QUANTITATIVE STABILITY OF HARMONIC MAPS FROM $\mathbb{R}^2$ TO $\mathbb{S}^2$ WITH HIGHER DEGREE

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ABSTRACT. For degree ± 1 harmonic maps from $\mathbb{R}^2$ (or $\mathbb{S}^2$) to $\mathbb{S}^2$, Bernand-Mantel, Muratov and Simon [2] recently establish a uniformly quantitative stability estimate. Namely, for any map $u : \mathbb{R}^2 \to \mathbb{S}^2$ with degree ± 1, the discrepancy of its Dirichlet energy and 4π can linearly control the $\dot{H}^1$-difference of u from the set of degree ± 1 harmonic maps. Whether a similar estimate holds for harmonic maps with higher degree is unknown. In this paper, we prove that a similar quantitative stability result for higher degree is true only in local sense. Namely, given a harmonic map, a similar estimate holds if u is already sufficiently near to it (modulo Möbius transform) and the bound in general depends on the given harmonic map. More importantly, we investigate an example of degree 2 case thoroughly, which shows that it fails to have a uniformly quantitative estimate like the degree ± 1 case. This phenomenon show the striking difference of degree ± 1 ones and higher degree ones. Finally, we also conjecture a new uniformly quantitative stability estimate based on our computation.

1. INTRODUCTION

1.1. **Motivation and main results.** The analysis of critical points of conformally invariant Lagrangians has drawn much attention since 1950, due to their important applications in physics and geometry. One of the prominent examples is harmonic maps $u : \mathbb{R}^2 \to \mathbb{S}^2$, which are critical points of the following Dirichlet energy

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx. \tag{1.1}$$

It is well-known that the critical points u will satisfy

$$\Delta u + |\nabla u|^2 u = 0, \quad \text{in } \mathbb{R}^2. \tag{1.2}$$

In the special case of mapping from $\mathbb{R}^2$ to $\mathbb{S}^2$, all the harmonic maps have been classified. For instance, see [9, 11.6] and [15, Section 2.2]. To state the result, we introduce

$$\mathcal{A}_d = \{(p, q) : p/q \text{ is an irreducible rational function of } z, \max\{\deg p, \deg q\} = d\} \tag{1.3}$$

for any $d \in \mathbb{Z}_{\geq 1}$. Here z is the complex variable in $\mathbb{C} = \mathbb{R}^2$. When $d \in \mathbb{Z}_{\leq -1}$, we also introduce the notation $\mathcal{A}_d = \{(\bar{p}, \bar{q}) : (p, q) \in \mathcal{A}_{|d|}\}$. Throughout this paper, we assume d is an integer with $|d| \geq 1$.

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Proposition 1.1. *A map u from $\mathbb{R}^2(=\mathbb{C})$ to $\mathbb{S}^2$ is harmonic if and only if u is holomorphic or anti-holomorphic. More precisely, if $\deg(u) = d$ then $u = \mathcal{S}(p/q)$ where $(p, q) \in \mathcal{A}_d$. Here $\mathcal{S}$ is the stereographic projection from $\mathbb{C} \to \mathbb{S}^2 \setminus \{N\}$ (see (2.5)).*

For any pair $(p, q) \in \mathcal{A}_d$, we can normalize p to be a monic polynomial such that p/q stays the same. We call such pair (p, q) to be canonical. Thus we define

$$\mathcal{A}_d^m = \{(p, q) \in \mathcal{A}_d : p \text{ is monic}\}. \tag{1.4}$$

Then for any harmonic map $\Phi : \mathbb{R}^2 \to \mathbb{S}^2$ it can be represented by a unique canonical pair $(p, q) \in \mathcal{A}_d^m$.

It is also known that harmonic maps from $\mathbb{R}^2$ to $\mathbb{S}^2$ achieve minimal Dirichlet energy within its homotopy class by the work of Lemaire [17] and Wood [23] (also see [9, (11.5)]).

Lemma 1.2. *Suppose that $u : \mathbb{R}^2 \to \mathbb{S}^2$ with $\mathcal{E}(u) < \infty$. Then $\mathcal{E}(u) \geq 4\pi|\deg(u)|$. The equality holds if and only if u is harmonic.*

A natural question is that whether the discrepancy $\mathcal{E}(u) - 4\pi|\deg(u)|$ can quantitatively control the difference of u from the harmonic maps. Such type of question has been raised for many other topics. For instance, Brezis and Lieb [4] ask a similar question to the classical Sobolev inequality on $\mathbb{R}^n$. Later Bianchi and Egnell [3] obtain a quantitative stability estimate in the spirit of (1.5). (See also recent work of Figalli and Zhang [12].) Fusco et al. [13] prove a sharp quantitative stability about isoperimetric inequality.

Recently Bernand-Mantel, Muratov, and Simon [2] prove a quantitative stability for degree ± 1 harmonic maps from $\mathbb{R}^2$ to $\mathbb{S}^2$ as reformulated in the following. The proof in [2] has been simplified by Hirsch and Zemas [16] and Topping [22].

Proposition 1.3. *There exists a universal constant C such that for every $u : \mathbb{R}^2 \to \mathbb{S}^2$ with $\mathcal{E}(u) < \infty$ and $\deg(u) = 1$, there exists $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ such that*

$$\int_{\mathbb{R}^2} \left| \nabla u - \nabla \mathcal{S}\left(\frac{az + b}{cz + d}\right) \right|^2 \leq C(\mathcal{E}(u) - 4\pi). \tag{1.5}$$

If $\deg(u) = -1$, then the above statement holds for $\bar{z}$.

The above theorem leaves us an intriguing question for harmonic maps with higher degree. We have addressed a similar question for half-harmonic maps in [7]. There we have shown that a similar quantitative stability for higher degree ones is only true in the local sense. In this paper, we shall prove that the similar phenomenon happens here. More precisely, given a harmonic map (or a compact set of harmonic maps), there is a local stability result near to it. The bound in general will depend on the given harmonic map (or the compact set).

To that end, let us introduce some notations. For any two complex polynomials $p, \tilde{p}$ on z (or $\bar{z}$), we define $|p - \tilde{p}|_\infty$ to be the maximum of all the modulus of each coefficient of $p - \tilde{p}$. Thus $\mathcal{A}_d^m$ becomes a metric space equipped with the distance $|\cdot|_\infty$. When we call Ω is a compact set of degree d harmonic maps from $\mathbb{R}^2$ to $\mathbb{S}^2$, it actually means there exists a compact set of $\mathcal{A}_d^m$, say $\mathcal{A}_\Omega^m$, such that $\Omega = \mathcal{S}(\mathcal{A}_\Omega^m)$. Here compact is in the sense of $|\cdot|_\infty$-topology. Namely, for any sequence of $(p_k, q_k) \in \mathcal{A}_\Omega^m$, it has a subsequence $(p_{k'}, q_{k'})$

and some $(p_*, q_*) \in \mathcal{A}_\Omega^m$ such that $|p_{k'} - p_*|_\infty + |q_{k'} - q|_\infty \to 0$ as $k' \to \infty$. For any map $u : \mathbb{R}^2 \to \mathbb{S}^2$, we define the $\dot{H}^1$-distance in the following way.

$$dist_{\dot{H}^1}(u, \Omega) = \inf_{\Phi \in \Omega} \left(\int_{\mathbb{R}^2} |\nabla u - \nabla \Phi|^2 \right)^{\frac{1}{2}}. \tag{1.6}$$

Theorem 1.4 (**local stability**). *Suppose Ω is a compact set of degree d harmonic maps. There exist two constants $\eta = \eta(\Omega)$ and $C = C(\Omega)$ such that if $u : \mathbb{R}^2 \to \mathbb{S}^2$ with $dist_{\dot{H}^1}(u, \Omega \circ F) < \eta$ for some Möbius transformation $F : \mathbb{R}^2 \to \mathbb{R}^2$, then there exists a harmonic map Φ which makes the following hold*

$$\int_{\mathbb{R}^2} |\nabla u - \nabla \Phi|^2 \leq C(\mathcal{E}(u) - 4\pi|\deg(u)|). \tag{1.7}$$

Moreover, $\Phi \circ F^{-1}$ is near to Ω in the sense of $|\cdot|_\infty$-topology.

