Nonexistence of Type II Blow-up for Nonlinear Energy-Critical Heat Equation

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Blow up for nonlinear heat equation

$$\begin{cases} \partial_t u - \Delta u = |u|^{p-1}u, p > 1, x \in \mathbb{R}^n \\ u(x,0) = u_0(x) \in L^\infty(\mathbb{R}^n). \end{cases} \tag{1}$$

- Local well-posedness: standard parabolic theory.

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- Local well-posedness: standard parabolic theory.
- Blow up in finite time: Fujita (1966), Quittner-Souplet ...
- **Type I** if there exists a constant C such that for any $t < T$,

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq C(T-t)^{-\frac{1}{p-1}}, \tag{2}$$

otherwise it is called **Type II**.

Classification of Blow-ups: $p < \frac{n+2}{n-2}$

- When $p < \frac{n+2}{n-2}$ (Sobolev subcritical), all blow ups are Type I. (Giga-Kohn (1987) (positive solutions), Giga-Matsui-Sasayama (2014) (sign-changing solutions)...)

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- What about $p = \frac{n+2}{n-2}$, energy-critical?

Main result

Theorem

(Wang-Wei 2022) If $n \geq 7$, $p = \frac{n+2}{n-2}$ *and* $u_0 \geq 0$, *then all (finite-time) blow ups are Type I.*

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(Wang-Wei 2022) If $n \geq 5$, $p = \frac{n+2}{n-2}$ *and* $u_0 = u_0(|x|) \geq 0$, *then all (finite-time) blow ups are Type I.*

Previous results

If $p = \frac{n+2}{n-2}$ only **type I blow-up** for Positive Solutions

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- Filippas-Herrero-Velázquez [2000]: true, for radially decreasing solutions, $n \geq 3$

- Collot-Merle-Raphaël [2017]: true for $n \geq 7$, provided that
$$\|u_0 - W\|_{\dot{H}^1} << 1,$$
W is the Aubin-Talenti bubble
$$\Delta W + W^{\frac{n+2}{n-2}} = 0, W > 0$$

Type II blow-up for $n \leq 6, p = \frac{n+2}{n-2}$

- Filippas-Herrero-Velázquez [2000] (formal computations, radial case)
- Schweyer [2012] (the radial case): $n = 4, p = 3$

$$\|u\|_{L^\infty} \sim \frac{|\log(T-t)|^2}{T-t}$$

  Merle-Raphael-Rodnianskii modulation approach
- del Pino-Musso-Wei-Zhou (2019): $n = 4, p = 3$, nonradial case, multiple bubbles

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  Inner-outer gluing approach

Type II Blow-ups for $n \leq 6, p = \frac{n+2}{n-2}$

- del Pino-Musso-Wei (2018): $n = 5, p = 7/3$, nonradial and multiple bubbles

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- del Pino-Musso-Wei-Zhou-Zhang (2021): $n = 3, p = 5$, nonradial case

$$\|u\|_{L^\infty} \sim (T - t)^{-k}, k = 1, 2, 3, ...$$

Type II Blow-up for $p > \frac{n+2}{n-2}$

- Herrero-Velazquez (1997), $p > p_{JL}(n)$, radial case; Mizoguchi, Seki, ...
- Collot (2017), $p > p_{JL}(n), p = 2m+1$, nonradial case; Collot-Merle-Raphael (2020), $p > p_{JL}(n-1), n \geq 13$;
- Matano-Merle (2004, 2009), No Type II blow-up for $u = u(r)$, $\frac{n+2}{n-2} < p < p_{JL}(n)$;
- del Pino-Musso-Wei (2020), Type II blow-up for $\frac{n+2}{n-2} < p = \frac{n+1}{n-3} < p_{JL}(n)$;
- Lai-del Pino-Musso-Zhou-Wei (arXiv 2021), Type II blow-up for $p = 3, 5 \leq n \leq 8$

Classification of Blow-ups when $p \geq \frac{n+2}{n-2}$

Based on the examples of Type II blow-ups, it is reasonable to

Conjecture 1: If $p = \frac{n+2}{n-2}$, all positive blow-ups are Type I, for $n \geq 3$

Wang-Wei: True if $n \geq 7$; True if $u = u(r), n \geq 5$

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Conjecture 3: If $\frac{n+2}{n-2} < p < p_{JL}(n)$ and $p \neq \frac{m+2}{m-2}$, all solutions (sign-changing) blow-ups are Type I.

