Non-simple Blowup solutions of Liouville equations with quantized singularities

joint work with Lei Zhang and Teresa D'Aprile

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The singular Liouville equation

In this talk I will talk about the following simple equation defined in two dimensional spaces:

$$\Delta v(x) + h(x)e^{v(x)} = 4\pi\alpha\delta_0, \quad \text{in} \quad B_1 \subset \mathbb{R}^2$$

where h is a positive smooth function and B_1 is the unit ball, δ_0 is a Dirac mass placed at the origin and $\alpha > -1$. Since

$$\Delta(\frac{1}{2\pi} \log |x|) = \delta_0,$$

Setting $u(x) = v(x) - 2\alpha \log |x|$ we have

$$\Delta u + |x|^{2\alpha} h(x)e^{u(x)} = 0.$$

Geometric background

Nirenberg problem: which smooth functions K on $\mathbb{S}^2$ are realized as the Gauss curvature of a metric g on $\mathbb{S}^2$ pointwise conformal to the standard round metric g_0 of $\mathbb{S}^2 \subset \mathbb{R}^3$? For $g = e^{2u} g_0$ the equation for the Gauss curvature h of g is

$$\Delta u + K e^{2u} = 1 \tag{1}$$

so that the Nirenberg problem asks to characterize for which K is the nonlinear PDE (1) solvable.

If in a neighborhood of one point, the metric can be written as

$$g = e^h |z|^{2\alpha} |dz|^2,$$

we say at this point it has a conical singularity of order α. The corresponding PDE to study is

$$\Delta u + K(x)|x|^{2\alpha} e^u = 0.$$

Physical Background

Mean Field Equation:

$$\Delta_g u + \rho\left(\frac{h(x)e^u}{\int_M h(x)e^u} - \frac{1}{|M|}\right) = 4\pi \sum_j \left(\delta_{p_j} - \frac{1}{|M|}\right), \text{ on } (M, g)$$

If the singular source is quantized, i.e. $\alpha_j \in \mathbf{N}$, the Liouville equation has close ties with Algebraic geometry, integrable system, number theory and complex Monge-Ampere equations.

Pioneering work by Ding-Jost-Li-Wang (1997).

Chern-Simons-Higgs theory or electro-weak theory:
Y. Yang. Solitons in Field Theory and Nonlinear Analysis, Springer-Verlag, New York, 2001

Case 1. No singularity ($\alpha = 0$)

$$\Delta u + h(x)e^u = 0.$$

$$\int h(x)e^u < +\infty$$

$$0 < C_1 \leq h(x) \leq C_2 < +\infty$$

Theorem: All bubbles are simple

- Brezis-Merle (CPDE91)
- Brezis-Li-Shafrir (IUMJ93)
- Li-Shafarir (IUMJ94)
- Li (CMP1995)

classification of global solutions

Theorem

(Chen-Li (Duke94)) Let u be a solution of

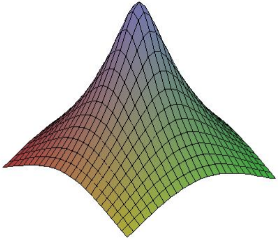
$$\Delta u + e^u = 0, \quad in \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^u < \infty,$$

then

$$u(x) = U_{\lambda, x_0} = \log \frac{e^\lambda}{(1 + \frac{e^\lambda}{8}|x - x_0|^2)^2}$$

for some $\lambda \in \mathbb{R}$ and $x_0 \in \mathbb{R}^2$. (Liouville 1836)

$\int_{\mathbb{R}^2} e^u = 8\pi$. If $v(y) = u(\delta y) + 2\log\delta$, then $\int_{\mathbb{R}^2} e^v = \int_{\mathbb{R}^2} e^u$.



local blowup for regular equation

Let u_k be a sequence of bubbling solutions of

$$\Delta u_k + he^{u_k} = 0, \quad \text{in} \quad B_1,$$

where h is a positive smooth function. If

1. $$\max_x u_k(x) = u_k(0) \to \infty, \quad \text{and} \quad \max_{K \subset\subset B_1 \setminus \{0\}} u_k \leq C(K)$$
2. $$\int_{B_1} he^{u_k} \leq C,$$
3. $$|u_k(x) - u_k(y)| \leq C, \quad \forall x, y \in \partial B_1,$$

Theorem

(Y.Y.Li, (CMP95)) Suppose $\lambda_k = u_k(0) = \max u_k \to \infty$*, then*

$$u_k(x) - \log \frac{e^{\lambda_k}}{(1 + \frac{e^{\lambda_k} h(0)}{8}|x|^2)^2} = O(1), \quad \forall x \in B_1.$$

Simple-vs-Non-simple blow-up

A blow-up is simple if after suitable rescaling

$$|u_k - U_{\lambda_k, p_k}| \leq C \text{ in } B_1$$

Equivalently

$$u_k + 2\log|x| \leq C \text{ in } B_1$$

Equivalently u satisfies spherical Harnack inequality around 0, which implies that, after scaling, the sequence u_k behaves as a single bubble around the maximum point.

