De Giorgi and Liouville Type Problems in Nonlinear Elliptic Problems

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Classification solutions

Understanding entire solutions of nonlinear elliptic equations in $\mathbb{R}^N$ such as

$$\Delta u + f(u) = 0 \text{in } \mathbb{R}^N,$$

is a basic problem in PDE research. This is the context of various classical results in literature like the Gidas-Ni-Nirenberg theorems on radial symmetry, Liouville type theorems, or the achievements around De Giorgis conjecture.

In the following I shall introduce various nonlinear elliptic problems and the associated De Giorgi/Liouville type problems

1. Problems in Allen-Cahn Equation

1.1 Allen-Cahn Equation

(AC) $$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N$$

The case $N = 1$. The function

$$w(t) := \tanh\left(\frac{t}{\sqrt{2}}\right)$$

connects monotonically -1 and $+1$ and solves

$$w'' + w - w^3 = 0, \quad w(\pm\infty) = \pm 1, \quad w' > 0.$$

Canonical Example

For any $p, \nu \in \mathbb{R}^N$, $|\nu| = 1, \nu_N > 0$, the functions

$$u(x) := w((x - p) \cdot \nu)$$

solve equation (AC) and connects -1 and $+1$ along x_N.

De Giorgi's conjecture (1978): *Let u be a bounded solution of equation*

$$\text{(AC)} \qquad \Delta u + u - u^3 = 0 \quad \textit{in } \mathbb{R}^N,$$

which is monotone in one direction, say $\partial_{x_N} u > 0$. Then, **at least** *when $N \leq 8$, there exist p, ν such that*

$$u(x) = w(\,(x-p)\cdot\nu).$$

Resolution of De Giorgi's conjecture

- True when $N = 2$, Ghoussoub and Gui (1998)
- True when $N = 3$, Ambrosio and Cabré (2000)
- False when $N \geq 9$, del Pino-Kowalczyk-Wei (2011)
- True when $4 \leq N \leq 8$, Savin (2009) under an additional assumption

$$\lim_{x_N \to \pm\infty} u(x^{'}, x_N) = \pm 1$$

Problem 1.1:

Can one remove or relax the additional assumption

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imposed by Savin?

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Extremely difficult !!!

Gibbons's Conjecture

The additional assumption is related to the following conjecture

Gibbons Conjecture: Let u be a bounded solution of Allen-Cahn equation such that

$$\lim_{x_N \to \pm\infty} u(x^{'}, x_N) = \pm 1, \text{ uniformly in } x^{'}$$

Then the level sets $\{u = \lambda\}$ are all hyperplanes.
Ghoussoub-Gui 1998: N=2, 3
Barlow-Bass-Gui 2000 : for all N
Farina 1999: for all N
Berestycki-Hamel-Moneau 2000: for all N.

1.2. Beyond De Giorgi Conjecture–Stable Solutions

The assumption of monotonicity in one direction for the solution u in De Giorgi conjecture implies a form of stability.
In general, given a bounded solution u to the semilinear elliptic equation

$$\Delta u + f(u) = 0 \quad \text{in } \mathbb{R}^N.$$

We say that u is stable if

$$\int_{\mathbb{R}^N} (|\nabla \varphi|^2 - f^{'}(u)\varphi^2) \geq 0, \ \forall \varphi \in C_0^\infty(\mathbb{R}^N)$$

Stability Conjecture

Stability Conjecture: Let u be a bounded stable solution of equation

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^N.$$

Then the level sets $\{u = \lambda\}$ are all hyperplanes.

- Ghoussoub-Gui, Ambrosio-Cabre: $N = 2$, Stability Conjecture is true
- Pacard-Wei: $N = 8$, Stability Conjecture is False

Problem 1.2: Classify stable solutions of Allen-Cahn Equation in dimensions $N = 3, 4, 5, 6, 7$

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Most likely Case: $N = 3$ (this will prove De Giorgi's conjecture when $N = 4$ without the additional conjecture)
Schoen, Schoen-Simon-Yau: stable minimal surfaces in $\mathbb{R}^3$ must be hyperplane

Question: can one borrow ideas from Schoen, Schoen-Simon-Yau to prove the stability conjecture?

1.3. Beyond De Giorgi Conjecture–Finite Morse Index Solution

After stable solutions, next we study solutions which are not too unstable–Finite Morse Index Solutions.

Morse index of a solution u of (AC), $m(u)$: roughly, the number of negative eigenvalues of the linearized operator, namely those of the problem

$$\Delta\phi + (1 - 3u^2)\phi + \lambda\phi = 0 \quad \phi \in L^\infty(\mathbb{R}^N).$$

Finite Morse Index: $m(u) < +\infty$. Easy to check: $m(u) < +\infty$ if and only if there exists a compact set K such that

$$\int_{\mathbb{R}^N} (|\nabla\varphi|^2 - f^{'}(u)\varphi^2) \geq 0, \ \forall\varphi \in C_0^\infty(\mathbb{R}^N \backslash K)$$

In other words, finite Morse index implies that u stable outside a finite region.

1.3.1: Unstable Solutions of Allen-Cahn Equation in $\mathbb{R}^2$

(AC) $$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^2$$

As we have discussed before, the only stable solution to (AC) in $\mathbb{R}^2$ is given by

$$u = \tanh\left(\frac{(x-p)\cdot\nu}{\sqrt{2}}\right)$$

Next we consider finite-Morse index solutions.

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Next we consider finite-Morse index solutions.
Multiple-end solutions are finite Morse index solutions.

$k-$end solutions

We say that u, a solution of (AC), is a multiple ends with k ends if there exist k oriented half lines $\{\mathbf{a}_j \cdot \mathbf{x} + b_j = 0\}$, $j = 1, \ldots, k$ (for some choice of $\mathbf{a}_j \in \mathbb{R}^2$, $|\mathbf{a}_j| = 1$ and $b_j \in \mathbb{R}$) such that along these half lines and away from a compact set K containing the origin, the solution is asymptotic to $H(\mathbf{a}_j \cdot \mathbf{x} + b_j)$, that is there exist positive constants C, c such that:

$$\|u(\mathbf{x}) - \sum_{j=1}^{k} w(\mathbf{a}_j \cdot \mathbf{x} + b_j)\|_{L^\infty(\mathbb{R}^2 \setminus K)} \leq Ce^{-c|\mathbf{x}|}.$$

$4-$end Solutions I: Saddle Solution

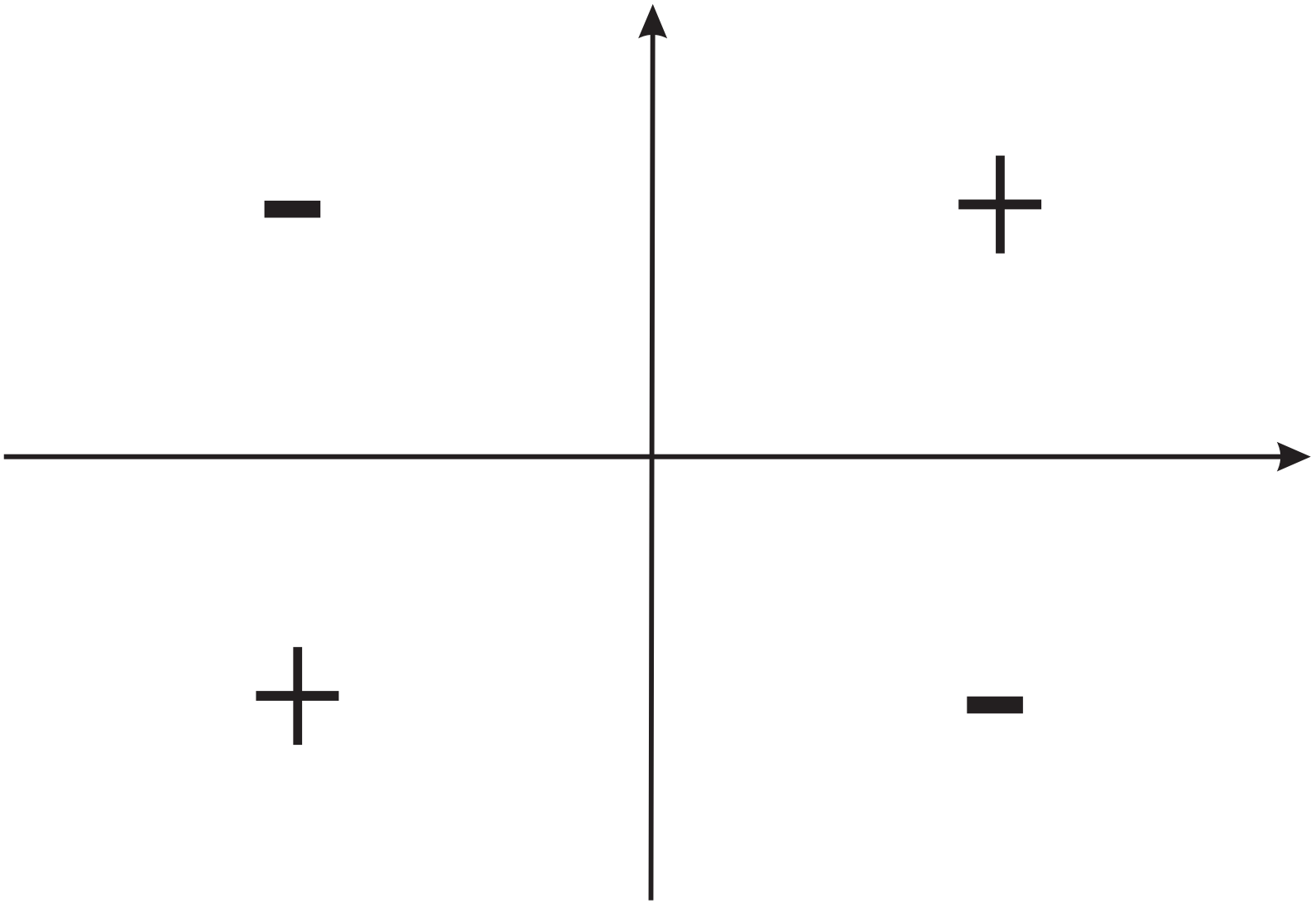
(AC) $$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^2$$

• Dang, Fife, Peletier (1992). The cross saddle solution: $u(x_1, x_2) > 0$ for $x_1, x_2 > 0$,

$$u(x_1, x_2) = -u(-x_1, x_2) = -u(x_1, -x_2).$$

Nodal set two lines (4 ends). Super-subsolutions in first quadrant.

