}$ (just the major part):

$$s^{H(1+\varepsilon(l+1))}\left(1,0,...,0\right).$$

When this same row operation is applied to A_{31}, the representative vector in D:

$$(s^{H(1+\varepsilon)},...,s^{H(1+\varepsilon l)})$$

becomes this after the row transformation:

$$(s^{H(1+\varepsilon)}(1+O(s^{-d})),...,s^{H(1+\varepsilon l)}(1+O(s^{-d})))$$

where we used $H<M$ again. After these elementary operations, B and F are turned into minor matrices. Thus the invertibility of $\mathbf{A_2}$ is reduced to the invertibility of the transformation of D, which is of the same nature of D. This method of reduction can be continued and the construction of p_1, ... p_{n^2+2n} is complete for matrix $\mathbf{M}$.

Since $\mathbf{M_1}$ is very similar to $\mathbf{M}$ and we only require N, s to be large and ε to be small in $\mathbf{M_1}$. Moreover the angles in $\mathbf{M_1}$ is the same as in $\mathbf{M}$. Thus $p_1,...,p_{n^2+2n}$ that make $\mathbf{M}$ invertible also make $\mathbf{M}$ invertible. The construction of $p_1,..,p_{n^2+2n}$ is complete.

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