Non-simple Blow-ups

$$\Delta u_k + h(x)|x|^{2\alpha} e^{u_k} = 0, \quad \text{in} \quad B_1.$$

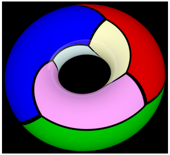
A blow-up is non-simple if after suitable rescaling

$$|u_k - U_{\lambda_k, p_k}| >> C \text{ in } B_1$$

Equivalently

$$\max_{B_1}(u_k + 2(1+\alpha)\log|x|) \to +\infty$$

Applications: Uniform Estimate

If we consider a mean field equation on a surface, say 

$$\Delta_g u + \rho(\frac{he^u}{\int_M he^u} - 1) = 0. \quad vol(M) = 1.$$

Since all bubbles are simple, the uniform estimate implies

1. Around each blowup point, there is only one bubble profile: $he_k^u \rightharpoonup 8\pi\delta_p$
2. The height of bubbles are roughly the same.
3. The energy ($\int_M he^{u_k}$) is concentrated around a few blowup points.
4. Further refined estimates are possible

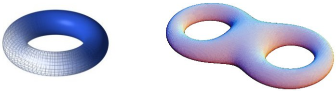
Application

Theorem

(C.C.Chen-C.S.Lin CPAM 03) Suppose u is a solution of the following mean field equation on (M, g) (volume of $M = 1$)

$$\Delta_g u + \rho\left(\frac{he^u}{\int_M he^u dV_g} - 1\right) = 0$$

If $\rho > 0$ is not a multiple of 8π and the genus of M is greater than 0, then the equation has a solution.



- if $8\pi N < \rho < 8\pi(N+1)$ we have $|u| < C$
- $$T_\rho = -\rho \Delta_g^{-1}\left(\frac{he^u}{\int_M he^u} - 1\right)$$
- $$d_\rho := deg(I - T_\rho, B_R, 0)$$

  is well defined for $\rho \neq 8N\pi$. (YY Li (2000))

Theorem

(Chen-Lin 02, 03)

$$d_\rho = \begin{cases} 1 & \rho < 8\pi, \\ \frac{(-\chi_M+1)\dots(-\chi_M+N)}{N!} & 8N\pi < \rho < 8(N+1)\pi. \end{cases}$$

$\chi(M) = 2 - 2g_e$, the g_e is the genus of the manifold, which is the number of handles.

$\rho = 8\pi$, Lin-Wang (Annals Math 2008).

Case 2. Non-quantized singularity ($\alpha \notin \mathbb{N}$)

$$\Delta u + h(x)|x|^{2\alpha} e^u = 0.$$

$$\int h(x)|x|^{2\alpha} e^u < +\infty$$

$$\alpha \notin \mathbb{N}$$

Theorem: All bubbles are simple

Classification Theorem

Theorem

(Prajapat-Tarantello 01) If $\alpha > -1$ is not an integer, all solutions to

$$\Delta u + |x|^{2\alpha} e^u = 0, \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^{2\alpha} e^u < \infty,$$

are radially symmetric and can be written as

$$u(x) = U_\lambda = \log \frac{e^\lambda}{(1 + \frac{e^\lambda}{8(1+\alpha)^2} |x|^{2+2\alpha})^2}$$

for some $\lambda \in \mathbb{R}$. The total integration is

$$\int_{\mathbb{R}^2} |x|^{2\alpha} e^u = 8\pi(1+\alpha).$$

Non-quantized singularity

Theorem

(Bartolucci-Chen-Lin-Tarantello (CPDE 04)) Let u_k be blowup solutions to

$$\Delta u_k + |x|^{2\alpha} h e^{u_k} = 0, \quad B_1$$

with $\alpha > -1$ and bounded oscillation on ∂B_1. Suppose 0 is the only blowup point in B_1, then

$$h e^{u_k} \rightharpoonup 8\pi(1+\alpha)\delta_0$$

and if α is not a positive integer

$$u_k(x) - \log \frac{e^{u_k(0)}}{(1 + \frac{h(0)}{8(1+\alpha)^2} e^{u_k(0)} |x|^{2\alpha+2})^2} = O(1) \quad B_1.$$

Chen-Lin (CPAM 07): Topological degree when $\alpha \notin N$.

Case 3. Quantized singularity ($\alpha \in \mathbb{N}$)

$$\Delta u + h(x)|x|^{2N} e^u = 0.$$

$$\int h(x)|x|^{2N} e^u < +\infty$$

$$\alpha = N \in \mathbb{N}$$

Classification Theorem

Theorem

(Prajapat-Tarantello 01) All solutions of

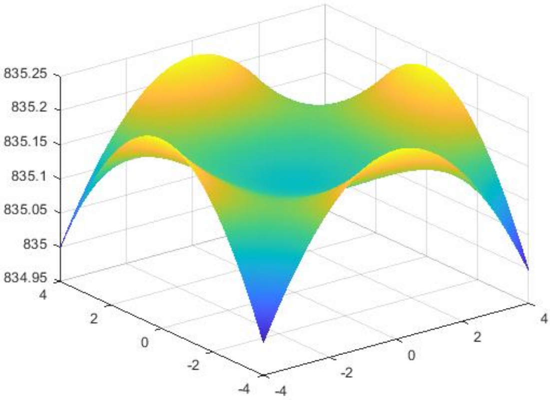
$$\Delta u + |x|^{2N} e^u = 0, \quad in \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} |x|^{2N} e^u < \infty,$$

are of the form

$$u(z) = \log \frac{e^\lambda}{(1 + \frac{e^\lambda}{8(1+N)^2} |z^{N+1} - \xi|^2)^2}$$

for some $\xi \in \mathbb{C}$. $\int_{\mathbb{R}^2} |x|^{2N} e^u = 8\pi(1+N)$.

Non-simple-blow-ups



If we choose $\xi_k \to 0$ and $\lambda_k \to \infty$ we can see non-simple blowup solutions.