Collot-Raphael-Merle (2020)

Theorem

(Wang-Wei 2022) If $n \geq 7$, $p = \frac{n+2}{n-2}$ *and* u *is positive, then all blow ups are Type I.*

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Why $n \geq 7$?: The scaling parameter $\lambda(t)$ (inverse height of $u(t)$) satisfies

$$\lambda'(t) \sim \lambda(t)^{\frac{n-4}{2}}.$$

Lipschitz nonlinearity if $n \geq 7 \implies$ Harnack inequality for λ...

Main Difficulties

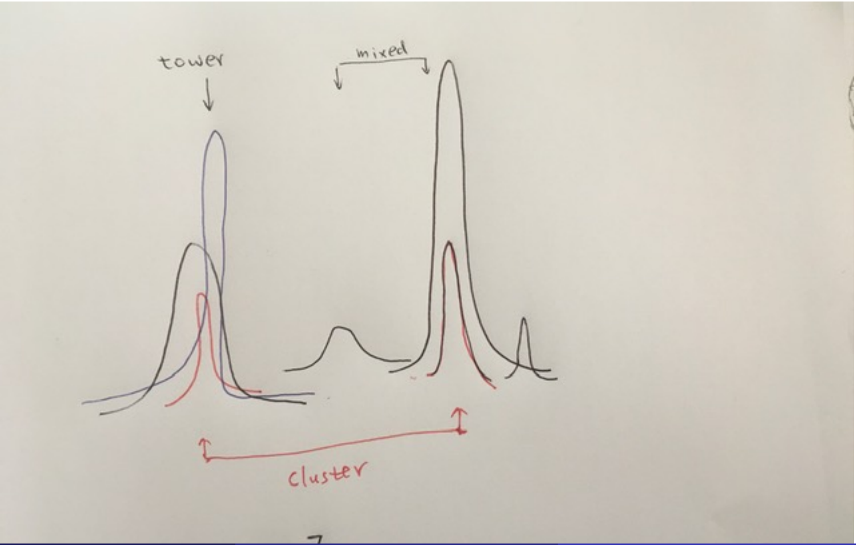
- Difficulty I: We only assume that $u_0 \in L^\infty$. There is no energy bound. The number of bubbles can approach $+\infty$.

$$u \sim \sum_{j=1}^{N(t)} \left(\frac{\lambda_j(t)}{\lambda_j^2 + |x - \xi_j|^2} \right)^{\frac{n-2}{2}}$$

$$\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \frac{|\xi_i - \xi_j|}{\lambda_i \lambda_j} \to +\infty$$

$$N(t) < +\infty; N(t) = +\infty?$$

- Difficulty II: Even if the energy is bounded, we don't know the relative scales of the bubbles: bubble towers+bubble clusters+mixed bubble towers and clusters.