Two orthogonal lines



cross-solution

$4-$end Solutions II: Two Parallel Lines

[del Pino, Kowalczyk, Pacard, Wei (2009)]If f satisfies

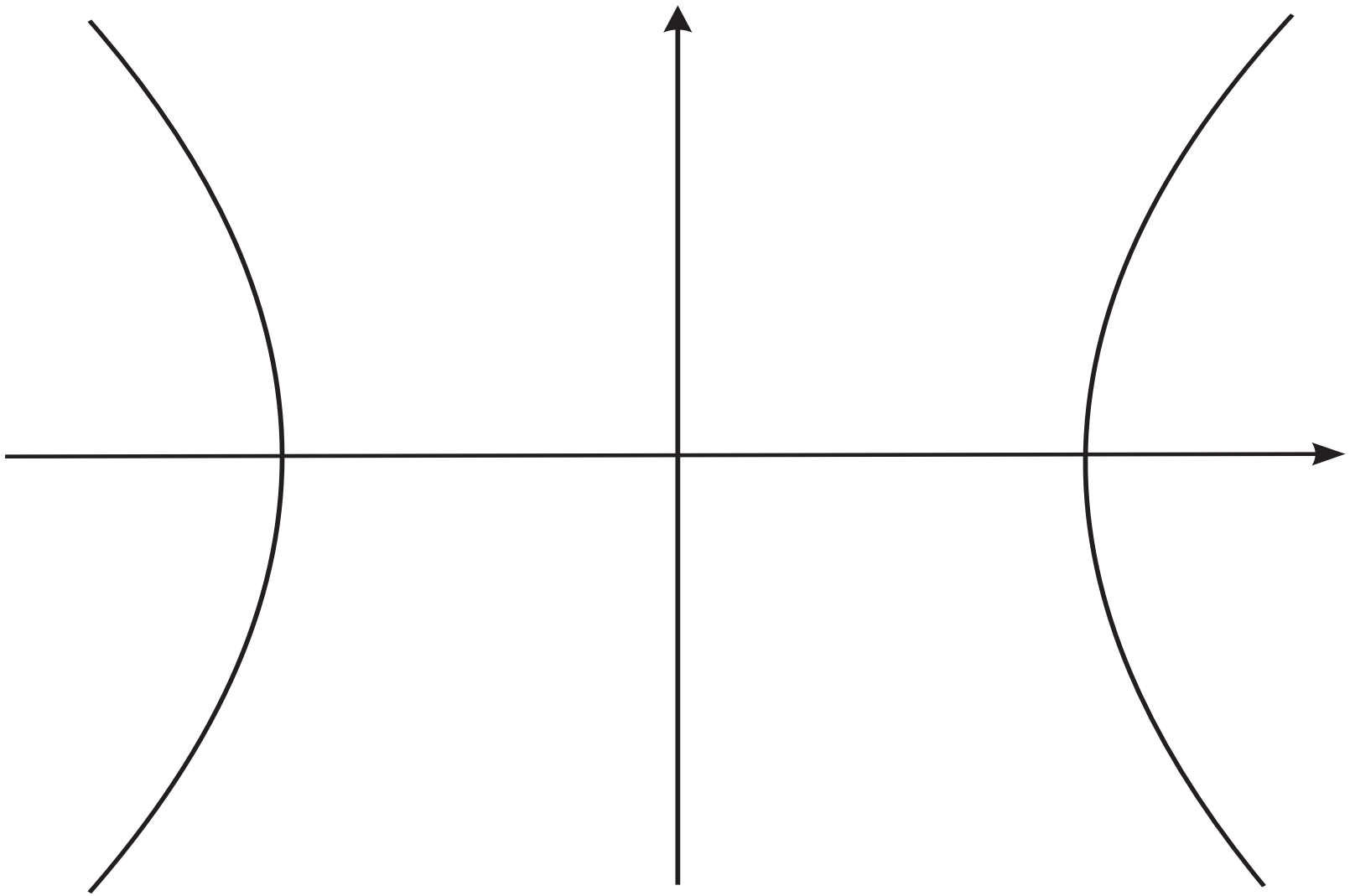
$$\frac{\sqrt{2}}{24} f''(z) = e^{-2\sqrt{2} f(z)}, \quad f'(0) = 0,$$

and $f_\alpha(z) := \sqrt{2} \log \frac{1}{\alpha} + f(\alpha z)$, then there exists a solution u_α to (AC) in $\mathbb{R}^2$ with

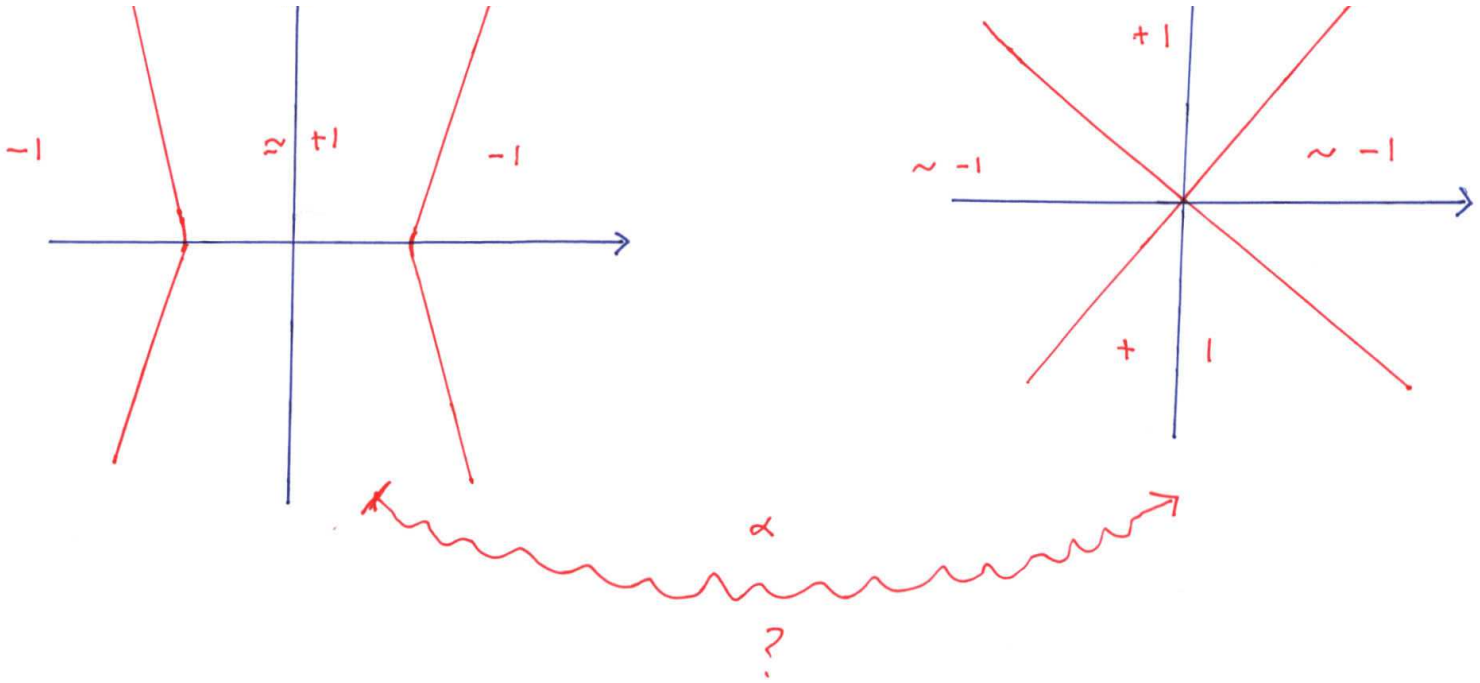
$$\boxed{u_\alpha(x_1, x_2) = w(x_1 + f_\alpha(x_2)\,) + w(x_1 - f_\alpha(x_2)\, - 1\, + o(1)}$$

as $\alpha \to 0^+$. Here $w(s) = \tanh(s/\sqrt{2})$.
This solution has 2 transition lines.

$$f(z) = A|z| + B + o(1) \quad \text{as } z \to \pm\infty.$$



Two-end solution



2-line transition layer and 4 end saddle: Do they connect?

Kowalczyk-Liu-Pacard (2012): Given any two lines in $\mathbb{R}^2$, one can find a solution to Allen-Cahn whose zero-level set approaches these two lines. The solution can be parametrized by the angle between the two lines.

This implies,in particular, that, given any two lines in the plane, there are solutions to the Allen-Cahn equation whose zero level sets approach the two lines.

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Question:

Given any k lines in $\mathbb{R}^2$, can one find a solution to Allen-Cahn whose zero-level set approaches these k lines?