Quantized singularity, Non-simple blowup

Let u_k be a sequence of solutions to

$$\Delta u_k + |x|^{2N} h(x) e^{u_k} = 0, \quad \text{in} \quad B_1 \subset \mathbb{R}^2,$$

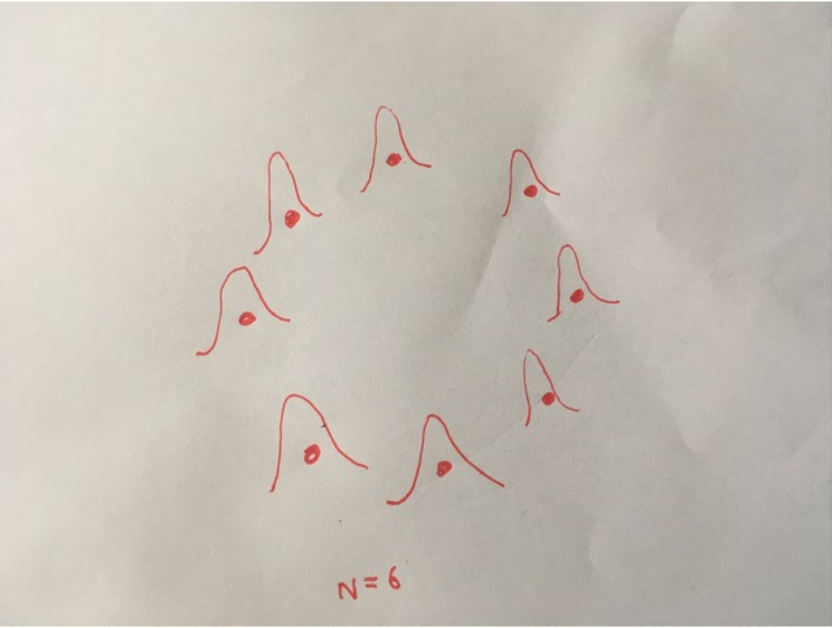
where $h > 0$ is smooth. Suppose 0 is the only blowup point and N is a positive integer, u_k has bounded oscillation on ∂B_1 and $\int_{B_1} |x|^{2N} h e^{u_k} < C$.

Theorem

(Kuo-Lin (16), Bartolucci-Tarantello (18)) For $N \in \mathbb{N}$, if u_k has a non-simple blowup point at 0:

$$\max_{x \in B_1} u_k(x) + 2(1+N) \log |x| \to \infty.$$

u_k has exactly $N+1$ local maximum points evenly distributed around 0.


N=6

Related questions

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- Is it possible to approximate bubbling solutions by global solutions?

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- How to analyze the non-simple blow-ups?
- Is it possible to approximate bubbling solutions by global solutions?
- Are there vanishing theorems? Especially the vanishing estimate of first and second derivatives of coefficient functions?

Vanishing Theorems

Theorem

(Wei-Zhang, (2022 preprint, 2022, 2021)) Let u_k *be non-simple blowup solutions to*

$$\Delta u_k + |x|^{2N} h_k(x) e^{u_k} = 0, \quad in \quad B_1 \subset \mathbb{R}^2,$$

under the usual assumptions. Then along a sub-sequence

$$\lim_{k\to\infty} \nabla h_k(0) = 0.$$

$$\lim_{k\to\infty} \Delta h_k(0) = 0.$$

More Vanishing Theorems when $N = 1$

Theorem

(D'Aprile-Wei-Zhang) Let u_k be non-simple blowup solutions

$$\Delta u_k + h_k |x|^2 e_k^u = 0 \text{ in } B_1$$

Then

$$\nabla h_k(0) = o(1)$$

$$\partial^2_{xx} h_k(0) = o(1), \partial^2_{xy} h_k(0) = \partial^2_{yx} h_k(0) = o(1), \partial^2_{yy} h_k(0) = o(1)$$

In other words, h_k vanishes all up to second order:

$$\nabla^\alpha h_k(0) = o(1), |\alpha| \leq 2$$

Conjecture on Vanishing Theorems when $N \geq 2$

Conjecture: Let u_k be non-simple blowup solutions

$$\Delta u_k + h_k |x|^{2N} e_k^u = 0 \text{ in } B_1$$

Then

$$\nabla^\alpha h_k(0) = o(1), |\alpha| \leq N + 1$$

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Then

$$\nabla^\alpha h_k(0) = o(1), |\alpha| \leq N + 1$$

Non-simple blow-ups do exist: D'Aprile (JMP 2022)

$$\Delta u + \lambda V(x)|x|^{2N} e^u = 0$$

$$V(x) = V(z^{N+1})$$

Non-existence theorems

Surprisingly in many general situations, non-simple blow-up does not happen. D'Aprile-Wei studied the following classical Liouville equation

$$\Delta u + \lambda e^u = \sum_{i=1}^{M} 4\pi\gamma_i\delta_{p_i} \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \qquad (2)$$
$$u = 0 \quad \text{on} \quad \partial\Omega,$$

where Ω is an open and bounded subset of $\mathbb{R}^2$, $p_1, ..., p_M \in \Omega$, $\partial\Omega$ is smooth, $\lambda > 0$ and $\gamma_i > -1$.

Existence of multiple bubbling solutions

- $\gamma_i = 0$ del Pino-Kowalczyk-Musso (2003)
- $\gamma_i \neq 0$ D'Aprile (2013)

Nonsimple blow-ups?

D'Aprile-Wei (JFA2020): Let $\Omega = B_1$ and $p_1 = 0$.

$$\begin{aligned} \Delta u + \lambda e^u = 4\pi N_\lambda \delta_0 \quad \text{in} \quad B_1, \\ u = 0 \quad \text{on} \quad \partial B_1, \end{aligned} \tag{3}$$

Then non-simple blow-ups exist if

$$N_\lambda - N \sim C\lambda \log^2 \lambda$$

Nonsimple blow-ups?