- Difficulty III: Even if there is only bubble, the outer part may be a Type I blow-up.

$$u \sim \left(\frac{\lambda(t)}{\lambda^2 + |x|^2}\right)^{\frac{n-2}{2}} + \psi_0(t)$$

$$\psi_0(t) \notin L^\infty$$

Key Ideas of Proofs

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- Parabolic second order estimates: Reverse Inner-outer parabolic gluing mechanism to exclude Multiplicity One case (Davila, del Pino, Musso, Wei)
  L^2 case: Carmen Cortazar, Manuel del Pino and Monica Musso, 2020.
  (L^2 and) Non-L^2 case: Manuel del Pino, Juan Davila and Juncheng Wei, 2020
  Elliptic second order estimates for Allen-Cahn (Wang-Wei (2019, 2020)

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  Elliptic second order estimates for Allen-Cahn (Wang-Wei (2019, 2020)
- Exclusion of Higher Multiplicity case (bubbling towering and bubbling clustering), parabolic compactness argument for Yamabe (Schoen, Khuri-Marques-Schoen, Y. Li, etc.)

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  Tools: Reverse Lyapunov-Schmidt reduction, inner-outer decoupling.

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Definition

If u is bounded in $Q_r^-(a,T) = B_r(a) \times (T - r^2, T)$, then the point a is a regular point, otherwise it is a blow up (singular) point.

Blow up analysis: energy-critical

- To determine the blow up profile, we zoom in the solution near a blow up point a:

$$u^{\lambda}(x,t) := \lambda^{\frac{2}{p-1}} u(a+\lambda x, T+\lambda^2 t), \quad \lambda \to 0.$$

$\Longleftarrow$ Scaling invariance of the equation.

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but it is a supercritical bubbling phenomena: parametrized bubbles

Energy concentration phenomena

Theorem

Given a sequence of smooth solutions u_i in Q_1, satisfying $\sup_i \int_{Q_1} |\nabla u_i|^2 + |u_i|^{p+1} < +\infty$.

- *A subsequence converges to a weak solution u_∞;*
- *there exists a defect measure μ such that*

$$|\nabla u_i|^2 dxdt \rightharpoonup |\nabla u_\infty|^2 dxdt + \mu, \quad |u_i|^{p+1} dxdt \rightharpoonup |u_\infty|^{p+1} dxdt + \mu;$$

- $\mu = \mu_t \otimes dt$, *and* $\mu_t = \sum_k m_k \delta_{\xi_k^*(t)}$.

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- Similar to Lin-Wang's work on harmonic map heat flow.
- Main tools: (i) ε-regularity theorem, (ii) monotonicity formula of Giga-Kohn.

Bubbling

- By Struwe's global compactness theorem, for a.e. t, the following bubble tree convergence holds for $u_i(t)$:

Theorem

There exist $N(t)$ points $\xi_{ik}^*(t)$, positive constants $\lambda_{ik}^*(t)$, $k = 1, \cdots, N(t)$, all converging to 0 as $i \to +\infty$, and $N(t)$ bubbles W^k, such that in $H^1(B_1)$,

$$u_i(x,t) = u_\infty(x,t) + \sum_{k=1}^{N(t)} W^k_{\xi_{ik}^*(t), \lambda_{ik}^*(t)}(x) + o_i(1).$$

- A bubble is an entire solution

$$-\Delta W = |W|^{p-1} W, \quad \int_{\mathbb{R}^n} |\nabla W|^2 < +\infty.$$

- If u_i is positive, all bubbles arising in this process are standard ones, thanks to Caffarelli-Gidas-Spruck.

Bubble clustering and towering

Bubble towering: bubbles are located at almost the same point (w.r.t. the bubble scales), but the height of one bubble is far larger than the other one's :

$$\limsup_{i\to+\infty} \frac{|\xi^*_{ik}(t)-\xi^*_{i\ell}(t)|}{\max\{\lambda^*_{ik}(t),\lambda^*_{i\ell}(t)\}} < +\infty, \quad \frac{\lambda^*_{ik}(t)}{\lambda^*_{i\ell}(t)}+\frac{\lambda^*_{i\ell}(t)}{\lambda^*_{ik}(t)} \to +\infty.$$

Bubble clustering: if for some $k \neq \ell$,

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Bubbling-towerings do exist: del Pino-Musso-Wei (2019)
($n \geq 7$, $T = +\infty$)

Application: tangent flow

Suppose (a, T) is a blow up point. Consider the blow up sequence

$$u^\lambda(x,t) := \lambda^{\frac{2}{p-1}} u\left(a+\lambda x, T+\lambda^2 t\right), \quad \lambda \to 0.$$

Theorem

Suppose $p = \frac{n+2}{n-2}$, and u is positive.