For any harmonic map, Proposition 1.1 says that there is a family of harmonic ones near to it. We take Ω to be a compact neighborhood of a given harmonic map, then the above theorem indicates that there is a local version of stability. Similar to degree ± 1 harmonic maps, one attempts to remove the dependence on Ω and get a **uniformly** quantitative stability estimate. However, the following theorem says that this is not true.

Theorem 1.5 (**non-uniform stability**). *For any large constant $M > 0$, we can find $u : \mathbb{R}^2 \to \mathbb{S}^2$ with $\deg(u) = 2$ such that for any $(p, q) \in \mathcal{A}_2$*

$$\int_{\mathbb{R}^2} |\nabla(u - \mathcal{S}(p/q))|^2 > M\left(\mathcal{E}(u) - 8\pi\right). \tag{1.8}$$

The above theorem indicates that there are some fundamental differences between the case of degree ± 1 and higher degree. To name one, for degree 1 (resp. -1) harmonic maps, one can compose it with a certain Möbius transformation of $\mathbb{R}^2$ such that it becomes the stereographic projection $\mathcal{S} : \mathbb{R}^2 \to \mathbb{S}^2$ (resp. $\bar{\mathcal{S}}$, i.e. the conjugate of $\mathcal{S}$). Since both sides of (1.5) is invariant under Möbius transformations, essentially (1.5) is a quantitative stability near $\mathcal{S}$ (resp. $\bar{\mathcal{S}}$). However, for higher degree ones, one can not transform a harmonic map to an arbitrary another one by Möbius transformations.

Of course, one wishes to have a uniformly quantitative stability like (1.5) for higher degree harmonic maps. Since Theorem 1.5 says that this is not possible with the naive extension, then one probably needs to minus more things in the square on the left hand side of (1.8), or strengthen the right hand side to some nonlinear expression of $\mathcal{E}(u) - 4\pi|\deg(u)|$. Actually we make the following conjecture in the higher degree case:

Conjecture. *Let $|d| \geq 2$. There exists a universal constant $C = C(d)$ such that for every $u : \mathbb{R}^2 \to \mathbb{S}^2$ with $\mathcal{E}(u) < \infty$ and $\deg(u) = d$, there exists $(p, q) \in \mathcal{A}_d$ such that*

$$\int_{\mathbb{R}^2} |\nabla(u - \mathcal{S}(p/q))|^2 \leq C\left(\mathcal{E}(u) - 4\pi|d|\right)\left(1 + \left|\log(\mathcal{E}(u) - 4\pi|d|)\right|\right). \tag{1.9}$$

This conjecture is based on the explicit computation of the example we construct in the proof of Theorem 1.5. In that example, one gets r^{-4} on the left hand side of (1.8) and

$r^{-4}|\log r|^{-1}$ on its right hand side for some $r \gg 1$ (see (4.41) and (4.42)). Plugging these two facts to (1.9), they exactly make two sides comparable.

To mention a few related works, the sharp quantitative stability of the Euler-Lagrange equation of Sobolev inequality on $\mathbb{R}^n$ also varies according to the number of bubbles and the dimension n, readers can consult [6, 11, 8]. In a different direction from ours, Topping [21] uses the torsion of an almost-harmonic map u to control its $\mathcal{E}(u) - 4\pi|\deg(u)|$, which can be considered as another notion of quantitative stability.

1.2. **Comment on proofs.** The locally quantitative stability theorem is proved by using the non-degeneracy result of the harmonic maps (see [14, 5]). The non-degeneracy (also called integrability) of the linearized operator implies that it has a spectral gap, which can be used to prove a local stability near one harmonic map. This is also how [2] proves (1.5) for $\deg(u) = \pm 1$. We generalize the approach of [2] to higher degree case.

The proof of Theorem 1.5 follows the general framework as the one in our work of half-harmonic maps [7] with new essential difficulties. First, the Jacobian of the kernels is uniformly non-degenerate in [7], while it degenerates as the parameter goes to infinity in the case of harmonic maps (cf. (4.9)). One needs to expand up to the third order in the computation to observe this fact, which makes the process substantially more involved. Second, the trick of choosing a vector field corresponds to the rotation in [7] does not work here, which is the heart of the construction. Fortunately, we leverage the degenerate tendency of the Jacobian to find a new one (cf. (4.17)). We have not gotten a satisfied explanation about why such a vector field works.

1.3. **Structure of the paper.** In the section 2, we give some preliminary of harmonic maps from $\mathbb{R}^2$ to $\mathbb{S}^2$. In the section 3, we prove the local stability result. In the section 4, we construct an example such that (1.8) holds, thus Theorem 1.5 is proved. In the section 5, we provide some computations which are needed in the previous section.

2. PRELIMINARY

Topological degree of a C^1 map u from $\mathbb{R}^2$ to $\mathbb{S}^2$ can be defined by the de Rham approach

$$\deg(u) = \frac{1}{4\pi}\int_{\mathbb{R}^2} u \cdot (u_y \times u_x) = \frac{1}{4\pi}\int_{\mathbb{R}^2} u_x \cdot (u \times u_y)\,. \tag{2.1}$$

It is well-known that (2.1) is equivalent to the Brouwer's degree for all C^1 maps. See, for instance, [19, Chapter III]. It is easy to know that the degree is continuous in $\dot{H}^1(\mathbb{R}^2;\mathbb{S}^2)$ topology. That is, fix any $u \in \dot{H}^1(\mathbb{R}^2;\mathbb{S}^2)$, there exists $\eta_1(u)$ such that if $v \in \dot{H}^1(\mathbb{R}^2;\mathbb{S}^2)$ with

$$\|u - v\|_{\dot{H}^1} \le \eta_1(u), \tag{2.2}$$

then $\deg(v) = \deg(u)$. Moreover, $\eta_1(u)$ can be made uniform to u if $u \in \Omega$ which is a compact set of degree d harmonic maps. In this case, one can replace $\eta_1(u)$ by $\eta_1(\Omega)$.

As discussed in the introduction, the following lemma is already known. For reader's convenience, we provide a direct proof here.

Lemma 2.1. *Suppose that $u : \mathbb{R}^2 \to \mathbb{S}^2$ with $\mathcal{E}(u) < \infty$. Then $\mathcal{E}(u) \geq 4\pi|\deg(u)|$. The equality holds if and only if u is harmonic.*

Proof. By completing square it is easy to establish the following identity

$$|\nabla u|^2 \pm 2u \cdot (u_x \times u_y) = |u_x \mp u \times u_y|^2. \tag{2.3}$$

Integrating on both sides and noticing (2.1), one obtains $\mathcal{E}(u) \geq 4\pi|\deg(u)|$. Since Brouwer's degree is invariant by homotopy deformations, we find that u is a minimizer in its degree class. If $\mathcal{E}(u) = 4\pi|\deg(u)|$ then u will be a critical point of $\mathcal{E}(u)$, thus it is harmonic.