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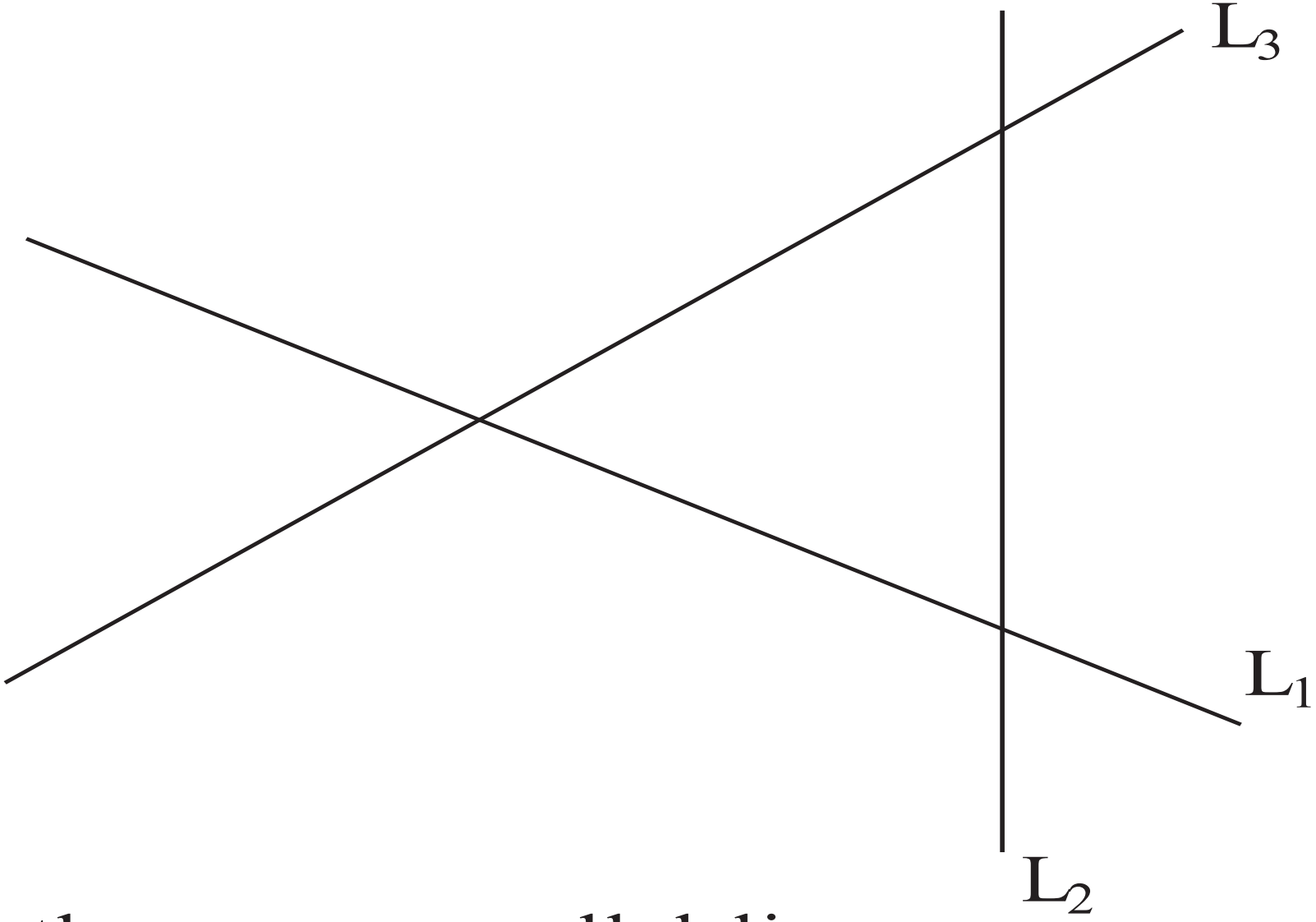
Question:

Given any k lines in $\mathbb{R}^2$, can one find a solution to Allen-Cahn whose zero-level set approaches these k lines?

Answer: Yes, for $k = 2$, and generic for $k > 2$.



three-non-parallel-lines

(Kowalczyk-Pacard-Liu-Wei 2012) Let Σ be a set of k straight lines, $k > 2$. Suppose any two lines in Σ are not parallel and the intersection of any three lines are empty. Also suppose that the angle between any two of these lines is not equal to some exceptional values $\theta_i, i = 1, ..., n$. Then there exists a family of $2k$-end solutions u_ε to (AC) such that the nodal sets of the functions $u_\varepsilon\left(\frac{z}{\varepsilon}\right)$ converge to Σ on any compact set of $\mathbb{R}^2$, as $\varepsilon \to 0$.

Problem 1.3.1.1 Given k lines, some of them may be parallel, construct solutions with zero-level set approaching these k lines.

Key idea: desingularizing using Toda system

Problem 1.3.1.2: It can be shown that finite ends solutions are finite Morse index. Show that finite Morse index solutions must have finite ends

Problem 1.3.1.3: Give an estimate of the Morse index in terms of the number of ends, or genus of the level set $\{u = 0\}$

1.3.2 Finite Morse Index Solutions in $\mathbb{R}^3$

We now consider three dimensional case:

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^3.$$

Finite Morse index solutions in $\mathbb{R}^3$ are generated by

- embedded minimal surfaces with finite total curvature in $\mathbb{R}^3$
- the Toda system in $\mathbb{R}^2$

(del Pino-Kowalczyk-Wei 2009, 2012) Let $N = 3$ and M be a minimal surface embedded, complete with finite total curvature which is nondegenerate. Then there exists a bounded solution u_α of Allen-Cahn equation, defined for all sufficiently small α, such that the level set $\{u_\alpha = 0\} \sim \frac{M}{\alpha}$.
Furthermore, for all sufficiently small α we have

$$m(u_\alpha) \;=\; i(M).$$

In the Costa-Hoffmann-Meeks surface it is known that $i(M) = 2l - 1$ where l is the genus of M.
Corollary: For each positive odd integer k, there exists a solution to the Allen-Cahn equation in $\mathbb{R}^3$ with Morse index k.

Finite Morse Index Solutions

Problem 1.3.2.1 Are there even Morse index solutions?

Solutions in $\mathbb{R}^3$ with Morse Index One

1. The catenoid solution has Morse index one.
2. (Agudelo-del Pino-Wei 2012) There exists an axially symmetric solution with nodal set made up of two components $\Gamma_\pm$ which are graphs of two functions

$$\varphi_\pm(r) \sim \pm 2\log(1+\epsilon r) \pm \log\frac{1}{\epsilon}$$

as $r \to +\infty$. This solution has Morse index 1 and is generated by the Liouville equation

$$\Delta u + e^u = 0 \quad \int_{\mathbb{R}^2} e^u < +\infty$$

Problem 1.3.2.2 Are these two family of Morse index one solutions connected?

Similar to the four-ended solutions in $\mathbb{R}^2$.

Toda System

The Morse index two solution is generated by Liouville equation in $\mathbb{R}^2$: $\Delta f + e^f = 0$ in $\mathbb{R}^2$) which is a special case of Toda system in $\mathbb{R}^2$

$$\begin{cases} \Delta u_1 + 2e^{u_1} - e^{u_2} = 0 \quad \text{in } \mathbb{R}^2, \\ \Delta u_2 + 2e^{u_2} - e^{u_1} - e^{u_3} = 0 \quad \text{in } \mathbb{R}^2, \\ \ldots\ldots \\ \Delta u_k + 2e^{u_k} - e^{u_{k-1}} - e^{u_{k+1}} = 0 \quad \text{in } \mathbb{R}^2, \\ \ldots. \\ \Delta u_N + 2e^{u_N} - e^{u_{N-1}} = 0 \quad \text{in } \mathbb{R}^2 \\ \int_{\mathbb{R}^2} e^{u_i} < +\infty, \quad i = 1, \ldots, N \end{cases}$$

A complete classification is given by Lin-Wei-Ye (2012)

Problem 1.3.2.3 Each family of solutions to the Toda system can generate solutions to (AC).

Open Questions

Problem 1.3.2.4. If u has finite Morse index and $\nabla u(x) \neq 0$ outside a bounded set, then each level set of u must have outside a large ball a finite number of components, each of them asymptotic to either a plane or to a catenoid. After a rotation of the coordinate system, all these components are graphs of functions of the same two variables.

Analogue results for minimal surfaces: Osserman, Schoen

Open Question Two

Problem 1.3.2.5 If u has Morse index equal to one. Then u must be axially symmetric, namely after a rotation and a translation, u is radially symmetric in two of its variables. Its level sets have two ends, both of them catenoidal.

Analogue results for minimal surfaces: Schoen

Allen-Cahn equation in $4 \leq N \leq 7$

Problem 1.3.2.7: Construct nontrivial solutions to the Allen-Cahn equation in the intermediate dimensions

$$\Delta u + u - u^3 = 0 \text{ in } \mathbb{R}^N, N = 3, 4, 5, 6, 7$$

1.4. Beyond de Giorgi Conjecture: Parabolic De Giorgi Conjecture

Next, we move on to the parabolic Allen-Cahn equation
parabolic Allen-Cahn equation (parabolic AC)

$$u_t = \Delta u + u - u^3 \quad \text{in } \mathbb{R}^N \times \mathbb{R} \quad .$$

Parabolic De Giorgi Conjecture:
Consider eternal solutions of parabolic Allen-Cahn equation

$$u_t = \Delta u + u - u^3, (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{1}$$

Assuming their monotonicity in the x_N direction:

$$\partial_{x_N} u > 0, \qquad \lim_{x_N \to \pm\infty} u(x', x_N, t) = \pm 1, \quad t \in \mathbb{R}$$

then u is one-dimensional.

This conjecture is false even in dimension $N = 2$.