D'Aprile-Wei (JFA2020): Let $\Omega = B_1$ and $p_1 = 0$.

$$\begin{aligned} \Delta u + \lambda e^u = 4\pi N_\lambda \delta_0 \quad \text{in} \quad B_1, \\ u = 0 \quad \text{on} \quad \partial B_1, \end{aligned} \tag{3}$$

Then non-simple blow-ups exist if

$$N_\lambda - N \sim C\lambda \log^2 \lambda$$

Conjecture: When $\Omega = B_1, N_\lambda = N$, there are no non-simple blow-up phenomena (3).

Theorem

(D'Aprile-Wei-Zhang-22) Let u_k be a sequence of blowup solutions of

$$\Delta u + \lambda e^u = \sum_{i=1}^{M} 4\pi\gamma_i \delta_{p_i} \quad in \quad \Omega \subset \mathbb{R}^2, \tag{4}$$

$$u = 0 \quad on \quad \partial\Omega,$$

with parameter λ_k that satisfies $\int_\Omega \lambda_k e^{u_k} < C$. Then u_k is simple around any blowup point in Ω.

Battaglia (PAMS2019): If Ω is simply connected and there is only one singularity, then

$$\lambda \int_\Omega e^u \leq C$$

As a result we have completely solved D'Aprile-Wei conjecture:
Corollary: When $\Omega = B_1, \gamma_i = N$, there are no non-simple blow-up phenomena (2).

$$\Delta u + \lambda e^u = 4\pi N \delta_0 \quad \text{in} \quad B_1, \qquad (5)$$
$$u = 0 \quad \text{on} \quad \partial B_1,$$

Non-simple blow-ups are lonely

$$\Delta u_k + e^{u_k} = \sum_{i=1}^{M} 4\pi\gamma_i\delta_{p_i} \quad \text{in} \quad \Omega \tag{6}$$

$$\int_\Omega e^{u_k} \le C \tag{7}$$

and

$$|u_k(x) - u_k(y)| \le C, \quad \forall x, y \in \partial\Omega. \tag{8}$$

Theorem

(D'Aprile-Wei-Zhang-22) Let u_k be a sequence of blowup solutions of (6) such that (7) and (8) hold. If there are at least two blowup points in Ω, each blowup point is simple.

Summary

$$\Delta u_k + h_k |x|^{2N} e^{u_k} = 0$$

- Vanishing Theorems

$$\nabla h_k(0) = 0, \Delta h_k(0) = 0$$

$$\nabla h_k = 0, D^2 h_k = 0 \quad (N = 1)$$

- 2. No-simple blow-ups does not exist

$$\Delta u + \lambda e^u = 4\pi \sum_{i=1}^{M} \gamma_i \delta_{p_i} \text{ in } \Omega; \; u = 0 \text{ on } \partial\Omega$$

- 3. Non-simple blow-ups are lonely
  If there are two blow-ups then non-simple blow-ups do not exist.

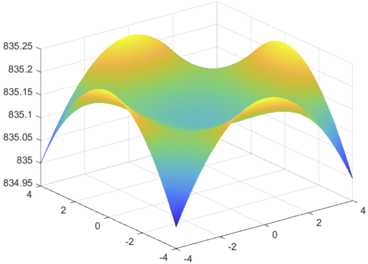
Local maximum points

Let p_0^k, ..., p_N^k be the $N+1$ local maximums of u_k

$$\Delta u_k + |x|^{2N} h_k(x) e^{u_k} = 0, \quad \text{in} \quad B_1.$$

Let

$$\delta_k = |p_0^k|, \quad \mu_k = u_k(p_0^k) + 2(1+N)\log \delta_k.$$



Trivial Observations

- The study of blowup solutions looks like that of a single Liouville equation near each local maximum point.
- The relations between these local maximums plays a crucial role.
- The blowup solutions look almost like a harmonic function away from the $N+1$ local maximums.
- If there is a perturbation on a global solution, there is a corresponding perturbation on each of its $N+1$ local maximumus:

$$V_k(x) = \log \frac{e^{\mu_k}}{(1 + \frac{e^{\mu_k}}{8(N+1)^2} |x^{N+1} - (1+p_k)|^2)^2}.$$