Conversely, suppose that u is a harmonic map from $\mathbb{R}^2$ to $\mathbb{S}^2$. Recall the Hopf differential

$$\mathcal{H}(z) = u_y \cdot u_y - u_x \cdot u_x + 2\mathrm{i}\, u_x \cdot u_y. \tag{2.4}$$

It is well-known that $\mathcal{H}$ is a holomorphic or anti-holomorphic function on $\mathbb{C}$ (one can use (1.2) to verify this directly). Since $\mathcal{E}(u) < \infty$, then $\mathcal{H}(z) \in L^1(\mathbb{C})$. It is easy to show that $\mathcal{H}(z) \equiv 0$. Thus $u_x \cdot u_y = 0$ and $|u_x| = |u_y|$. Since $|u| = 1$, then $u \perp u_x$ and $u \perp u_y$. Combining these facts, we must have $u_x = u \times u_y$ or $u_x = -u \times u_y$. In any case, it holds that $\mathcal{E}(u) = 4\pi|\deg(u)|$. □

Let $z = (x, y) \in \mathbb{R}^2 = \mathbb{C}$ and $s = (s_1, s_2, s_3) \in \mathbb{S}^2$. Define the stereographic projection

$$\mathcal{S} : \mathbb{C} \to \mathbb{S}^2 \setminus \{N\} \quad \text{by } s_1 = \frac{2x}{1+|z|^2}, s_2 = \frac{2y}{1+|z|^2} \text{ and } s_3 = \frac{|z|^2-1}{1+|z|^2}. \tag{2.5}$$

Alternatively, in complex variable form,

$$\mathcal{S} = \frac{1}{1+|z|^2}(2z, |z|^2 - 1). \tag{2.6}$$

Suppose $u = \mathcal{S}(\psi)$ where ψ is a meromorphic function on $\mathbb{C}$. Then we have

$$\partial u = \frac{1}{(1+|\psi|^2)^2}\left(2(1+|\psi|^2)\partial\psi - 2\psi\partial|\psi|^2, 2\partial|\psi|^2\right). \tag{2.7}$$

Here ∂ could be ∂_x, ∂_y or with respect to a real parameter which ψ depends on. It is easy to see that

$$|\partial u|^2 = \frac{4|\partial\psi|^2}{(1+|\psi|^2)^2}. \tag{2.8}$$

In particular, for a harmonic map, i.e. $\psi = p/q$ where $(p, q) \in \mathcal{A}_d$, one has

$$|\nabla\mathcal{S}(p/q)|^2 = \frac{4(|\partial_x(p/q)|^2 + |\partial_y(p/q)|^2)}{(1+|p/q|^2)^2} = \frac{4|\partial_x pq - p\partial_x q|^2 + 4|\partial_y pq - p\partial_y q|^2}{(|p|^2+|q|^2)^2}. \tag{2.9}$$

Since p, q satisfies (1.3), there exists a constant $C(p, q)$ such that

$$|\nabla\mathcal{S}(p/q)|^2 \leq C(p,q)(1+|z|)^{-4}, \quad \forall\, z \in \mathbb{C}. \tag{2.10}$$

Moreover, if (p, q) belongs to a compact set $\mathcal{A}_\Omega$ of $\mathcal{A}_d$, then $C(p, q)$ can be replaced by some uniform constant $C(\Omega)$.

The linear equation of (1.2) is

$$\mathcal{L}[u](v) := \Delta v + 2(\nabla u : \nabla v)u + |\nabla u|^2 v \tag{2.11}$$

where

$$\nabla u : \nabla v = \sum_{i=1}^{3} \nabla u^i \cdot \nabla v^i.$$

We shall use this notation throughout this paper.

Take any harmonic map $\mathcal{S}(p/q)$ where $(p,q) \in \mathcal{A}_d$. Apparently, changing the coefficients of p or q continuously yields a family of harmonic maps. Therefore it generates kernel maps of the linearized operator $\mathcal{L}[\mathcal{S}(p/q)]$. For instance, using (2.7) and taking derivative with respect to the real (or imaginary) part of a coefficient of p or q will produce a kernel map of $\mathcal{L}[\mathcal{S}(p/q)]$. Conversely, this is also true. It is called the non-degeneracy (or integrability) of harmonic maps.

Proposition 2.2 ([14, 5])**.** *Suppose $\Phi : \mathbb{R}^2 \to \mathbb{S}^2$ is a harmonic map of degree d. Then all bounded maps in the $\ker \mathcal{L}[\Phi]$ are generated by harmonic maps near Φ. In particular,* $\dim_{\mathbb{R}} \ker \mathcal{L}[\Phi] = 4|d| + 2$.

Remark 2.3. *Furthermore, if $\Phi = \mathcal{S}(p/q)$, using* (2.7)*, one can see that each kernel map $K \in \ker \mathcal{L}[\Phi]$ is smooth and depends on p, q smoothly. In addition,* (2.8) *implies that*

$$|\nabla^j K|(z) \le C_{K,j} |z|^{-|d|-j}. \tag{2.12}$$

We have the expansion of Dirichlet energy in the following. One can compare it with the second variation of Dirichlet energy in [18, page 7] in the smooth setting.

Lemma 2.4. *Suppose that $\Phi : \mathbb{R}^2 \to \mathbb{S}^2$ is a harmonic map. Assume $v \in \dot{H}^1(\mathbb{R}^2;\mathbb{R}^3) \cap L^\infty(\mathbb{R}^2;\mathbb{R}^3)$ with $v \cdot \Phi = 0$. Then for $\varepsilon > 0$ small*

$$\mathcal{E}(\sqrt{1-\varepsilon^2|v|^2}\Phi + \varepsilon v) = \mathcal{E}(\Phi) + \frac{1}{2}\varepsilon^2 \int_{\mathbb{R}^2} |\nabla v|^2 - |\nabla\Phi|^2|v|^2 + O_{v,\Phi}(\varepsilon^3) \tag{2.13}$$

where $|O_{v,\Phi}(\varepsilon^3)| \le C(|v|_{L^\infty}, |v|_{\dot{H}^1}, \Phi)\varepsilon^3$ as $\varepsilon \to 0$.