- In 2007 Chen, Guo, Hamel, Ninomiya, Roquejoffre showed the existence of solutions to (1) of the form $u(x', x_N - ct) = U(r, x_{N+1} - ct)$, $r = |x'|$, $N \geq 1$. Functions U have paraboloid-like profiles of their nodal sets Γ.
- U satisfies the traveling wave Allen-Cahn

$$(AC)_{TW} \qquad \Delta u + u - u^3 + c u_{x_N} = 0 \quad \text{in } \mathbb{R}^N.$$

- the asymptotic profiles of the fronts are given:

$$\lim_{\substack{x_N \to +\infty \\ (x', x_N) \in \Gamma}} \frac{r^2}{2x_N} = \frac{N-1}{c}, \quad \text{if } N > 1.$$

Traveling Wave De Giorgi Conjecture

Problem 1.4.1

Let u be a bounded solution of equation

$$(AC)_{TW} \qquad \Delta u + u - u^3 + cu_{x_N} = 0 \quad \text{in } \mathbb{R}^N.$$

which satisfies

$$\partial_{x_N} u > 0$$

Then, u must be radially symmetric in $x^{'}$.

Parabolic Allen-Cahn Equation and Mean Curvature Flow

Consider the mean curvature flow for a graph $x_{N+1} = F(x)$:

$$\frac{\partial F}{\partial t} = \sqrt{1+|\nabla F|^2}\nabla\left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}}\right) \tag{2}$$

Mean Curvature Solitons: Graphs which are translated by the mean curvature (MC) flow with constant velocity (say 1) in a fixed direction satisfy:

$$\nabla\left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}}\right) = \frac{1}{\sqrt{1+|\nabla F|^2}}. \tag{3}$$

Rotationally symmetric eternal solution to the MC flow

Altschuler-Wu, Clutterbuck-Schnurer-Schulze (CVPDE 2003).
There exists a unique radially symmetric solution F of (3):

$$F(r) = \frac{r^2}{2(N-1)} - \log r + 1 + O(r^{-1}), \quad r \gg 1. \tag{4}$$

The first term in this asymptotic behavior coincides with the asymptotic behavior of the nodal set of solutions to (AC-TW) found by Chen, Guo, Hamel, Ninomiya, Roquejoffre.

Bernstein Type Conjecture for MC Solitons

Let F be a solution of

$$\nabla\left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}}\right) = \frac{1}{\sqrt{1+|\nabla F|^2}} \text{ in } \mathbb{R}^N. \tag{5}$$

Then F is rotationally or cylindrically symmetric.

A natural critical dimension seems to be $N = 8$. However

X-J Wang , Annals Math. 2012, showed that the existence of non-radial eternal convex graphs when $N \geq 3$.

Revised Traveling Wave De Giorgi Conjecture

Problem 1.4.2

Let u be a bounded solution of equation

$$(AC)_{TW} \qquad \Delta u + u - u^3 + cu_{x_N} = 0 \quad \text{in } \mathbb{R}^N.$$

which satisfies

$$\partial_{x_N} u > 0$$

Then, u must be radially symmetric in $x^{'}$, at least when $N \leq 3$.

Theorem (P.Daskalopoulos, Davila, M. del Pino, Wei, 2014)

- For dimensions $N \geq 8$, there exists a nonconvex nonradial traveling graph to
$$\nabla\left(\frac{\nabla F}{\sqrt{1+|\nabla F|^2}}\right) = \frac{1}{\sqrt{1+|\nabla F|^2}} \text{ in } \mathbb{R}^N.$$
- For dimensions $N \geq 9$, the parabolic De Giorgi Conjecture is not true

2. De Giorgi Type Problems for Allen-Cahn Elliptic Systems

Part 2: An Allen-Cahn System

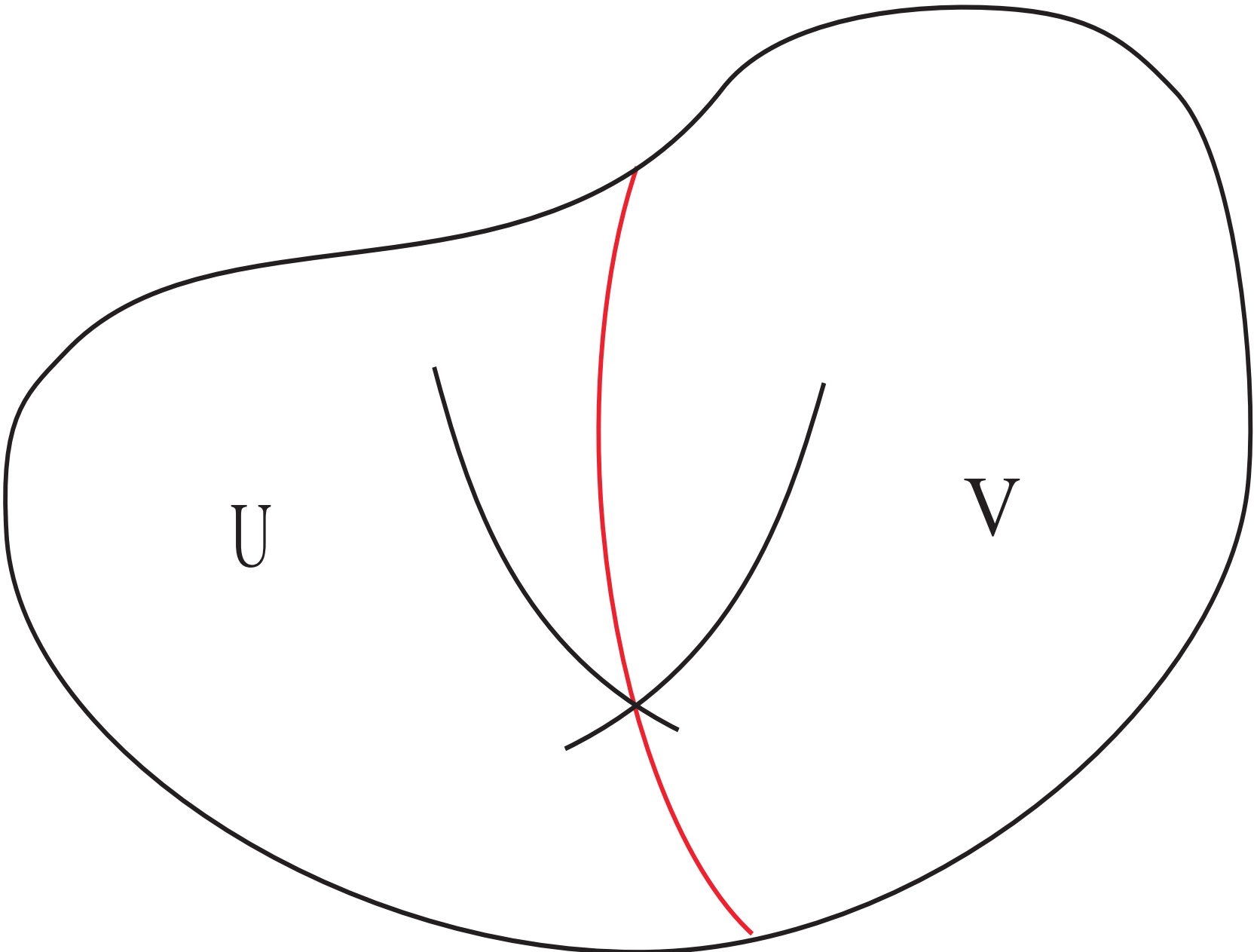
Consider the following Bose-Einstein competition system

$$
\begin{aligned}
&-\Delta u + \alpha u^3 + \Lambda v^2 u = \lambda_1 u && \text{in } \Omega, \\
&-\Delta v + \beta v^3 + \Lambda u^2 v = \lambda_2 v && \text{in } \Omega, \\
&u > 0, \quad v > 0 && \text{in } \Omega, \\
&u = 0, \quad v = 0 && \text{on } \partial\Omega.
\end{aligned}
\tag{6}
$$

Asymptotic Behavior when $\Lambda \to +\infty$

•• Noris-Tavares-Terracini-Verzini (2009): uniform Hölder convergence.

Phase Separation



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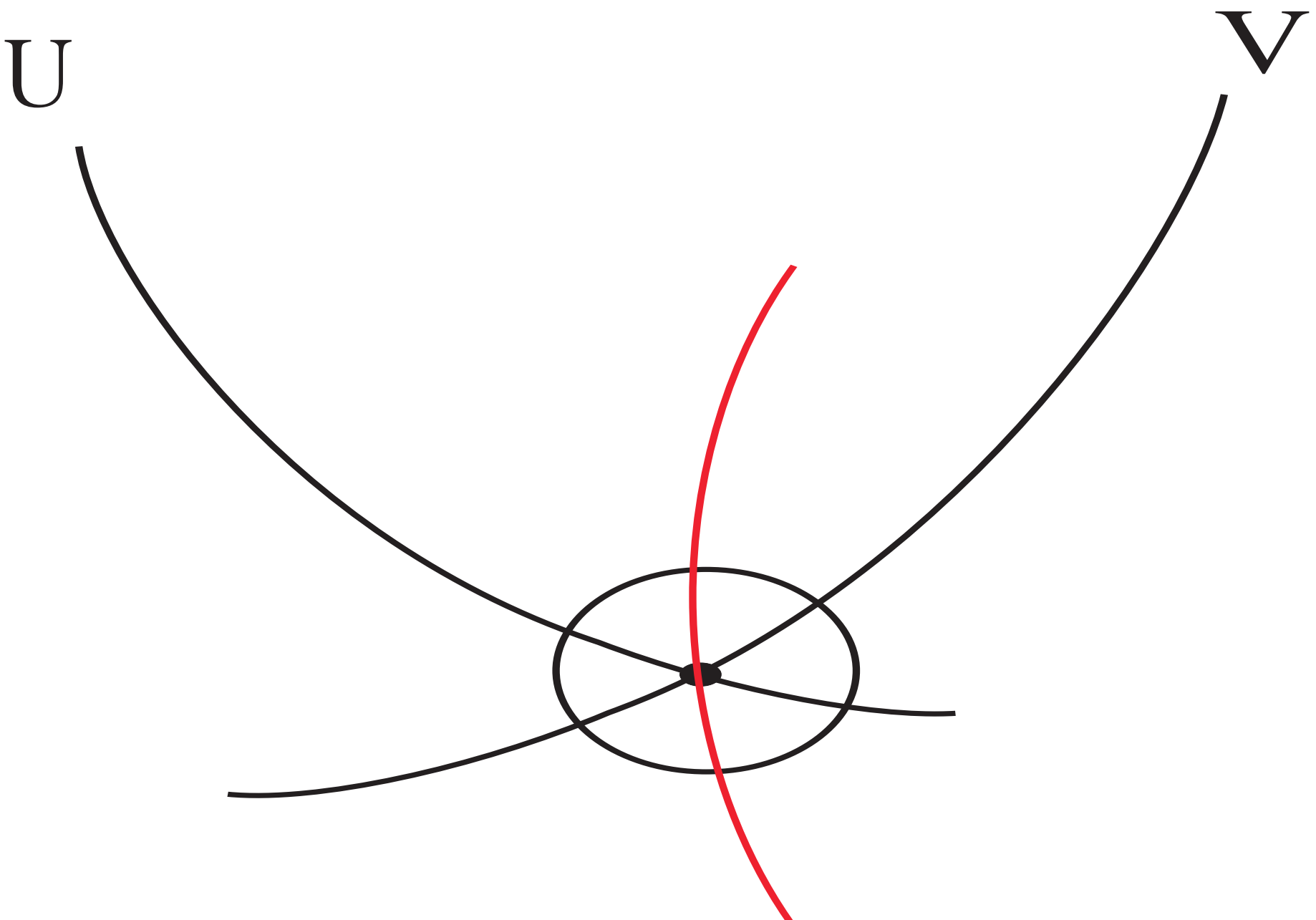