Difficulty and Problem

- The main difficulty: a priori we don't have any relation between

$$\delta_k \quad \text{(the distance between small bubbles)}$$

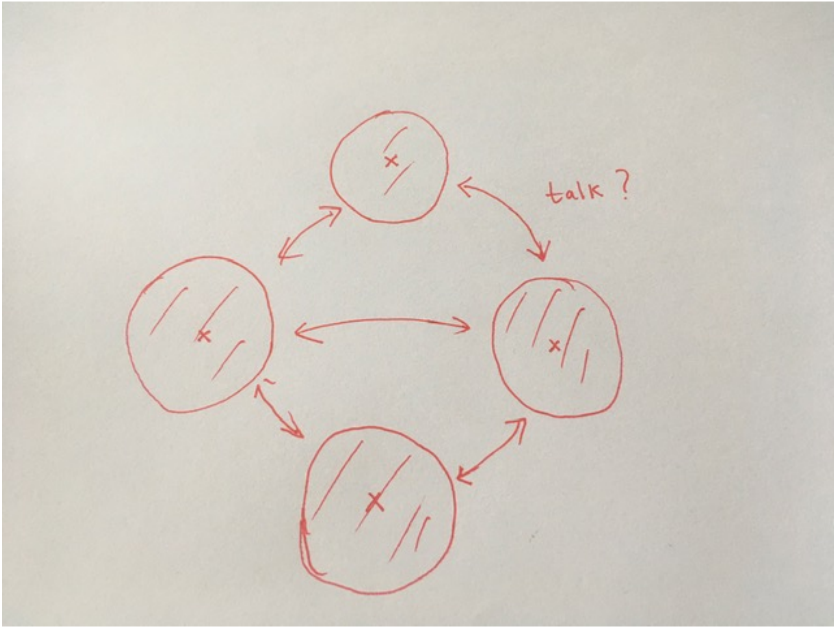
  and

$$\mu_k = u_k(p_0^k) + 2(1+N)\log\delta_k. \text{(the height of bubbles)}$$

  In fact it should be no relation at all, from the ground state solution:

$$U(z) = \log \frac{e^{\lambda}}{(1 + \frac{e^{\lambda}}{8(1+N)^2}|z^{N+1} - \xi|^2)^2}$$

- The main problem: how do different bubbles talk to each other?


talk ?

Stage 1: First Vanishing Theorems

Theorem

(Wei-Zhang, (Proc. LMS 21)) Let ϕ_k be the harmonic function that eliminates the oscillation of u_k on ∂B_1, then

$$|\nabla(\log h_k + \phi_k)(0)| = O(\delta_k^{-1}\mu_k e^{-\mu_k}) + O(\delta_k).$$

$$\Delta(\log h_k)(0) = O(\delta_k^{-2}\mu_k e^{-\mu_k}) + O(\delta_k), \quad N \geq 2.$$

Obviously we don't know if $\nabla(\log h_k + \phi_k)(0) = o(1)$ when $\delta_k \leq C\mu_k e^{-\mu_k}$. We cannot tell if $\Delta h_k(0) = o(1)$ if $\delta_k \leq C\mu_k^{\frac{1}{2}} e^{-\mu_k/2}$ even for $N \geq 2$. The conclusion for $N = 1$ is even weaker.

Theorem

(Wei-Zhang 21) If $\delta_k \leq Ce^{-\mu_k/4}$, *then there exists a sequence of global solutions* U_k *such that*

$$|u_k(x) - U_k(x)| \leq C, \quad x \in B_1.$$

For $|x| \sim 1$, $u_k(x) = -u_k(p_0^k) + O(1)$.

Linearized equation

Let U be the solution of

$$\Delta U + e^U = 0, \quad \text{in} \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^U < \infty,$$

with $\max_x U(x) = 1 = U(0)$. By Chen-Li, $U(x) = \log \frac{1}{(1+\frac{1}{8}|x|^2)^2}$. Let ϕ be a solution of

$$\Delta\phi + e^U \phi = 0, \quad \text{in} \quad \mathbb{R}^2$$

with $\phi(x) = o(|x|)$ at infinity. Then $\phi(x) = c_0\phi_0 + c_1\phi_1 + c_2\phi_2$ where

$$\phi_0 = \frac{1-\frac{1}{8}|x|^2}{1+\frac{1}{8}|x|^2}, \quad \phi_1(x) = \frac{x_1}{1+\frac{1}{8}|x|^2}, \quad \phi_2 = \frac{x_2}{1+\frac{1}{8}|x|^2}.$$

key ideas of the proof

Step one: A lot of Pohozaev identities.

- A Pohozaev identity for $\Delta u_k + h_k e^{u_k} = 0$ on B_σ is

$$\int_{B_\sigma} (\nabla h_k \cdot x) e^{u_k} = \int_{\partial B_\sigma} \left(\frac{\sigma}{2}(|\partial_\nu u_k|^2 - |\partial_\tau u_k|^2) + \sigma h_k e^{u_k} + 2\partial_\nu u_k \right) dS.$$

- $$\delta_k \nabla(\log h_k)(\delta_k Q_l^k) + 2N \frac{Q_l^k}{|Q_l^k|^2} + \nabla \phi_{l,k}(Q_l^k) = O(\mu_k e^{-\mu_k}).$$

- $$\nabla \phi_l^k(Q_l^k) = -4 \sum_{m \neq l} \frac{Q_l^k - Q_m^k}{|Q_l^k - Q_m^k|^2} + O(\delta_k^2) + O(\mu_k e^{-\mu_k}).$$

- Denoting $Q_l^k = e^{i\frac{2\pi l}{N+1}}(1+m_l^k)$ and use this in the long computation of each Pohozaev identity, we have
- $$\begin{pmatrix} m_1^k \\ m_2^k \\ \vdots \\ m_N^k \end{pmatrix} = A^{-1}\delta_k \bar{\nabla}(\log h_k)(0) \begin{pmatrix} e^{i\beta_1} \\ e^{i\beta_2} \\ \vdots \\ e^{i\beta_N} \end{pmatrix} + O(\delta_k^2) + O(\mu_k e^{-\mu_k})$$

  where $\beta_l = 2\pi l/(N+1), l = 0, ..., N$.

$$A = \begin{pmatrix} D & -d_1 & \dots & -d_{N-1} \\ -d_1 & D & \dots & -d_{N-2} \\ \vdots & \vdots & \dots & \vdots \\ -d_{N-1} & -d_{N-2} & \dots & D \end{pmatrix}$$

where

$$d_i = \frac{1}{\sin^2(\frac{i\pi}{N+1})}, \quad i = 1, ..., N, \quad D = d_1 + ... + d_N.$$

key ideas

1.Let $v_k(y) = u_k(\delta_k y) + 2\log\delta_k$. Since δ_k is the distance from a local maximum of v_k to the origin, and Δ is invariant under rotation of coordinates, we can assume that v_k has a local maximum at e_1. Then we use a global solution V_k that agrees with v_k at e_1:

$$\Delta V_k + h_k(\delta_k e_1)|y|^{2N} e^{V_k} = 0.$$

$$V_k(y) = \log \frac{e^{\mu_k}}{\left(1 + \frac{e^{\mu_k} h_k(\delta_k e_1)}{8(1+N)^2}|y^{N+1} - e_1|^2\right)^2}.$$

V_k has $N+1$ local maximums located at exactly $e^{2\pi i l/(N+1)}$ for $l = 0, ..., N$.