Proof. We shall choose ε small so that $\varepsilon|v| < \frac{1}{2}$. Denote $f = \sqrt{1-\varepsilon^2|v|^2}$. Then

$$\partial_\alpha(\varepsilon v^j + f\Phi^j) = \varepsilon\partial_\alpha v^j + f\partial_\alpha\Phi^j + \partial_\alpha f \Phi^j$$

for $\alpha \in \{x,y\}$ and $j \in \{1,2,3\}$. Thus

$$|\nabla(\varepsilon v + f\Phi)|^2 = \varepsilon^2|\nabla v|^2 + f^2|\nabla\Phi|^2 + |\nabla f|^2 + 2\varepsilon f \nabla v : \nabla\Phi + 2\varepsilon\partial_\alpha v^j \partial^\alpha f \Phi_j. \tag{2.14}$$

Here we have used Einstein summation convention and $\partial_\alpha\Phi^j\Phi_j = 0$ for $\alpha \in \{x,y\}$. We shall integrate the above equation and estimate them one by one. First, note that

$$\int_{\mathbb{R}^2} f^2|\nabla\Phi|^2 = \int_{\mathbb{R}^2} |\nabla\Phi|^2 - \varepsilon^2 \int_{\mathbb{R}^2} |\nabla\Phi|^2|v|^2. \tag{2.15}$$

Second,

$$\int_{\mathbb{R}^2} |\nabla f|^2 \le \varepsilon^4 \int_{\mathbb{R}^2} \frac{|v|^2|\nabla v|^2}{1-\varepsilon^2|v|^2} \le C(|v|_{L^\infty}, |v|_{\dot{H}^1})\varepsilon^4. \tag{2.16}$$

Third, since (2.10), we can integrate by parts to get

$$\int_{\mathbb{R}^2} f\nabla v : \nabla\Phi = -\int_{\mathbb{R}^2} v : \nabla(f\nabla\Phi) = -\int_{\mathbb{R}^2} fv\cdot\Delta\Phi + v^j\partial_\alpha f\partial^\alpha\Phi_j. \tag{2.17}$$

The first term on the right hand side is zero because $\Delta\Phi = -|\nabla\Phi|^2\Phi$. For the second one we apply Hölder's inequality to get

$$\left|\int_{\mathbb{R}^2} v^j\partial_\alpha f\partial^\alpha\Phi_j\right| \le \left(\int_{\mathbb{R}^2}|\nabla f|^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}|\nabla\Phi|^2\right)^{\frac{1}{2}}|v|_{L^\infty} \le C(|v|_{L^\infty})\varepsilon^2. \tag{2.18}$$

Fourth, since $\partial_\alpha v^j\Phi_j = -\partial_\alpha\Phi_j v^j$, we have

$$\left|\int_{\mathbb{R}^2}\partial_\alpha v^j\partial^\alpha f\Phi_j\right| = \left|\int_{\mathbb{R}^2} v^j\partial^\alpha f\partial_\alpha\Phi_j\right| \le C(|v|_{L^\infty})\varepsilon^2. \tag{2.19}$$

Combining the above four points, we obtain the conclusion. □

For any u with $\deg(u) = \pm 1$, Bernand-Mantel et al. [2, Lemma 4.3] prove that the $\dot{H}^1$-distance of u to the set of degree ± 1 harmonic maps can be achieved. For higher degree, we show that this is also true when u is already near to them.

Lemma 2.5. *Suppose Ω is a compact set of degree d harmonic maps. There exists a constant $\eta_2(\Omega)$ such that if $u : \mathbb{R}^2 \to \mathbb{S}^2$ with $dist_{\dot{H}^1}(u,\Omega) < \eta_2(\Omega)$, then the following infimum can be achieved by some harmonic map $\Phi \in \mathcal{S}(\mathcal{A}_d)$ satisfying*

$$\inf_{\forall(p,q)\in\mathcal{A}_d}\int_{\mathbb{R}^2}|\nabla u - \nabla\mathcal{S}(p/q)|^2 = \int_{\mathbb{R}^2}|\nabla u - \nabla\Phi|^2. \tag{2.20}$$

Proof. Since the degree is continuous on $\dot{H}^1$-topology, we choose $\eta_2(\Omega) < \eta_1(\Omega)$ so that $\deg(u) = d$.

Taking a minimizing sequence Φ_k such that $\|u - \mathcal{S}(p_k/q_k)\|_{\dot{H}^1}$ converges to the infimum. Either the coefficients of p_k, q_k are all uniformly bounded, or there exists a subsequence of them that goes to infinity as $k \to \infty$. In the former case, the infimum is apparently achieved by Lebesgue's dominating convergence theorem. In the latter case, since there are only finitely many coefficients of each p_k and q_k, there exists a subsequence that one of the coefficients of p_k and q_k grows the fastest. Dividing both p_k and q_k by such a coefficient, we have three possible consequences, namely there exists a subsequence (which we still denote as p_k, q_k) such that $|p_k/q_k|(z) \to \infty$ a.e. $\mathbb{C}$ as $k \to \infty$, or a subsequence $|p_k/q_k|(z) \to 0$ a.e. as $k \to \infty$, or a subsequence and a rational function p_*/q_* such that $p_k/q_k \to p_*/q_*$ locally uniformly on $z \in \mathbb{R}^2 \setminus \{\text{zeros of } q_*\}$. In the first case, $\mathcal{S}(p_k/q_k) \to (0,0,1)$ for almost every z. For any $v \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^2}\nabla\mathcal{S}(p_k/q_k) : \nabla v = -\int_{\mathbb{R}^2}\mathcal{S}(p_k/q_k) : \Delta v \to 0, \quad \text{as } k \to \infty. \tag{2.21}$$

Note that $\|\nabla\mathcal{S}(p_k/q_k)\|^2_{L^2(\mathbb{R}^2;\mathbb{R}^3)} = 8\pi|d|$ is uniformly bounded. Thus it has a subsequence weakly converges to 0 in $L^2(\mathbb{R}^2;\mathbb{R}^3)$. However, this leads to

$$\|\nabla u\|_{L^2} = \liminf_{k\to\infty}\|\nabla(u - \mathcal{S}(p_k/q_k))\|_{L^2} < \eta_2(\Omega) \tag{2.22}$$

which contradicts to the fact that $\mathcal{E}(u) \geq 4\pi|d| \geq 4\pi$ if $\eta_2(\Omega)$ is chosen small enough.

In the second case, one can prove $\nabla\mathcal{S}(p_k/q_k) \to 0$ weakly in $L^2(\mathbb{R}^2;\mathbb{R}^3)$ similar to the first case. Thus we get contradiction again.

In the third case, we must have $0 \leq |\deg(\mathcal{S}(p_*/q_*))| \leq |d|$ (see [5, Prop. 2] for how to get the degree from $\deg p$ and $\deg q$). Clearly $\nabla\mathcal{S}(p_k/q_k) \to \nabla\mathcal{S}(p_*/q_*)$ weakly in $L^2(\mathbb{R}^2;\mathbb{R}^3)$.

If $|\deg(\mathcal{S}(p_*/q_*))| < |d|$, by the lower semi-continuity of weak convergence,

$$\int_{\mathbb{R}^2} |\nabla u - \nabla\mathcal{S}(p_*/q_*)|^2 \leq \liminf_{k\to\infty} \int_{\mathbb{R}^2} |\nabla u - \nabla\mathcal{S}(p_k/q_k)|^2 < \eta_2^2(\Omega). \tag{2.23}$$

However,

$$\begin{aligned}\|u - \mathcal{S}(p_*/q_*)\|_{\dot{H}^1(\mathbb{R}^2)} \geq& \|u\|_{\dot{H}^1(\mathbb{R}^2)} - \|\mathcal{S}(p_*/q_*)\|_{\dot{H}^1(\mathbb{R}^2)}\\ =&\sqrt{8\pi|d|} - \sqrt{8\pi|\deg(\mathcal{S}(p_*/q_*))|} \geq \sqrt{8\pi|d|} - \sqrt{8\pi(|d|-1)},\end{aligned}$$

because $|\deg(u)| = |d| > |\deg(\mathcal{S}(p_*/q_*))|$. One clearly has a contradiction when $\eta_2(\Omega)$ is small enough.