Limiting System

Let $x_\Lambda \in \Omega$ be a point where u_Λ and v_Λ meet, i.e., $u_\Lambda(x_\Lambda) = v_\Lambda(x_\Lambda) = m_\Lambda$.
Then as $\Lambda \to +\infty, x_\Lambda \to x_0 \in \{u_0 = v_0 = 0\}$. Suppose $x_0 \in \Omega$ and we do the following scaling

$$\tilde{u}_\Lambda(y) = \frac{1}{m_\Lambda} u_\Lambda(m_\Lambda y + x_\Lambda), \tilde{v}_\Lambda(y) = \frac{1}{m_\Lambda} v_\Lambda(m_\Lambda y + x_\Lambda). \tag{7}$$

Then $(\tilde{u}_\Lambda, \tilde{v}_\Lambda)$ satisfies

$$-\Delta \tilde{u}_\Lambda + m_\Lambda^4 \alpha \tilde{u}_\Lambda^3 + m_\Lambda \Lambda^4 \tilde{v}_\Lambda^2 \tilde{u}_\Lambda = m_\Lambda^2 \lambda_1 \tilde{u}_\Lambda \quad \text{in } \Omega_\Lambda, \tag{8}$$

$$-\Delta \tilde{v}_\Lambda + m_\Lambda^4 \beta \tilde{v}^3 + m_\Lambda \Lambda^4 \tilde{u}_\Lambda^2 \tilde{v}_\Lambda = m_\Lambda^2 \lambda_1 \tilde{v}_\Lambda \quad \text{in } \Omega_\Lambda. \tag{9}$$

where $\Omega_\Lambda = \frac{\Omega - x_\Lambda}{m_\Lambda}$.

Limiting Elliptic System

Letting $\Lambda \to +\infty$ and assuming that

$$m_\Lambda^4 \Lambda \to C_0 > 0 \tag{10}$$

we derive formally the following system of equations (after rescaling)

$$\boxed{\Delta U = UV^2,\ \Delta V = VU^2,\ U, V \geq 0,\ \text{in } \mathbb{R}^N.} \tag{11}$$

Limiting System

$$\begin{cases} \Delta U = UV^2 \ \text{ in } \mathbb{R}^N, \\ \\ \Delta V = VU^2 \ \text{ in } \mathbb{R}^N \\ \\ U > 0, V > 0 \end{cases}$$

Complete Analysis in One-Dimensional Case

(Berestycki-Lin-Wei-Zhao 2009): Let $N = 1$ and (U, V) be a solution to (11).
(1) (Symmetry) There exists $x_0 \in \mathbb{R}$ such that

$$V(y - x_0) = U(x_0 - y). \tag{12}$$

(2) (Nondegeneracy) (U, V) is nondegenerate,
(3) (Uniqueness) There exists a unique solution (U, V), up to translation and scaling
(Berestycki-Terracini-Wang-Wei 2012)

New De Giorgi Conjecture

New De Giorgi Conjecture: Under the following monotone condition

$$\frac{\partial U}{\partial y_N} > 0, \ \frac{\partial V}{\partial y_N} < 0, \tag{13}$$

the solutions to the system

$\Delta U = UV^2, \Delta V = U^2 V, U, V > 0$ *in* $\mathbb{R}^N$

are one-dimensional.

New Stability Conjecture

For a solution (U, V) to the limiting system, we say it is stable if

$$\int_{\mathbb{R}^N} |\nabla\phi|^2 + |\nabla\psi|^2 + \int_{\mathbb{R}^N} V^2\phi^2 + U^2\psi^2 + 4UV\phi\psi \geq 0, \tag{14}$$

for any compactly supported smooth functions ϕ, ψ.

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For a solution (U, V) to the limiting system, we say it is stable if

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for any compactly supported smooth functions ϕ, ψ.
New Stability Conjecture: The stable solutions to the system

$$\Delta U = UV^2, \Delta V = U^2V, U, V > 0 \text{ in } \mathbb{R}^N$$

$$are$$

$$one - dimensional.$$

Results

•• Berestycki-Lin-Wei-Zhao (2011): Let $N = 2$. De Giorgi Conjecture holds if U and V has linear growth.

•• Berestycki-Terracini-Wang-Wei (2012): Let $N = 2$. Stability Conjecture holds if U and V has linear growth.

•• Farina-Soave (2013): $N = 2$, De Giorgi Conejcture holds if U and V have polynomial growth

•• Farina (2013): $N = 2$, polynomial growth, then

$$\frac{\partial U}{\partial x_N} > 0 \implies \frac{\partial V}{\partial x_N} < 0$$

Saddle Solutions: Entire Solutions with Polynomial Growth

$$\Delta U = UV^2, \ \Delta V = VU^2, \ \text{in } \mathbb{R}^N$$

For Allen-Cahn equation, u is bounded between -1 and $+1$. On the other hand, the system (11) has unbounded one-dimensional solutions with linear growth. Is there a growth estimate for (11)

$$U(x) + V(x) = O(|x|)???$$

Saddle Solutions: Entire Solutions with Polynomial Growth

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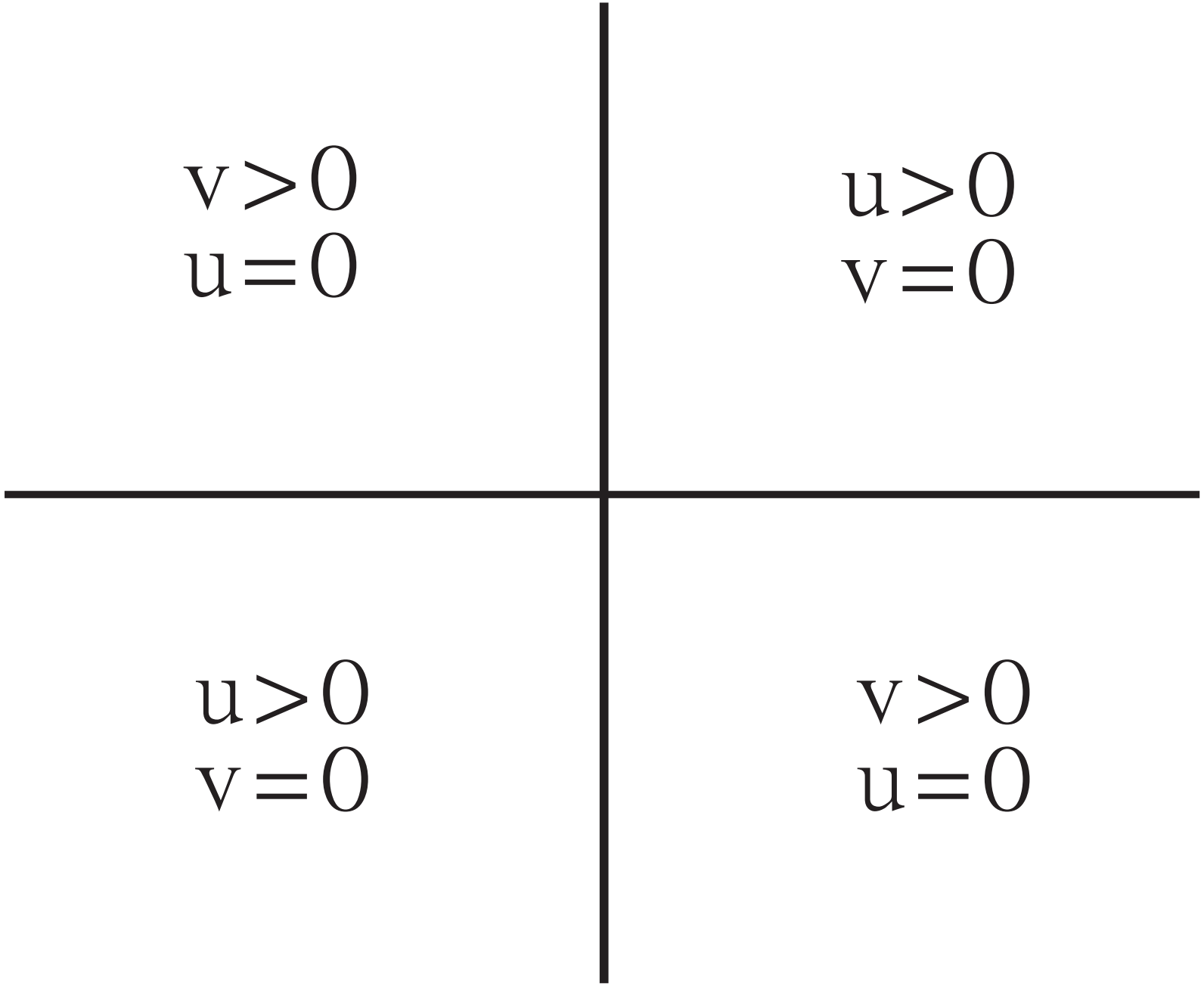
$$U(x) + V(x) = O(|x|)???$$