- *u^λ sub-sequentially converges to (u^∞, μ) in $\mathbb{R}^n \times \mathbb{R}^-$.*
- *(u^∞, μ) is backwardly self-similar, which implies that*
  - *either $u^\infty = 0$ or $u^\infty \equiv [-(p-1)t]^{-\frac{1}{p-1}}$ (Giga-Kohn's Liouville theorem);*
  - *there exists a constant $M > 0$ such that $\mu = M\delta_0 \otimes dt$.*
- *Bubble tree convergence for u^λ...*

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- Modulate parameters to get an optimal approximation:

  $$\phi_i(t) := u_i(t) - \sum_k W_{\xi_{ik}(t),\lambda_{ik}(t)}.$$

  This gives an orthogonal condition (note that the linearized equation has nontrivial kernels), which leads to reduction equations (here some ODEs) for the modulated parameters.

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  This gives an orthogonal condition (note that the linearized equation has nontrivial kernels), which leads to reduction equations (here some ODEs) for the modulated parameters.
- Because now we are looking at the next order term (i.e. the error function ϕ_i), we need to take care of
  - interaction between bubbles and the background;
  - interaction between different bubbles.

Inner-outer decomposition

The naive linearization is not efficient:

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In recent years, Davila,Del Pino, Musso, Wei used the inner-outer gluing method to construct blow up solutions with bubbling behavior. Here we take a reverse view, using this decomposition to analyse the behavior of the error function,

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- $\phi_{i,inn}$ satisfies the linearized equation around standard bubble;
- $\phi_{i,out}$ satisfies a parabolic equation with small Hardy potential ($\Longleftarrow$ fast decay of bubbles);
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In the following we look at the one bubble case more closely.

The standard bubble

By Caffarelli-Gidas-Spruck, all entire positive solutions to the stationary equation are given by Aubin-Talenti bubbles

$$W_{\xi,\lambda}(x) := \left(\frac{\lambda}{\lambda^2 + \frac{|x-\xi|^2}{n(n-2)}} \right)^{\frac{n-2}{2}}, \qquad \lambda > 0, \quad \xi \in \mathbb{R}^n.$$

They have finite energy, which are always equal to

$$\Lambda := \int_{\mathbb{R}^n} |\nabla W_{\xi,\lambda}|^2 = \int_{\mathbb{R}^n} W_{\xi,\lambda}^{p+1}.$$

Linearization around the standard bubble

Theorem (Non-degeneracy)

(i) *There exists one and only one negative eigenvalue for* $-\Delta - pW^{p-1}$, *denoted by* $-\mu_0$, *for which there exists a unique (up to a constant), positive, radially symmetric and exponentially decaying eigenfunction* Z_0.

(ii) *There exist exactly* $(n+1)$*-eigenfunctions* Z_i *in* $L^\infty(\mathbb{R}^n)$ *corresponding to eigenvalue* 0, *given by*

$$\begin{cases} Z_i = \dfrac{\partial W}{\partial x_i}, \quad i = 1, \cdots, n, \\ Z_{n+1} = \dfrac{n-2}{2} W + x \cdot \nabla W. \end{cases}$$

One bubble case: blow up profile

Assume u_i is a sequence of smooth, positive solutions in Q_1 s.t.

$$|\nabla u_i|^2 dxdt \sim |\nabla u_\infty|^2 dxdt + \Lambda \delta_0 dt,$$

where u_∞ is a smooth solution, Λ is the energy of the standard bubble.