If $|\deg(\mathcal{S}(p_*/q_*))| = |d|$, then $\|\nabla\mathcal{S}(p_k/q_k)\|_{L^2} = \sqrt{8\pi|d|} = \|\nabla\mathcal{S}(p_*/q_*)\|_{L^2}$ for any $k \geq 1$. Therefore $\nabla\mathcal{S}(p_k/q_k) \to \nabla\mathcal{S}(p_*/q_*)$ strongly in $L^2(\mathbb{R}^2;\mathbb{R}^3)$. Then $\mathcal{S}(p_*/q_*)$ achieves the infimum. □

Lemma 2.6. *Suppose Ω is a compact set of degree d harmonic maps. For any $\varepsilon > 0$, there exists $\eta_3(\Omega,\varepsilon)$ such that if $\tilde{\Phi}$ is a harmonic map with canonical pair $(\tilde{p},\tilde{q}) \in \mathcal{A}_d^m$ satisfies $\|\Phi - \tilde{\Phi}\|_{\dot{H}^1} \leq \eta_3(\Phi,\varepsilon)$ for some $\Phi \in \Omega$ with canonical pair $(p,q) \in \mathcal{A}_d^m$, then*

$$|\tilde{p} - p|_\infty + |\tilde{q} - q|_\infty < \varepsilon. \tag{2.24}$$

Proof. We just prove the case when Ω consists of one harmonic map Φ. The general case follows from minor modification.

Argue by contradiction. Suppose there exists ε_0 such that for any $k \geq 1$ there exists $\Phi_k = \mathcal{S}(p_k/q_k)$ with canonical $(p_k,q_k) \in \mathcal{A}_d^m$ satisfying $|p_k - p|_\infty + |q_k - q|_\infty \geq \varepsilon_0$ and $\|\Phi_k - \Phi\|_{\dot{H}^1} < 1/k$.

Since p_k, q_k are all complex polynomials, as we did for the previous lemma, either all the coefficients of p_k, q_k are all uniformly bounded, or at least one of them goes to infinity as $k \to \infty$. In the former case, there must exist a subsequence $(p_{k'}, q_{k'})$ and a canonical pair $(p_*, q_*) \in \mathcal{A}_d$ such that $|p_{k'} - p_*|_\infty + |q_{k'} - q_*|_\infty \to 0$. Letting $k' \to \infty$ in $\|\Phi_{k'} - \Phi\|_{\dot{H}^1} < 1/k'$, one obtains $\Phi_* = \Phi$ and consequently $p_* = p$ and $q_* = q$. This is a contradiction. In the latter case, we have three possible consequences, namely there exists a subsequence (which we still denote p_k, q_k) such that $|p_k/q_k| \to \infty$ a.e. as $k \to \infty$, or a subsequence $|p_k/q_k| \to 0$ a.e. as $k \to \infty$, or a subsequence and a rational function p_*/q_* such that $p_k/q_k \to p_*/q_*$ locally uniformly on $\mathbb{R}^2 \setminus \{\text{zeros of } q_*\}$. Moreover, $\max\{\deg p_*, \deg q_*\} < |d|$.

In the first two cases, we have $\nabla\Phi_k \to 0$ weakly in $L^2(\mathbb{R}^2;\mathbb{R}^3)$. Then using the trick in (2.22), we also obtain a contradiction.

In the third case, we must have $|\deg(\mathcal{S}(p_*/q_*))| < |d|$. It can be excluded as before by making η_3 small enough. □

Roughly speaking, the following corollary shows that the minimizers of (2.20) should be near to Ω.

Corollary 2.7. *Suppose Ω is a compact set of degree d harmonic maps. Then for any $\varepsilon > 0$. There exists a constant $\eta_4(\Omega, \varepsilon)$ such that if $u : \mathbb{R}^2 \to \mathbb{S}^2$ with $dist_{\dot{H}^1}(u, \Omega) < \eta_4(\Omega, \varepsilon)$, then any minimizer $\tilde{\Phi}$ of (2.20) has canonical representation $(\tilde{p}, \tilde{q}) \in \mathcal{A}_d^m$ satisfying*

$$|\tilde{p} - p|_\infty + |\tilde{q} - q|_\infty < \varepsilon$$

for some canonical pair $(p, q) \in \mathcal{A}_d^m$ such that $\mathcal{S}(p/q) \in \Omega$.

Proof. Given any $\varepsilon > 0$, we take $\eta_4(\Omega, \varepsilon) = \min\{\eta_2(\Omega), \frac{1}{3}\eta_3(\Phi, \varepsilon)\}$. It follows from Lemma 2.5 that there exist a minimizer of (2.20). For any minimizer $\tilde{\Phi}$ of the infimum, we have

$$\|\tilde{\Phi} - \Phi\|_{\dot{H}^1} \leq \|\tilde{\Phi} - u\|_{\dot{H}^1} + \|u - \Phi\|_{\dot{H}^1} \leq 2\eta_4 < \eta_3. \tag{2.25}$$

Thus we can apply Lemma 2.6 to get the conclusion. $\square$

3. Local stability

In this section, we shall prove the local stability result, i.e. Theorem 1.4. Throughout this section, we will assume Ω is a compact set of degree d harmonic maps from $\mathbb{R}^2$ to $\mathbb{S}^2$. Moreover, Φ always denotes a harmonic map. Define

$$\begin{aligned}
\mathcal{W}_\Phi(\mathbb{R}^2) &= \{v \in L^1_{loc}(\mathbb{R}^2; \mathbb{R}^3) : \int_{\mathbb{R}^2} |v|^2 |\nabla\Phi|^2 < \infty, v \cdot \Phi = 0\},\\
\mathcal{H}_\Phi(\mathbb{R}^2) &= \{v \in H^1_{loc}(\mathbb{R}^2; \mathbb{R}^3) : \int_{\mathbb{R}^2} |\nabla v|^2 + |\nabla\Phi|^2 |v|^2 < \infty, v \cdot \Phi = 0\},\\
\dot{\mathcal{H}}_\Phi(\mathbb{R}^2) &= \{v \in H^1_{loc}(\mathbb{R}^2; \mathbb{R}^3) : \int_{\mathbb{R}^2} |\nabla v|^2 < \infty, v \cdot \Phi = 0\}.
\end{aligned}$$

Then $\mathcal{W}_\Phi$ is a Hilbert space with inner product $(v_1, v_2)_{\mathcal{W}_\Phi} = \int_{\mathbb{R}^2} v_1 \cdot v_2 |\nabla\Phi|^2$. $\mathcal{H}_\Phi$ is a Hilbert space with inner product $(v_1, v_2)_{\mathcal{H}_\Phi} = \int_{\mathbb{R}^2} \nabla v_1 : \nabla v_2 + |\nabla\Phi|^2 v_1 \cdot v_2$. Similar to Proposition A.1 and A.2 in [11], we can prove the following two lemmas.

Lemma 3.1. *There exists a constant $C(\Phi)$ such that*

$$\|v\|_{\mathcal{W}_\Phi(\mathbb{R}^2)} \leq C(\Phi)\|v\|_{\mathcal{H}_\Phi(\mathbb{R}^2)}.$$

Consequently $\mathcal{H}_\Phi(\mathbb{R}^2) \hookrightarrow \mathcal{W}_\Phi(\mathbb{R}^2)$. Moreover, this embedding is compact. If Φ belongs to a compact set Ω, then $C(\Phi)$ can be replaced by $C(\Omega)$.