Answer: NO

Result: For any harmonic function with polynomial growth, there exists a solution (U, V) with

$$U(x) + V(x) \sim |x|^d$$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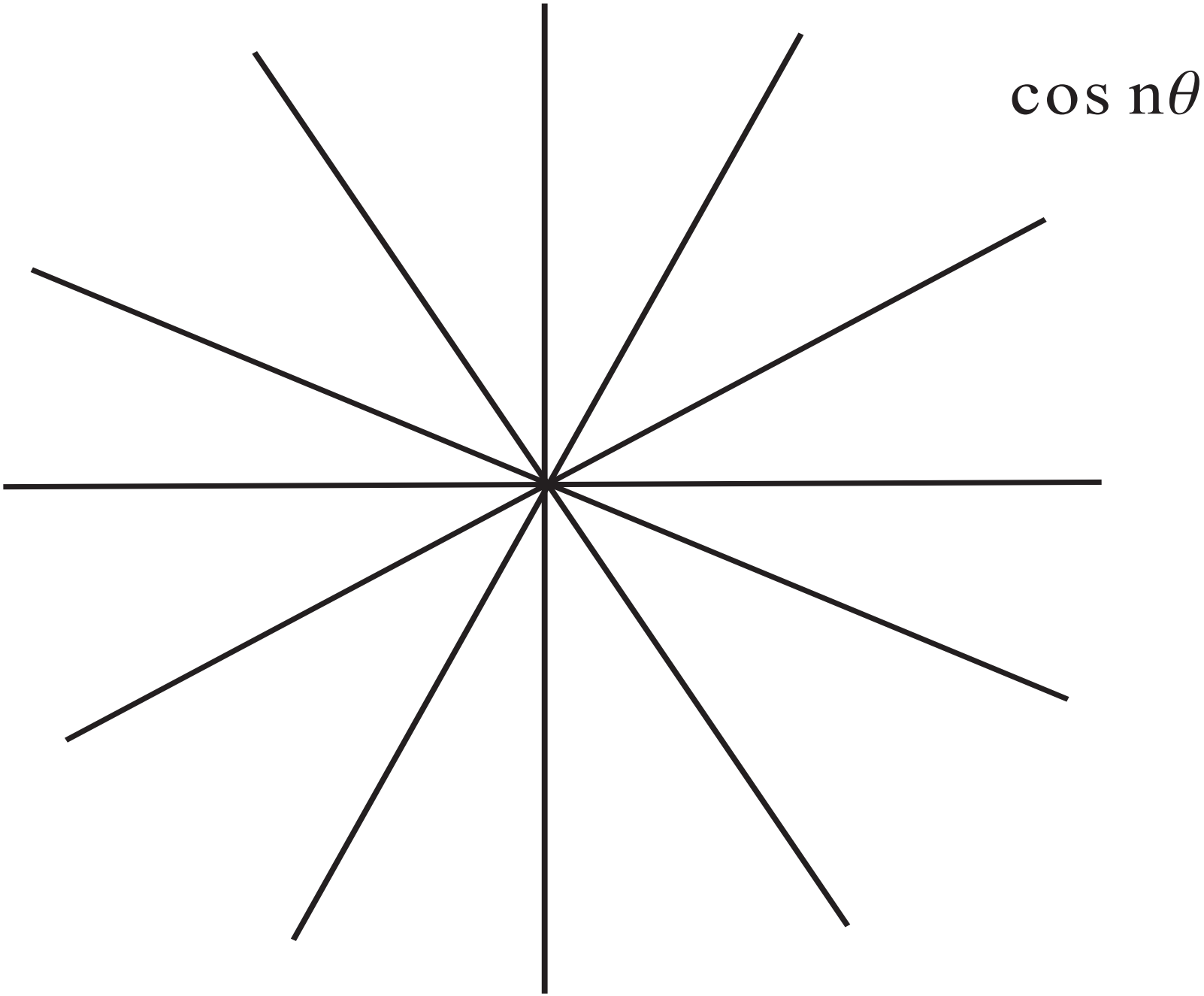
v>0
u=0


cos nθ

Questions

Problem 2.1. Is there a "Γ-Convergence" theory for Bose-Einstein system?

Problem 2.2. The key estimate

$$m_\Lambda^4 \Lambda \to C_0 > 0 \tag{15}$$

in higher dimensions is still missing. It will require some extra techniques. (Even in the radially symmetric case, it is unclear.)

Problem 2.3. The De Giorgi type result for the system is completely open, except in dimension two. What is the underlying geometry? We tend to believe that minimal surface is the underlying geometry.
Problem 2.4 Let us recall that in one space dimension, there exists a unique solution having linear growth at infinity and, in the Almgren monotonicity formula, they satisfy

$$\lim_{r \to +\infty} N(r) = 1. \tag{16}$$

It is natural to conjecture that, in any space dimension, a solution of elliptic system satisfying (17) is actually one dimensional, that is, there is a unit vector a such that $(u(x), v(x)) = (U(a \cdot x), V(a \cdot x))$ for $x \in \mathbb{R}^n$. However this result seems to be difficult to obtain at this stage.

Problem 2.5. A further step would be to prove uniqueness of the (family of) solutions having polynomial asymptotics in two space dimension. A more challenging question is to classify all solutions with

$$\lim_{r \to +\infty} N(r) = d. \tag{17}$$

Problem 2.6. For the Allen-Cahn equation $\Delta u + u - u^3 = 0$ in $\mathbb{R}^2$, there are solutions with multiple disjoint fronts. Are there solutions to the elliptic system such that the set $\{u = v\}$ contains disjoint multiple curves?

Problem 2.7. Are there counterexamples to the De Giorgi Conjecture and Stability Conjecture for the system?

Problem 2.8. Extension of results to multiple-component elliptic systems

Ghoussoub-Fazly (2012-2013), Ghoussoub: relation with optimal transport

Problem 2.9: Similar to the Gibbons's Conjecture for Allen-Cahn, we also have the following conjecture

New Gibbons' Conjecture: Let (U, V) be a solution of system (11) satisfying

$$\lim_{x_N \to -\infty} U(x^{'}, x_N) = 0, \quad \lim_{x_N \to +\infty} U(x^{'}, x_N) = +\infty, \ \textit{uniformly in } x', \tag{18}$$

$$\lim_{x_N \to -\infty} V(x^{'}, x_N) = +\infty, \quad \lim_{x_N \to +\infty} V(x^{'}, x_N) = 0, \ \textit{uniformly in } x'. \tag{19}$$

Then (U, V) are one-dimensional.
Farina-Soave (2013): true if u and v have polynomial growth.

This conjecture is completely open.

De Giorgi Type Conjecture for Lane-Emden Equation

3. Second Order Lane-Emden Equations

We start with the simplest second order nonlinear Lane-Emden equation:

$$(I) \qquad \Delta u + u^p = 0, u > 0 \quad \text{in } \mathbb{R}^N, \quad p > 1,\ N \geq 2$$

Known Results

- (Gidas-Spruck 1981; Chen-Li 1993) If $p < \frac{N+2}{N-2}$, then $u \equiv 0$
- (Cafferalli-Gidas-Spruck 1989; Chen-Li 1993) If $p = \frac{N+2}{N-2}$, all solutions are given by

$$u_{\epsilon,\xi}(x) = C_N \left(\frac{\epsilon}{\epsilon^2 + |x - \xi|^2} \right)^{\frac{N-2}{2}}.$$

Classification Result of Farina

$$(I) \qquad -\Delta u = |u|^{p-1}u, \text{ in } \mathbb{R}^N$$

$$p > 1$$

Theorem: Farina (2007) Let u be solution to (I) with finite Morse index.

- If $p \in (1, p_{JL}(N))$, $p \neq \frac{N+2}{N-2}$, then $u \equiv 0$;
- If $p = \frac{N+2}{N-2}$, then u has finite energy i.e.

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \int_{\mathbb{R}^N} |u|^{\frac{2N}{N-2}} < +\infty.$$

If in addition u is stable, then in fact $u \equiv 0$.

De Giorgi-type Conjecture

Problem 3.1
"De Giorgi-type Conjecture": If $p_{JL}(N) \leq p < p_{JL}(N-1)$, all stable solutions to

$$(I) \qquad \Delta u + u^p = 0 \text{ in } \mathbb{R}^N$$

are radially symmetric around some point.

To prove or disprove the De Giorgi's Conjecture, a possible approach (following the theory of minimal surfaces) is to find stable nonradial singular solutions:

$$u(r,\theta) = r^{-\frac{2}{p-1}} w(\theta)$$

Dancer-Guo-Wei 2012: for

$$\frac{N+1}{N-3} < p < p_{JL}(N-1)$$

there are nonradial singular solutions. Unfortunately they are unstable.

Classifying homogeneous solutions

Let $u = r^{-\frac{2}{p-1}} w(\frac{x}{|x|})$. Then w satisfies

$$\Delta_{S^{N-1}} w - \frac{2}{p-1}(N-2-\frac{2}{p-1})w + w^p = 0 \text{ on } S^{N-1}$$

u stable means:

$$p\int_{S^{N-1}} w^{p+1} \leq \int_{S^{N-1}} |\nabla w|^2 + \frac{(N-2)^2}{4}\int_{S^{N-1}} w^2$$

Problem 3.2. Classifying all stable homogeneous singular solutions.