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Lemma (Blow up profile)

For any t, there exists a unique maxima point of $u_i(\cdot, t)$ in the interior of $B_1(0)$. Denote this point by $\xi_i^*(t)$ and let $\lambda_i^*(t) := u_i(\xi_i^*(t), t)^{-\frac{2}{n-2}}$.

$$u_i(x,t) \sim \lambda_i^*(t)^{-\frac{n-2}{2}} W\left(\frac{x - \xi_i^*(t)}{\lambda_i^*(t)}\right).$$

One bubble case: orthogonal decomposition

Lemma

For any t, there exists a unique $(a_i(t), \xi_i(t), \lambda_i(t)) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^+$ *with*

$$\frac{|\xi_i(t) - \xi_i^*(t)|}{\lambda_i(t)} + \left| \frac{\lambda_i(t)}{\lambda_i^*(t)} - 1 \right| + \left| \frac{a_i(t)}{\lambda_i(t)} \right| = o(1),$$

such that for each $k = 0, \cdots, n+1$,

$$\int_{B_1} \left[u_i(x,t) - W_{\xi_i(t),\lambda_i(t)}(x) - a_i(t) Z_{0,\xi_i(t),\lambda_i(t)}(x) \right]$$

$$\times \eta\left(\frac{x - \xi_i(t)}{K\lambda_i(t)} \right) Z_{k,\xi_i(t),\lambda_i(t)}(x) dx = 0.$$

Here we take a cut-off at $K\lambda_i(t)$-scale and a scaling preserving the L^2 norm:

$$Z_{k,\xi,\lambda}(x) := \lambda^{-\frac{n}{2}} Z_k \left(\frac{x-\xi}{\lambda} \right).$$

Reduction procedure

The error function ϕ_i satisfies

$$\partial_t \phi_i - \Delta \phi_i = p W_i^{p-1} \phi_i + \left(-a_i' + \mu_0 \frac{a_i}{\lambda_i^2}, \xi_i', \lambda_i' \right) \cdot Z_i + \text{h.o.t.} \qquad (3)$$

Together with the orthogonal condition

$$\int_{B_1} \phi_i(x,t) \eta \left(\frac{x - \xi_i(t)}{K \lambda_i(t)} \right) Z_{k, \xi_i(t), \lambda_i(t)}(x) dx = 0,$$

we can (and we need to) get at the same time

- equations for λ_i' ... $\Longrightarrow$ blow up rate;
- a priori estimates on ϕ_i.

Inner-outer decomposition

Keep K as the large constant used in the orthogonal decomposition. Take another constant L satisfying $1 \ll L \ll K$. Denote

$$\eta_{i,in}(x,t) := \eta\left(\frac{x-\xi_i(t)}{K\lambda_i(t)}\right), \quad \eta_{i,out}(x,t) := \eta\left(\frac{x-\xi_i(t)}{L\lambda_i(t)}\right).$$

Set

$$\phi_{i,in}(x,t) := \phi_i(x,t)\eta_{i,in}(x,t), \quad \phi_{out}(x,t) := \phi(x,t)\left[1-\eta_{i,out}(x,t)\right].$$

We analyse the inner and outer equations separately.

Reverse Inner-outer Gluing Scheme

- Inner problem estimate:

$$\mathcal{I} \leq A\mathcal{O} + \text{higher order terms from scaling parameters etc.}$$

  where $\mathcal{I}$ is a quantity measuring the inner component, $\mathcal{O}$ is a quantity measuring the outer component.