Let $w_k = v_k - V_k$. Then w_k is very small near e_1.

2. By Harnack inequality, this smallness will be passed to control all the regions away from the N other bubbling disks.

key ideas of the proof

3. The difference between the Pohozaev identities. Let Ω_s be the region about Q_s. Then the Pohozaev identity for v_k in this region is

$$\int_{\Omega_s} \partial_\xi(|y|^{2N} h_k(\delta_k y)) e^{v_k} - \int_{\partial\Omega_s} e^{v_k} |y|^{2N} h_k(\delta_k y)(\xi \cdot \nu) = \int_{\partial\Omega_s} (\partial_\nu v_k \partial_\xi v_k - \frac{1}{2}|\nabla v_k|^2 (\xi \cdot \nu)) dS.$$

$$\int_{\Omega_s} \partial_\xi(|y|^{2N} h_k(\delta_k e_1)) e^{V_k} - \int_{\partial\Omega_s} e^{V_k} |y|^{2N} h_k(\delta_k e_1)(\xi \cdot \nu) = \int_{\partial\Omega_s} (\partial_\nu V_k \partial_\xi V_k - \frac{1}{2}|\nabla V_k|^2 (\xi \cdot \nu)) dS.$$

Main ideas

Using these in the computation of the $N+1$ Pohozaev identities we have

$$\nabla(\log h_k + \phi_k)(0) = O(\delta_k^{-1}\mu_k e^{-\mu_k}) + O(\delta_k) \quad N \geq 1$$
$$\Delta \log h_k(0) = O(\delta_k^{-2}\mu_k e^{-\mu_k}) + O(\delta_k), \quad N \geq 2.$$

and a corresponding estimate for $N = 1$.

Stage 2, better first order estimates

Key ideas to prove the vanishing rate of $\nabla h_k(0)$ (for simplicity ϕ_k is ignored). Let $w_k = v_k - V_k$, then we have this key estimate:

$$|w_k(y)| \leq C(|\nabla h_k(0)|\delta_k + \delta_k^2\mu_k).$$

Only need to consider $\delta_k \leq o(\epsilon_k)$. The equation of w_k can be written as

$$\Delta w_k + h_k(\delta_k y)|y|^{2N}e^{\xi_k}w_k = \delta_k \nabla h_k(\delta_k e_1)\cdot(e_1 - y)|y|^{2N}e^{V_k} + E$$

where

$$E = O(\delta_k^2)|y - e_1|^2|y|^{2N}e^{V_k}.$$

It is important to observe that the right hand side is zero when $y = e_1$. *The analysis is first carried out near e_1 and pass to other regions by Harnack inequality*

First order vanishing estimate

Let $M_k = \max |w_k|$ and let $\tilde{w}_k = w_k / M_k$. It is crucial to observe that we still have

$$\tilde{w}_k(e_1) = |\nabla \tilde{w}_k(e_1)| = 0.$$

This important information will make us obtain

$$\tilde{w}_k(e_1 + \epsilon_k z) \le C\epsilon_k^\sigma (1 + |z|)^\sigma, \quad |z| < \epsilon_k^{-1}$$

where $\epsilon_k = e^{-\mu_k/2}$ and $\sigma \in (0, 1)$. Because of the smallness of $|Q_s^k - e^{i\beta_s}|$, $\tilde{w}_k$ is supposed to converge to a kernel of

$$\Delta\phi + e^U \phi = 0$$

around each Q_s^k. The same argument can also be applied around each Q_s.

At Q_s^k, v_k is very close to another global solution V_s^k which agrees with v_k at Q_s^k and $\nabla V_s^k(Q_s^k) = 0$. The expression of V_s^k, which satisfies

$$\Delta V_s^k + h_k(\delta_k Q_s^k)|y|^{2N} e^{V_s^k} = 0, \quad \text{in} \quad \mathbb{R}^2,$$

is

$$V_s^k(y) = \log \frac{e^{\mu_s^k}}{(1 + \frac{e^{\mu_s^k} h_k(\delta_k Q_s^k)}{8(1+N)^2}|y^{N+1} - (e_1 + p_s^k)|^2)^2}.$$

The function $\tilde{w}_k$ is supposed to converge

$$c_1 \frac{1 - \frac{1}{8}|y|^2}{1 + \frac{1}{8}|y|^2} + c_2 \frac{y_1}{1 + \frac{1}{8}|y|^2} + c_3 \frac{y_2}{1 + \frac{1}{8}|y|^2}.$$

All these coefficients are determined by $V_s^k - V_k$. It is standard to prove $c_1 = 0$ (this implies that the differences on the magnitudes don't matter too much). To prove c_2 and c_3 zero we need to use p_s.