Proof. Fix $R > 0$ and denote $B_R = B(0, R)$ the ball of radius R centered at the origin. It is easy to see $\dot{\mathcal{H}}_\Phi \to \mathcal{W}_\Phi(B_R)$ is a compact embedding.

Fix any bounded sequence $\{f_k\} \subset \dot{\mathcal{H}}_\Phi(\mathbb{R}^2)$. Using a diagonal argument, we can extract a subsequence and a function $f \in \dot{\mathcal{H}}_\Phi(\mathbb{R}^2)$ such that for any $R > 0$ it holds $f_k \to f$ in

$\mathcal{W}_\Phi(B_R)$-norm. We want to prove that $f_k \to f$ in $\mathcal{W}_\Phi(\mathbb{R}^2)$-norm. By Hölder and Sobolev inequalities, we get

$$\begin{aligned}\limsup_{k\to\infty}\|f_k-f\|^2_{\mathcal{W}_\Phi} =&\limsup_{k\to\infty}\|f_k-f\|^2_{\mathcal{W}_\Phi(B_R)}+\limsup_{k\to\infty}\|f_k-f\|^2_{\mathcal{W}_\Phi(B_R^c)}\\ \le&\limsup_{k\to\infty}\|f_k-f\|^{1/2}_{L^4(\mathbb{R}^2)}\|\nabla\Phi\|^{1/2}_{L^4(B_R^c)}\\ \le&\|\nabla f_k-\nabla f\|^{1/2}_{L^2(\mathbb{R}^2)}\|\nabla\Phi\|^{1/2}_{L^4(B_R^c)}\to 0\quad\text{as } R\to\infty.\end{aligned}\tag{3.1}$$

Here we have used (2.10) to get $\int_{\mathbb{R}^2}|\nabla\Phi|^4<\infty$. Therefore the embedding is compact. The rest conclusion is easy to see. □

Lemma 3.2 (Poincaré type inequality). *Fix any harmonic map Φ, $1\le p<\infty$, there exists a constant $C_{\Phi,p}>0$ such that for any $v\in \dot H^1(\mathbb{R}^2;\mathbb{R}^3)=\{v\in H^1_{loc}(\mathbb{R}^2;\mathbb{R}^3):\int_{\mathbb{R}^2}|\nabla v|^2<\infty\}$ with $\int_{\mathbb{R}^2}v|\nabla\Phi|^2=0$ one has*

$$\left(\int_{\mathbb{R}^2}|v|^p|\nabla\Phi|^2\right)^{\frac{1}{p}}\le C_{\Phi,p}\left(\int_{\mathbb{R}^2}|\nabla v|^2\right)^{\frac12}.\tag{3.2}$$

Moreover, if $\Phi\in\Omega$ a compact set of harmonic maps, then $C_{\Phi,p}$ can be replaced by some uniform constant $C_{\Omega,p}$.

Proof. We can repeat the proof of Lemma 3.1 to see that $\dot H^1(\mathbb{R}^2;\mathbb{R}^3)$ is compactly embedded in $\{v\in H^1_{loc}(\mathbb{R}^2;\mathbb{R}^3):\int_{\mathbb{R}^2}|v|^p|\nabla\Phi|^2<\infty\}$. Our conclusion follows from a standard contradiction argument. For instance, one can see [10, section 5.8.1] and the argument in the following Lemma 3.4. □

Lemma 3.3. *The inverse operator $(|\nabla\Phi|^{-2}\Delta)^{-1}$ is a well-defined and continuous mapping from $\mathcal{W}_\Phi(\mathbb{R}^2)$ into $\mathcal{H}_\Phi(\mathbb{R}^2)$.*

Proof. Let $g\in\mathcal{H}_\Phi(\mathbb{R}^2)$ and $f\in\mathcal{W}_\Phi(\mathbb{R}^2)$. Applying Hölder and Sobolev inequalities, we obtain,

$$\begin{aligned}\langle f,g\rangle_{\mathcal{W}_\Phi}=\int_{\mathbb{R}^2}f\cdot g|\nabla\Phi|^2\le&\left(\int_{\mathbb{R}^2}f^2|\nabla\Phi|^2\right)^{1/2}\left(\int_{\mathbb{R}^2}|\nabla\Phi|^4\right)^{1/4}\left(\int_{\mathbb{R}^2}g^4\right)^{1/4}\\ \le& C\|f\|_{\mathcal{W}_\Phi}\|\nabla g\|_{L^2(\mathbb{R}^2)}.\end{aligned}\tag{3.3}$$

As a consequence,

$$f\mapsto\langle f,\cdot\rangle_{\mathcal{W}_\Phi}\in(\mathcal{H}_\Phi)'$$

is continuous and injective. By Riesz theorem, there exists a unique continuous linear map $T:\mathcal{W}_\Phi(\mathbb{R}^2)\to\mathcal{H}_\Phi(\mathbb{R}^2)$ such that for any $f\in\mathcal{W}_\Phi$ and $g\in\mathcal{H}_\Phi$

$$\int_{\mathbb{R}^2}f\cdot g|\nabla\Phi|^2=\int_{\mathbb{R}^2}\nabla T(f):\nabla g=-\int_{\mathbb{R}^2}\Delta T(f)\cdot g.\tag{3.4}$$

Thus $-\Delta T(f)=f|\nabla\Phi|^2$, which implies $(|\nabla\Phi|^{-2}\Delta)^{-1}=-T$ satisfies the conclusion. □

Lemma 3.4. *Suppose Ω is a compact set of degree d harmonic maps. There exists $\mu = \mu(\Omega) > 1$ such that for any $\Phi \in \Omega$*

$$\int_{\mathbb{R}^2} |\nabla v|^2 \geq \mu \int_{\mathbb{R}^2} |\nabla \Phi|^2 |v|^2, \quad v \in (\ker \mathcal{L}[\Phi])^\perp \subset \mathcal{H}_\Phi. \tag{3.5}$$

Here the orthogonality is with respect to $(\cdot,\cdot)_{\mathcal{H}_\Phi}$.

Proof. Lemma 2.1 says that each harmonic map achieves minimal Dirichlet energy in its homotopy class, therefore the second variation of $\mathcal{E}(u)$ is non-negative. It follows from (2.13) that a density argument that

$$\int_{\mathbb{R}^2} |\nabla v|^2 - |\nabla \Phi|^2 |v|^2 \geq 0, \quad v \in \mathcal{H}_\Phi(\mathbb{R}^2). \tag{3.6}$$

The equality holds if and only if $v \in \ker \mathcal{L}[\Phi]$. Thus

$$\inf_{v \in \mathcal{H}_\Phi} \frac{\int_{\mathbb{R}^2} |\nabla v|^2}{\int_{\mathbb{R}^2} |\nabla \Phi|^2 |v|^2} = 1. \tag{3.7}$$

Using Lemma 3.1 and Lemma 3.3, $(|\nabla \Phi|^{-2}\Delta)^{-1} : \mathcal{W}_\Phi \to \mathcal{W}_\Phi$ is a compact self-adjoint operator, thus its spectrum is discrete. By the min-max characterization of eigenvalues, there exists $\mu_2(\Phi) > 1$ such that

$$\int_{\mathbb{R}^2} |\nabla v|^2 \geq \mu_2(\Phi) \int_{\mathbb{R}^2} |\nabla \Phi|^2 |v|^2, \quad v \in (\ker \mathcal{L}[\Phi])^\perp. \tag{3.8}$$

Here the orthogonality is with respect to $(\cdot,\cdot)_{\mathcal{H}_\Phi}$.