- Outer problem estimate:

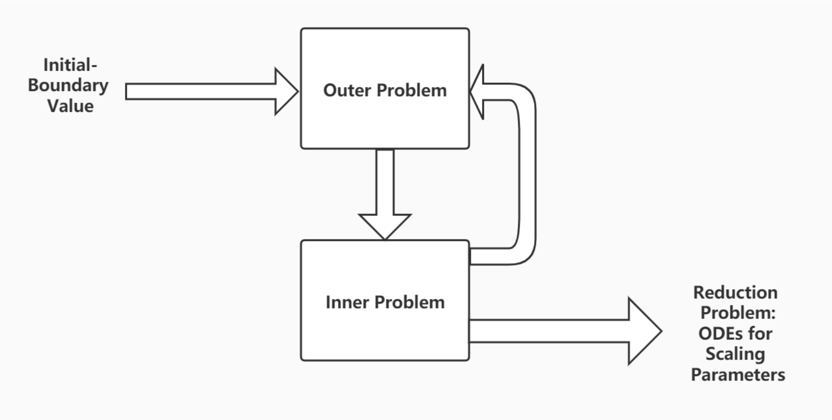
$$\mathcal{O} \leq B\mathcal{I} + \text{effect from initial-boundary value}$$
$$+ \text{higher order terms from scaling parameters etc.}$$

The inner-outer gluing mechanism works thanks to the fact that

$$A\ B < 1$$

This follows from a fast decay estimate away from the bubble domains, where we mainly rely on a Gaussian bound on heat kernels associated to a parabolic operator with small Hardy term

Feedback between inner and outer components



Inner problem

Introduce an inner coordinate system around $\xi_i(t)$ by

$$y := \frac{x - \xi_i(t)}{\lambda_i(t)}, \qquad \tau = \tau(t),$$

where

$$\tau'(t) = \lambda_i(t)^{-2}, \qquad \tau(0) = 0.$$

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Let $\varphi_{i,K}(y,\tau) := \lambda_i(t)^{\frac{n-2}{2}} \phi_i(x,t)\eta_{i,K}(x,t)$. Then

$$\begin{cases} \partial_\tau \varphi_K - \Delta_y \varphi_K = pW^{p-1}\varphi_K + \lambda^{-1}\left(-\dot{a} + \mu_0 a, \dot{\xi}, \dot{\lambda}\right) \cdot Z + E_K, \\ \int_{\mathbb{R}^n} \varphi_K(y,\tau) Z_i(y) dy = 0, \quad \forall \tau. \end{cases}$$

Non-degeneracy of the linearized operator $\Longrightarrow$ exponential decay in τ.

Outer problem

The outer component satisfies

$$\begin{aligned}\partial_t \phi_{i,out} - \Delta \phi_{i,out} &= O\left(\frac{\delta}{|x-\xi_i(t)|^2}\right)\phi_{i,out} + \\ &+ \text{ terms from inner component and } \lambda_i' ...\end{aligned}$$

where $\delta \ll 1$, thanks to the fast decay away from bubble point.

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where $\delta \ll 1$, thanks to the fast decay away from bubble point.

Key Heat Kernel Estimates: The heat kernel $G(x, y; t, s)$ of the operator $\partial_t - \Delta - (\frac{\delta}{|x-\xi(t)|^2} + C) + \xi'(t)\nabla$

$$G(x, y; t, s) \leq C(t-s)^{-\frac{n}{2}} e^{-c\frac{|x-y|^2}{t-s}} (1 + \frac{\sqrt{t-s}}{|x|})^\gamma (1 + \frac{\sqrt{t-s}}{|y|})^\gamma$$

$$\gamma = \frac{n-2}{2} - \sqrt{(\frac{n-2}{2})^2 - 4\delta}$$

(Saloff-Coste (2012), Moschini-Tesei (2007))

Then $\phi_{i,out} = \phi_{i1} + \phi_{i2} + \phi_{i3} + \cdots$, where

- ϕ_{i1} solves the Cauchy-Dirichlet problem, and it is almost regular in the interior;
- ϕ_{i2} is determined by $\phi_{i,inn}$, ϕ_{i3} is determined by λ_i' ..., all enjoying a fast decay away from bubble point;

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- the last property gives us a small transmission coefficient (inner to outer), which closes our estimates on ϕ_i.