If we take Q_s^k as a base and consider the kernel function around Q_l^k, then the limit function is supposed to be

$$c_{1,s,t}\frac{y_1}{1+\frac{1}{8}|y|^2}+c_{2,s,t}\frac{y_2}{1+\frac{1}{8}|y|^2}.$$

After some computations we have

$$c_{1,s,t}=\lim_{k\to\infty}\frac{|p_s^k-p_t^k|}{2(N+1)M_k\epsilon_k}\cos(\frac{2\pi s}{N+1}+\theta_{st}).$$

$$c_{2,s,t}=\lim_{k\to\infty}\frac{|p_s^k-p_t^k|}{2(N+1)M_k\epsilon_k}\sin(\frac{2\pi s}{N+1}+\theta_{st}).$$

where $p_s^k-p_t^k=|p_s^k-p_t^k|e^{i\theta_{ts}}$. If limit has to exist, p_1^k,....,p_N^k have to satisfy certain relations, which will lead to a contradiction if we observe the second order terms.

After proving

$$|w_k(y)| \leq C\delta_k |\nabla h_k(0)| + C\delta_k^2 \mu_k,$$

we use this estimate in the computation of Pohozaev identities around each Q_s^k to obtain

$$|\nabla h_k(0)| \leq C\delta_k \mu_k.$$

Stage 3: Laplace Vanishing Theorem

The first order estimate leads to a better estimate on the difference function:

$$|w_k(y)| \leq C\delta_k^2 \mu_k,$$

Then we use Gluck's estimate for single Liouville equation around each Q_s^k ($s \neq 1$) to obtain the vanishing rate for $\Delta h_k(0)$. Recall the expansion of a blowup solution for a single Liouville equation:

$$\begin{aligned} u_k(x) &= \log \frac{e^{u_k(0)}}{(1 + \frac{h(0)}{8} e^{u_k(0)} |x - q_k|^2)^2} + \psi_k \\ &\quad -8 \frac{(\Delta \log h)(0)}{h(0)} \epsilon_k^2 (\log(2 + \epsilon_k^{-1}|x|))^2 + O(\epsilon_k^2 \log \epsilon_k^{-1}) \end{aligned}$$

Main Idea for the proof of D'Aprile-Wei conjecture

Theorem (D'Aprile-Wei-Zhang): All blow-ups for the following problem is simple:

$$-\Delta u = \lambda e^u - 4\pi \sum_{i=1}^{M} \gamma_i \delta_{p_i} \text{ in } \Omega$$

$$\lambda \int_\Omega e^u < C$$

$$u = 0 \text{ on } \partial\Omega$$

We found that when non-simple blowup happens, the oscillation on the boundary has to be very special. This is the main reason that we can prove the conjecture in a very general setting. Basically, as long as we know the behavior of the blowup solutions on the boundary and it is different from that of a non-simple blowup global solutions, we can capture this difference and say that non-simple blowup cannot happen.

D'Aprile-Wei conjecture. Key Theorem

Basic set-up: Let u_k be a sequence of solutions of the following equation that blows up at 0:

$$\Delta u_k + |x|^{2N} e^{u_k} = 0, \quad \text{in} \quad B_1 \tag{9}$$

Suppose the oscillation of u_k on the boundary of B_1 is finite:

$$|u_k(x) - u_k(y)| \le C, \quad \forall x, y \in \partial B_1 \tag{10}$$

for some $C > 0$ independent of k, and there is a uniform bound on the integration of $|x|^{2N} e^{u_k}$:

$$\int_{B_1} |x|^{2N} e^{u_k} < C. \tag{11}$$

D'Aprile-Wei-Conjecture, key theorem

Set

$$\Phi_k(x) = u_k(x) - \frac{1}{2\pi}\int_{\partial B_1} u_k, \quad x \in B_1,$$

and let Φ be the limit of Φ_k over any fixed compact subset of B_1. Then our assumption of Φ_k is

$$\text{Either } \Phi \neq 0 \quad \text{or} \quad \Phi_k \equiv 0. \tag{12}$$

Theorem

Let 0 be the only blowup point of u_k in B_1, which has a uniformly bounded integration. Suppose (12) holds. Then u_k is a simple blowup sequence:

$$u_k(x) + 2(1+N)\log|x| \leq C$$

for some $C > 0$.

key-ideas

Let v_k be the scaled u_k with $p_0^k = e_1$:

$$v_k(y) = u_k(\delta_k y e^{i\theta_k}) + 2(N+1)\log\delta_k, \quad |y| < \delta_k^{-1}.$$

Other local maximums are very close to $e^{\frac{2i\pi l}{N+1}}$ for $l = 1, ..., N$. Let

$$V_k(x) = \log \frac{e^{\bar{\mu}_k}}{(1 + \frac{e^{\bar{\mu}_k}}{8(1+N)^2}|y^{N+1} - e_1|^2)^2}.$$

that agrees with v_k at e_1 as a common local maximum. Now we use the following expansion of V_k for $|y| = L_k$ ($L_k = \delta_k^{-1}$)

$$\begin{aligned} V_k(y) = & -\bar{\mu}_k + 2\log(8(N+1)^2) - 4(N+1)\log L_k + \frac{2}{L_k^{2N+2}} \\ & + \frac{4\cos((N+1)\theta)}{L_k^{N+1}} + \frac{4}{L_k^{2N+2}}\cos((2N+2)\theta) \\ & + O(L_k^{-3N-3}) + O(e^{-\bar{\mu}_k} L_k^{-2N-2}). \end{aligned}$$

key ideas

The oscillating part of V_k is mainly

$$4\cos((N+1)\theta)\delta_k^{N+1} + 4\delta_k^{2N+2}\cos((2N+2)\theta).$$

based on this we set $\phi_{v,k}(\delta_k \cdot)$ to be the harmonic function that is equal to 0 at 0 and represents the oscillation of V_k $\partial\Omega_k$:

$$\phi_{v,k}(\delta_k y) = 4\delta_k^{2N+2} r^{N+1}\cos((N+1)\theta) + 4\delta_k^{4N+4} r^{2N+2}\cos((2N+2)\theta) + ...$$