Next, we want to show that $\exists\, \mu > 1$ such that $\mu_2(\Phi) > \mu$ for all $\Phi \in \Omega$. Suppose not, then there exists a sequence of $\Phi_k = \mathcal{S}(p_k/q_k) \in \Omega$, $v_k \in (\ker \mathcal{L}[\Phi_k])^\perp$ such that

$$\int_{\mathbb{R}^2} |\nabla v_k|^2 \leq (1 + \frac{1}{k}) \int_{\mathbb{R}^2} |\nabla \Phi_k|^2 |v_k|^2. \tag{3.9}$$

Since Ω is a compact set, going to a subsequence if necessary, we can assume $p_k \to p_*$ and $q_k \to q_*$. After rescaling, we assume that $\|v_k\|_{\mathcal{H}_{\Phi_k}} = 1$ and $\Phi_* = \mathcal{S}(p_*/q_*)$. Then (3.9) implies $v_k \in \dot{H}^1(\mathbb{R}^2)$ for any $k \geq 1$. Similar to Lemma 3.1, $\dot{H}^1(\mathbb{R}^2)$ is compactly embedded in $\mathcal{W} = \{v \in L^1_{loc}(\mathbb{R}^2; \mathbb{R}^3) : \int_{\mathbb{R}^2} (1+|z|)^{-4}|v|^2 < \infty\}$. Therefore v_k all belong to the weighted space $\mathcal{H}$ defined in (4.10). Then there exists $v_* \in \mathcal{H}$ such that, going to a subsequence if necessary, $v_k \to v_*$ weakly in $\mathcal{H}$ and strongly in $\mathcal{W}$. Recall that (2.9) implies $|\nabla \Phi| \leq C(1+|z|)^{-2}$ uniformly for all $\Phi \in \Omega$. Then $v_* \in \mathcal{H}_{\Phi_*}$. Since $p_k \to p_*$ and $q_k \to q_*$ then $\nabla \Phi_k \to \nabla \Phi_*$ a.e. Therefore by dominated convergence theorem,

$$\int_{\mathbb{R}^2} |\nabla \Phi_*|^2 |v_*|^2 = \lim_{k\to\infty} \int_{\mathbb{R}^2} |\nabla \Phi_k|^2 |v_k|^2. \tag{3.10}$$

Taking limit in (3.9), we obtain from the above fact that

$$\int_{\mathbb{R}^2} |\nabla v_*|^2 = \int_{\mathbb{R}^2} |\nabla \Phi_*|^2 |v_*|^2 = \frac{1}{2}. \tag{3.11}$$

Thus $v_* \in \ker \mathcal{L}[\Phi_*]$ and $v_* \neq 0$. On the other hand, it follows from the non-degeneracy of $\mathcal{L}[\Phi_k]$ (see Lemma 2.2) that $\ker \mathcal{L}[\Phi_k]$ consists of smooth vector fields obtained from taking derivative of coefficients of p_k and q_k. Moreover, Remark 2.3 concludes that these vector fields depend smoothly on p_k, q_k and thus converges to a vector field in $\ker \mathcal{L}[\Phi_*]$ as $k \to \infty$. Since $v_k \in (\ker \mathcal{L}[\Phi_k])^\perp$ means

$$\int_{\mathbb{R}^2} \nabla v_k : \nabla K_k + |\nabla \Phi_k|^2 v_k \cdot K_k = 0, \quad K_k \in \ker \mathcal{L}[\Phi_k]. \tag{3.12}$$

Letting $k \to \infty$, we obtain $(v_*, K_*)_{\mathcal{H}_{\Phi_*}} = 0$ for some $K_* \in \ker \mathcal{L}[\Phi_*]$. Conversely for any K_*, we can choose a sequence of vector fields in $\ker \mathcal{L}[\Phi_k]$ converging to it. Therefore $v_* \in (\ker \mathcal{L}[\Phi_*])^\perp$. This contradicts the previous fact. The lemma is proved. □

The following Lemma is crucial for our local stability theorem. The proof here is a slight modification of that in [2] for degree ± 1 case.

Lemma 3.5. *Suppose Ω is a compact set of degree d harmonic maps and $p > 1$. There exist two constants $\eta_5(\Omega)$ and $C_{\Omega,p}$ such that if $u \in \dot{H}^1(\mathbb{R}^2; \mathbb{S}^2)$ and $\|u - \Phi\|_{\dot{H}^1} \leq \eta_5(\Omega)$ for some harmonic map $\Phi \in \Omega$, then*

$$\int_{\mathbb{R}^2} |u - \Phi|^p |\nabla \Phi|^2 \leq C_{\Omega,p} \left(\int_{\mathbb{R}^2} |\nabla(u - \Phi)|^2 \right)^{\frac{p}{2}}. \tag{3.13}$$

Proof. We will prove the theorem assuming u is smooth. By a density result of Schoen and Uhlenbeck [20] it holds for any $u \in \dot{H}^1(\mathbb{R}^2; \mathbb{S}^2)$. Using (2.3), we have $|\nabla u|^2 \geq 2|u \cdot (u_x \times u_y)|$. Thus we obtain

$$\begin{aligned} &\left| \int_{\mathbb{R}^2} u \left(2|u \cdot (u_x \times u_y)| - |\nabla u|^2 \right) \right| \leq \int_{\mathbb{R}^2} \left(|\nabla u|^2 - 2|u \cdot (u_x \times u_y)| \right) \\ &\leq \min \left\{ \int_{\mathbb{R}^2} \left(|\nabla u|^2 + 2u \cdot (u_x \times u_y) \right), \int_{\mathbb{R}^2} \left(|\nabla u|^2 - 2u \cdot (u_x \times u_y) \right) \right\} \\ &= 2[\mathcal{E}(u) - 4\pi |\deg(u)|]. \end{aligned} \tag{3.14}$$

Take $\eta_5(\Omega) = \min\{\eta_1(\Omega), 1/100\}$ (see (2.2)), then we have $\deg(u) = \deg(\Phi)$. Notice that

$$2[\mathcal{E}(u) - 4\pi |\deg(u)|] = \int_{\mathbb{R}^2} |\nabla u|^2 - |\nabla \Phi|^2 = \int_{\mathbb{R}^2} |\nabla(u - \Phi)|^2 + 2\nabla(u - \Phi) : \nabla \Phi.$$

Since $|\nabla \Phi| \leq C(1 + |z|)^{-2}$ as $z \to \infty$, we can apply integration by parts for the last term to see that

$$2\int_{\mathbb{R}^2} \nabla(u - \Phi) : \nabla \Phi = 2\int_{\mathbb{R}^2} \Phi \cdot (u - \Phi) |\nabla \Phi|^2 = -\int_{\mathbb{R}^2} |u - \Phi|^2 |\nabla \Phi|^2, \tag{3.15}$$

where we have used the fact $2\Phi \cdot (u - \Phi) = -|u - \Phi|^2$ in the last step. It follows that

$$2[\mathcal{E}(u) - 4\pi |\deg(u)|] = \int_{\mathbb{R}^2} |\nabla(u - \Phi)|^2 - |u - \Phi|^2 |\nabla \Phi|^2. \tag{3.16}$$