Reduction equation

- ϕ_i are uniformly bounded in L^∞ ...
- Linearzing local Pohozaev identity $\Longrightarrow$

$$\begin{cases} \lambda_i'(t) = c_1(n)\left[\phi_i(\xi_i(t),t) + h.o.t.\right]\lambda_i^{\frac{n-4}{2}}, \\ \xi_i'(t) = -c_2(n)\left[\nabla\phi_i(\xi_i(t),t) + h.o.t.\right]\lambda_i^{\frac{n-2}{2}}. \end{cases}$$

Corollary

For positive solutions of (1), bubble towers are unstable.

Proof: By a rescaling, we can choose $u_\infty = W$. Then λ_i increases, which forces $\xi_i(t)$ to move to infinity, so bubble towers will be transformed into bubble clusters.

Analysis of bubble cluster

Setting: $\forall t$, there are exactly N (≥ 2) bubbles located at $\xi_{ij}^*(t)$, with height $\lambda_{ij}^*(t)^{-\frac{n-2}{2}}$, satisfying for some large M,

$$|\xi_{ij}^*(t) - \xi_{ik}^*(t)| \geq M \max\left\{\lambda_{ij}^*(t) + \lambda_{ik}^*(t)\right\}.$$

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- We can still take localized orthogonal and inner-outer decomposition for each bubble.
- A new term describing the interaction between different bubbles: $\sum_{k\neq j} W_{ij}^{p-1} W_{ik}$.
- The interaction term is positive ($\sim$ repulsive force between bubbles).

Unstable mechanism in bubble clusters

A simple case: distances between different bubbles are comparable. Then

$$\begin{cases} \sum_{j=1}^{N} \lambda'_{ij} \geq -C \left(\sum_{j=1}^{N} \lambda_{ij} \right)^{\frac{n-4}{2}}, \\ \left| \sum_{j=1}^{N} \xi'_{ij} \right| \lesssim \sum_{j=1}^{N} \lambda'_{ij} + C \left(\sum_{j=1}^{N} \lambda_{ij} \right)^{\frac{n-4}{2}}. \end{cases}$$

1. The location of bubbles doesn't move too much unless λ_{ij} changes a lot;

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1. The location of bubbles doesn't move too much unless λ_{ij} changes a lot;
2. Inter-distances between different bubbles can't decrease too much unless λ_j has changed a lot;
3. Bubble clusters are unstable, too.

General case: Mutli-scale structure

1. Divide $\{1, \ldots, N\} = \cup_a \mathcal{G}_a$, where
$$\mathrm{diam}(\mathcal{G}_a) \ll \mathrm{dist}(\mathcal{G}_a, \mathcal{G}_b) \quad \forall b \neq a.$$
2. Define
$$\lambda_a = \left(\sum_{j \in \mathcal{G}_a} \lambda_j^2 \right)^{1/2}, \quad \xi_a = \frac{1}{|\mathcal{G}_a|} \sum_{j \in \mathcal{G}_a} \xi_j.$$

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3. Similar relations between λ_a' and ξ_a' by an inductive analysis $\Longrightarrow$ Inter-distances between different subgroups can't decrease too much unless λ_a change a lot.

Conclusion: eventually there is only one bubble near the blow up time.

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Applying all of the above analysis to

$$\tilde{u}^t(y,s) := (T-t)^{\frac{n-2}{4}} u\left(\sqrt{T-t}y, T-ts\right), \quad \forall t \in (-1/4, 0)$$

gives an ODE for $\lambda(t) = \min u(x,t)^{-\frac{2}{n-2}}$:

$$\left|\lambda'(t)\right| \lesssim (T-t)^{\frac{n-2}{4}} \lambda(t)^{\frac{n-4}{2}}.$$

$$\Longrightarrow \lambda(t)^{-\frac{n-6}{2}} \lesssim (T-t)^{-\frac{n-6}{4}},$$

so the blow up is Type I.

Thanks for your attention!