Recall that the oscillation of v_k is $\Phi_k(\delta_k \cdot)$. Thus if we set

$$\phi_{0,k}(y) = \Phi_k(\delta_k y) - \phi_{v,k}(\delta_k y)$$

and $v_{0,k} = v_k - \phi_{0,k}$, then $v_{0,k} - V_k$ is a constant on the boundary, but the equation of $v_{0,k}$ is

$$\Delta v_{0,k} + h_{0,k}|y|^{2N} e^{v_{0,k}} = 0, \quad \text{in} \quad \Omega_k$$

where $h_{0,k} = e^{\phi_{0,k}}$.

key-ideas

Because of the difference on the oscillations, we can prove that $\nabla h_{0,k}(e^{\frac{2\pi is}{N+1}})$ is different from zero to some extent (based on the Fourier expansions of these harmonic functions):

Lemma

There exist an integer $L > 0$,$\delta_k^* \in (\delta_k^L, \delta_k)$ an integer $0 \leq s \leq N$ such that

$$\nabla h_{0,k}(e^{\frac{2\pi is}{N+1}})/\delta_k^* \neq 0.$$

This lemma eventually leads to a contradiction: If non-simple blowup does exist, and v_k is so close to a global solution V_k, there is no way for $v_{0,k}$ to have a coefficient function different from 1. This part of the proof is similar to that of the vanishing theorems: two functions are extremely close near one local maximum, using Harnack this closeness can be passed to the neighborhood of other local maximums, then Pohozaev identities say this is not possible.

Application to Mean Field Equation

Corollary

(Wei-Zhang-22) Let u be a solution of

$$\Delta_g u + \rho(\frac{he^u}{\int_M he^u} - 1) = 4\pi \sum_{j=1}^{d} \alpha_j(\delta_{p_j} - 1).$$

If all $\alpha_j \in \mathbb{N}$ and

$$\Delta(\log h)(p_j) - 2K(p_j) \notin 4\pi\mathbb{N}, \quad j = 1, .., d.$$

Then any blowup solutions u^k satisfy a spherical Harnack inequality around any blowup point.

Application to Toda systems

$$\Delta u_1 + 2\rho_1\left(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1\right) - \rho_2\left(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1\right) = 0,$$
$$\Delta u_2 - \rho_1\left(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1\right) + 2\rho_2\left(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1\right) = 0,$$

Theorem

(Lin-Wei-Yang-Zhang 18 APDE) For $(\rho_1, \rho_2) \in (4\pi m, 4\pi(m+1)) \times (4\pi n, 4\pi(n+1))$ *($n, m \in \mathbb{N}$) and* $u = (u_1, u_2)$ *in certain Sobolev space, the following a priori estimate holds*

$$|u_i| \leq C, \quad i = 1, 2.$$

This theorem leads to a huge degree counting program for Toda systems.

Theorem

(Wei-Wu-Zhang (JLMS23)) If $u_k = (u_1^k, u_2^k)$ *is a sequence of blowup solutions of* $SU(3)$ *Toda system corresponding to* $(\rho_1^k, \rho_2^k) \to (4\pi m, 4\pi n)$*, if one blowup point is a fully bubbling blowup point and*

$$\Delta_g \log h_i^k(x) - 2K(x) \notin 4\pi\mathbb{Z}, \quad i = 1, 2.$$

then the spherical Harnack inequality holds around each blowup point.

Applications

1. Toda system:

$$\Delta u_1 + 2\rho_1\left(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1\right) - \rho_2\left(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1\right) = 0$$
$$\Delta u_2 - \rho_1\left(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1\right) + 2\rho_2\left(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1\right) = 0.$$

2. Liouville system: Let $A = (a_{ij})_{n\times n}$ be a symmetric, non-negative matrix:

$$\Delta u_1 + a_{11}\rho_1\left(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1\right) + a_{12}\rho_2\left(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1\right) = 0$$
$$\Delta u_2 + a_{12}\rho_1\left(\frac{h_1 e^{u_1}}{\int_M h_1 e^{u_1}} - 1\right) + a_{22}\rho_2\left(\frac{h_2 e^{u_2}}{\int_M h_2 e^{u_2}} - 1\right) = 0.$$

More Applications

1. Fourth order equation: Q curvature equation on 4−manifold:

$$P_g u + 2Q_g = 2he^{4u} - 8\pi^2\gamma(\delta_q - \frac{1}{vol_g(M)})$$

$$P_g\phi = \Delta_g^2\phi + div_g((\frac{2}{3}R_g g - 2Ric_g)\nabla\phi)$$

$$Q_g = -\frac{1}{12}(\Delta_g R_g - R_g^2 + 2|Ric_g|^2)$$

Classification theorems were proved for

$$\Delta^2 u = 6e^{4u} - 8\pi^2\gamma\delta_0 \quad \text{in} \quad \mathbb{R}^4, \quad \int_{\mathbb{R}^4} e^{4u} < \infty.$$

If $\gamma = 0$ the classification theorem was proved by Chang-shou Lin, Wei-Xu. For $-1 < \gamma < 0$, the classification was done by Ahmedou-Wu-Zhang (22).

2. Many other equations and situations.

THANKS FOR YOUR ATTENTION