4. De Giorgi/Liouville Type Theorems for Fractional Laplacian Problems

Fractional Laplacian:
We apply similar procedure to the classification of stable/finite Morse index solutions for fractional Laplace problem

$$(III) \qquad (-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^N.$$

$$0 < s < 1, p > 1$$

Here we have assumed that $u \in C^{2\sigma}(\mathbb{R}^N), \sigma > s$ and

$$\int_{\mathbb{R}^N} \frac{|u(y)|}{(1+|y|)^{N+2s}}\, dy < +\infty, \tag{20}$$

so that the fractional Laplacian of u

$$(-\Delta)^s u(x) := \mathcal{A}_{N,s} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\, dy$$

is well-defined (in the principal-value sense) at every point $x \in \mathbb{R}^N$.

Caffarelli-Silvestre extension: this is equivalent to

$$(III) \qquad \begin{cases} -\nabla \cdot (t^{1-2s} \nabla \bar{u}) = 0 & \text{in } \mathbb{R}^{N+1}_{+} \\ -\lim\limits_{t \to 0} t^{1-2s} \partial_t \bar{u} = \kappa_s | f(\bar{u}) & \text{on } \partial \mathbb{R}^{N+1}_{+} \end{cases}$$

Fractional Allen-Cahn Equation

$$(-\Delta)^2 u = u - u^3, \text{ in } \mathbb{R}^N$$

$N = 1$: existence and uniqueness of hetereoclinic solution (Cabre-Sire 2012)

$$w = \pm 1 + O(\frac{1}{|x|^{2s}})$$

De Giorgi Conjecture: True for $N = 2, 3$ and $\frac{1}{2} \leq s < 1$. (Cabre-Cinti 2012)

Problem 4.1. De Giorgi Conjecture for $0 < s < 1/2$ and $N = 2$

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Problem 4.2: De Giorgi Conjecture for $0 < s < 1$ and $N \leq 7$ (more comments later)

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Problem 4.5: Stability Conjecture???

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Problem 4.6: Gibbons Conjecture???

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Problem 4.4: Counter-examples to De Giorgi Conjecture for $0 < s < 1/2, N \geq 8$
Problem 4.5: Stability Conjecture???
Problem 4.6: Gibbons Conjecture???
Problem 4.7: Finite Morse index solutions, saddle solutions, Toda type solutions ....

fractional Allen-Cahn system

Consider

$$(-\Delta)^s u = -uv^2,\ (-\Delta)^2 v = vu^2, u, v > 0 \text{ in } \mathbb{R}^N$$

$s = \frac{1}{2}$: Terracini-Verzi-Zillio 2013
Problem 5.8: existence, uniqueness and nondegeneracy of one-dimensional solutions $N = 1$.
Completely nontrivial

Problem 5.9: De Giorgi, Stability, Gibbons type conjectures

Liouville Type Results
for
Nonlinear Schrodinger Equations

Part II: NLS

The second part of my talk deals with entire solutions of the Nonlinear Schrodinger Equation

(NLS) $$\Delta u - u + u^p = 0 \quad \text{in } \mathbb{R}^N$$

Nonlinear Schrodinger equation (NLS)

standing wave equation of Nonlinear Schrodinger equation (NLS)

$$-iv_t = \Delta v + |v|^{p-1}v, \quad v = e^{it}u$$

If $p = 3$, Klein-Gordon equation (KG)

Positive Solutions

We first consider positive solutions of (NLS). For simplicity we consider $N = 2$ only.

$$\begin{cases} \triangle\, u - u + u^p = 0, \text{ in } \mathbb{R}^2 \\ u > 0 \text{ in } \mathbb{R}^2 \end{cases}$$

$$p > 1$$

II.1. Ground States

$$\boxed{\triangle\, u - u + u^p = 0, u > 0 \text{ in } \mathbb{R}^2 \quad (NLS)}$$

It is well-known that if we impose $\boxed{\text{decaying}}$ assumption

(1) $u(x) \to 0$ as $|x| \to +\infty$ uniformly

then Gidas-Ni-Nirenberg theorem implies that

(2) u is radially symmetric around some point x_0.
That is

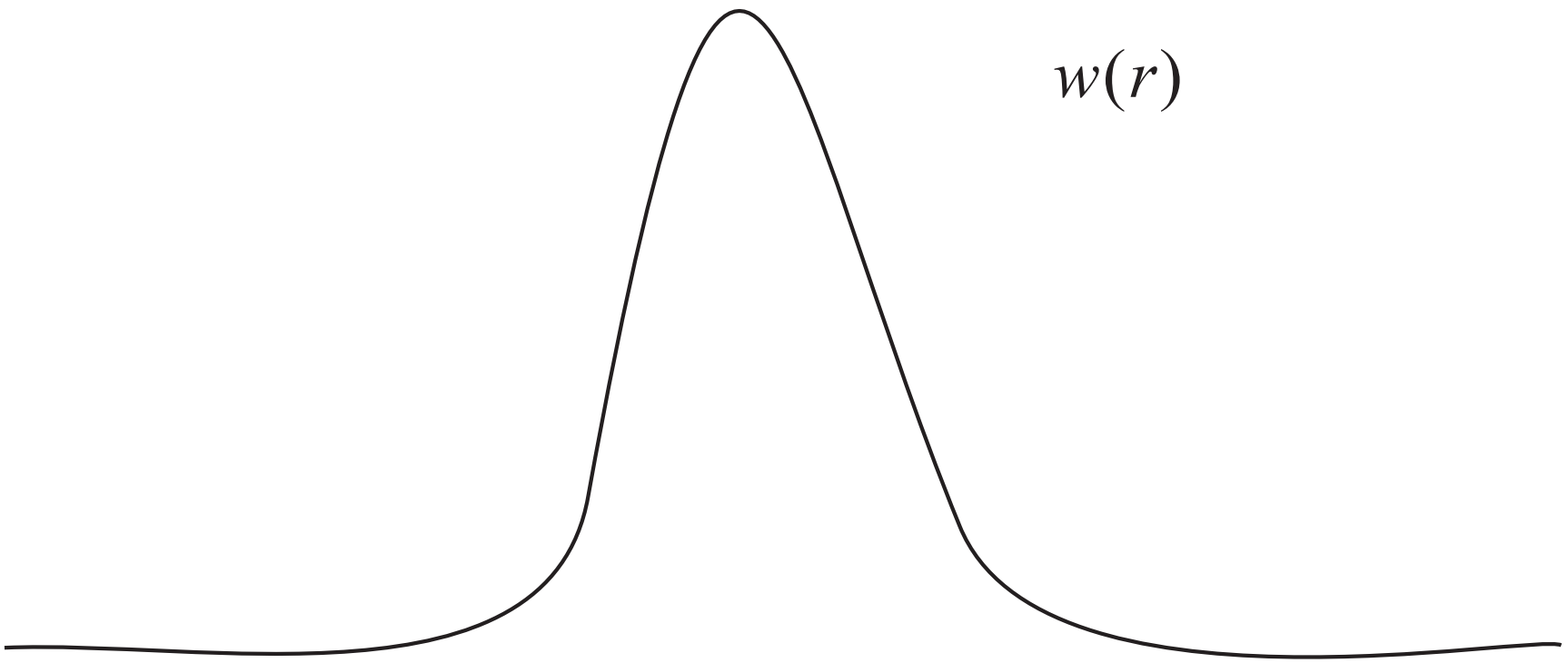
$$u(x) = w(|x - x_0|), \qquad w = w(r)$$

where w satisfies

$$w'' + \frac{1}{r} w' - w + w^p = 0$$

$w = w(r)$ – "ground state"
This solution is called spike

Ground State



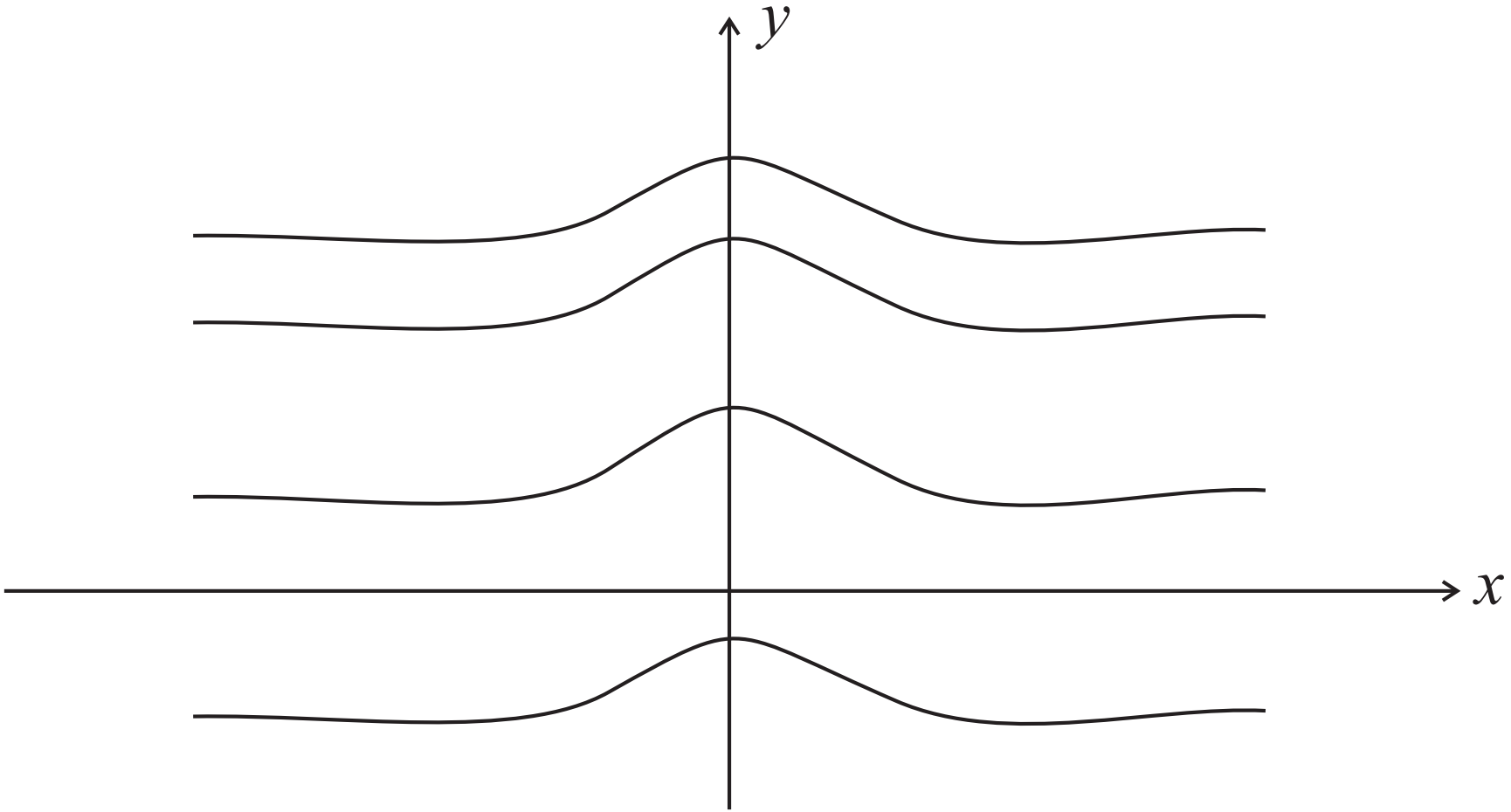
Second Solution

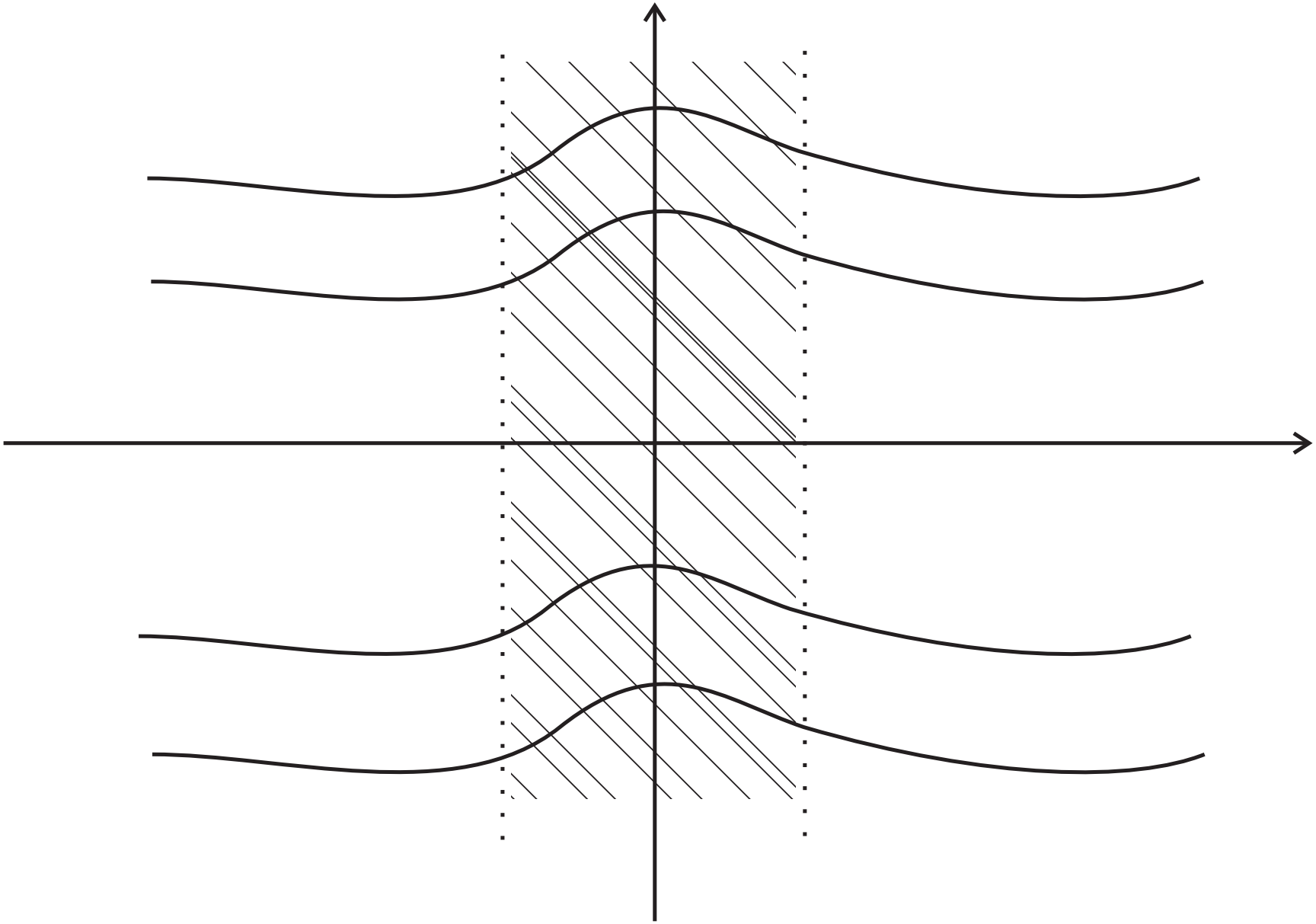
Another solution, which is obvious, canonical one, is the one-dimensional profile.

$$(3)\quad \begin{cases} w'' - w + w^p = 0, & w > 0 \text{ in } \mathbb{R} \\ w \to 0 \text{ at } +\infty \end{cases}$$

Let $u(x, y) = w(x)$ – solution to (NLS) with two "ends"

This solution is called front


y
x



Third Solution:Dancer's Solutions

The third type of solution is the so-called Dancer's solution. Such solutions satisfy

$$\begin{cases} u(x, y+T) = u(x, y) \\ u(x, y) \to 0 \text{ as } |x| \to +\infty \end{cases}$$

for some $T > 0$

$$w_\delta(x, y) := w(x) + \delta Z(x) \cos \sqrt{\lambda_1} y + O(\delta^2 e^{-2|x|})$$

for $|\delta| \ll 1$.

$T = T_1 + O(\delta^2) = \frac{2\pi}{\sqrt{\lambda_1}} + O(\delta^2)$

More generally, the Dancer solutions have two parameters

$$w_{\delta,\tau}(x,y) := w(x) + \delta Z(x)\cos\sqrt{\lambda_1}y + \tau Z(x)\sin\sqrt{\lambda_1}y + O((\delta^2+\tau^2)e^{-2|x|})$$

for $|\delta| \ll 1, |\tau| \ll 1.$

Least-energy Solutions

Dancer's solution actually continues as $T > T_1$. Another way of obtaining Dancer's solutions is to consider the following problem in a stripe:

$$\begin{cases} \Delta u - u + u^p = 0 \text{ in } \Sigma := \mathbb{R}^{N-1} \times (0, L), \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Sigma, \ u > 0, \ u \in H^1(\Sigma). \end{cases} \tag{21}$$

Here L is the parameter. We consider the so-called least energy solutions. More precisely, let

$$c(L) := \inf_{u \in H^1(\Sigma), u \not\equiv 0} \frac{\int_\Sigma (|\nabla u|^2 + u^2)}{\left(\int_\Sigma u^{p+1}\right)^{\frac{2}{p+1}}}. \tag{22}$$

- If $p < \frac{N+2}{N-2}$ when $N \geq 3$ and $p < +\infty$ when $N = 2$, then there exists a unique $L_* = \frac{\pi}{\sqrt{\lambda_1}}$ such that for $L \leq L_*$, $c(L)$ is attained by a trivial solution and for $L > L_*$, $c(L)$ is attained by a nontrivial non-one-dimensional solution.

- If $p < \frac{N+2}{N-2}$ when $N \geq 3$ and $p < +\infty$ when $N = 2$, then there exists a unique $L_* = \frac{\pi}{\sqrt{\lambda_1}}$ such that for $L \leq L_*$, $c(L)$ is attained by a trivial solution and for $L > L_*$, $c(L)$ is attained by a nontrivial non-one-dimensional solution.
- There exist $L_2 \geq L_*$ such that the least energy solution is unique and nondegenerate for any $L \geq L_2$. As $L \to +\infty$, the least energy solution approaches the spike solution in $\mathbb{R}^N$.

Berestycki-Wei 2010

$N = 2, T = 2L$ gives Dancer's solution.

Problem 5.1: Is the solution u_L continuous in L?

Problem 5.2: Are the solutions u_L nondegenerate?

Gibbons's Conjecture for NLS

Problem 5.3: Consider the following problem in $\mathbb{R}^2$:

$$\Delta u - u + u^3 = 0, u > 0 \text{ in } \mathbb{R}^2$$

Suppose

$$\lim_{|x| \to +\infty} u(x, y) = 0 \text{ uniformly in } y$$

Can one show that u is periodic in T and even in x?

Fractional NLS

$$-(-\Delta)^s u - u + u^p = 0 \text{ in } \mathbb{R}^N$$

Frank-Lenzman, Frank-Lenzman-Silvstre, proved the uniqueness of ground state solution:

$$w = w(r) \in H^s$$

Problem 5.4. Existence of Dancer's solutions for fractional NLS?

Problem 5.5. Existence of multiple-front solutions for fractional NLS?