Plugging this back to (3.14), we obtain

$$\left| \int_{\mathbb{R}^2} u \left(2|u \cdot (u_x \times u_y)| - |\nabla u|^2 \right) \right| \leq \int_{\mathbb{R}^2} |\nabla(u - \Phi)|^2. \tag{3.17}$$

Since $u \cdot u_x = 0$ and $u \cdot u_y = 0$, the two vectors u and $u_x \times u_y$ are parallel. Therefore, we have

$$|u \cdot (u_x \times u_y)|^2 = |u_x \times u_y|^2 = |u_x|^2 |u_y|^2 - (u_x \cdot u_y)^2. \tag{3.18}$$

Thus if we think of u as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$, then $|u \cdot (u_x \times u_y)|$ is the modulus of the Jacobian of u. By the coarea formula (see Theorem 2.17 in [1]), one has

$$\int_{\mathbb{R}^2} u|u \cdot (u_x \times u_y)| = \int_{\mathbb{S}^2} z\mathcal{H}^0(\{u^{-1}(z)\}) d\mu_{\mathbb{S}^2} \tag{3.19}$$

where $\mathcal{H}^0$ is the 0-dimensional Hausdorff measure, i.e. counting the number of points. By Sard's theorem, for almost all $z \in \mathbb{S}^2$, z is a regular value of u. For such z, one has $\mathcal{H}^0(\{u^{-1}(z)\}) \geq |\deg(u)|$ (see [19, pg 95]). By symmetry of sphere we get

$$\begin{aligned} \left| \int_{\mathbb{S}^2} z\mathcal{H}^0(\{u^{-1}(z)\}) d\mu_{\mathbb{S}^2} \right| &= \left| \int_{\mathbb{S}^2} z \left(\mathcal{H}^0(\{u^{-1}(z)\}) - |\deg u| \right) d\mu_{\mathbb{S}^2} \right| \\ &\leq \int_{\mathbb{S}^2} \left(\mathcal{H}^0(\{u^{-1}(z)\}) - |\deg u| \right) d\mu_{\mathbb{S}^2}. \end{aligned} \tag{3.20}$$

Using coarea formula again

$$\begin{aligned} \int_{\mathbb{S}^2} \left(\mathcal{H}^0(\{u^{-1}(z)\}) - |\deg u| \right) d\mu_{\mathbb{S}^2} &= \int_{\mathbb{R}^2} |u \cdot (u_x \times u_y)| dx - 4\pi |\deg(u)| \\ &\leq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 - 4\pi |\deg(u)| \end{aligned} \tag{3.21}$$

where we used (2.3). Now one concatenates (3.19), (3.20), (3.21) and (3.16) to get

$$\left| \int_{\mathbb{R}^2} u|u \cdot (u_x \cdot u_y)| \right| \leq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla(u - \Phi)|^2. \tag{3.22}$$

Using (3.22) and (3.17), we get

$$\left| \int_{\mathbb{R}^2} u|\nabla u|^2 \right| \leq 2 \int_{\mathbb{R}^2} |\nabla(u - \Phi)|^2. \tag{3.23}$$

Applying $|\nabla u|^2 - |\nabla \Phi|^2 = 2\nabla\Phi : \nabla(u - \Phi) + |\nabla(u - \Phi)|^2$ and Cauchy-Schwarz inequality, one gets

$$\left| \int_{\mathbb{R}^2} u(|\nabla u|^2 - |\nabla \Phi|^2) \right| \leq 4\sqrt{2\pi} \left(\int_{\mathbb{R}^2} |\nabla(u - \Phi)|^2 \right)^{\frac{1}{2}} + \int_{\mathbb{R}^2} |\nabla(u - \Phi)|^2. \tag{3.24}$$

Since $\int_{\mathbb{R}^2} |\nabla(u - \Phi)|^2 < \eta_5(\Omega) \leq 1/100$, then

$$\left| \int_{\mathbb{R}^2} u|\nabla \Phi|^2 \right| \leq 8\sqrt{\pi} \left(\int_{\mathbb{R}^2} |\nabla(u - \Phi)|^2 \right)^{\frac{1}{2}}. \tag{3.25}$$

Since $\int_{\mathbb{R}^2} \Phi|\nabla\Phi|^2 = \int_{\mathbb{R}^2} \Delta\Phi = 0$, then

$$\left| \int_{\mathbb{R}^2} (u - \Phi)|\nabla \Phi|^2 \right| \leq 8\sqrt{\pi} \left(\int_{\mathbb{R}^2} |\nabla(u - \Phi)|^2 \right)^{\frac{1}{2}}. \tag{3.26}$$

Now we apply Lemma 3.2 to get

$$\int_{\mathbb{R}^2} |u-\Phi|^p |\nabla\Phi|^2 \le C_{\Omega,p}\left(\int_{\mathbb{R}^2} |\nabla(u-\Phi)|^2\right)^{\frac{p}{2}} + C_{\Omega,p}\left|\int_{\mathbb{R}^2}(u-\Phi)|\nabla\Phi|^2\right|^p. \tag{3.27}$$

Inserting (3.26), one gets (3.13). □

Now we can prove the local stability as claimed in the introduction. Our proof follows closely to the one in [2, Lemma 4.4].

Proof of Theorem 1.4. Since the Dirichlet energy is invariant under Möbius transformations, without loss of generality we assume $F = id_{\mathbb{R}^2}$. Otherwise one can work with $u \circ F^{-1}$.

Since Ω is a compact, then $\mathcal{A}^m_\Omega = \{(p,q) : p \text{ is monic and } \mathcal{S}(p/q) \in \Omega\}$ is a compact subset of $\mathcal{A}_d$. Therefore there exists a $\varepsilon_1(\Omega)$ such that the following subset is also compact.

$$(\mathcal{A}^m_\Omega)_{\varepsilon_1} = \{(\tilde{p}/\tilde{q}) : \exists\,(p,q) \in \mathcal{A}^c_\Omega \text{ such that } |\tilde{p}-p|_\infty + |\tilde{q}-q|_\infty \le \varepsilon_1\}. \tag{3.28}$$

Denote $\Omega_{\varepsilon_1} = \mathcal{S}((\mathcal{A}^m_\Omega)_{\varepsilon_1})$. It is a compact set of degree d harmonic maps. We shall take

$$\tilde{\eta}(\Omega) = \min\{\eta_2(\Omega_{\varepsilon_1}), \eta_3(\Omega_{\varepsilon_1}, \varepsilon_1), \eta_4(\Omega_{\varepsilon_1}, \varepsilon_1), \eta_5(\Omega_{\varepsilon_1})\}. \tag{3.29}$$

It follows from Lemma 2.5 that the infimum can be achieved. Let us assume it is achieved at Φ. Then Corollary 2.7 and Lemma 2.6 imply that $\Phi \in \Omega_\varepsilon$. Denote $\delta = \int_{\mathbb{R}^2} |\nabla(u-\Phi)|^2$. We decompose $\zeta = u - \Phi$ into three parts

$$\zeta_\parallel = (\zeta\cdot\Phi)\Phi, \quad \zeta_K = proj_{\ker\mathcal{L}[\Phi]}(\zeta - \zeta_\parallel), \quad \zeta^* = \zeta - \zeta_\parallel - \zeta_K. \tag{3.30}$$

Here the projection is with respect to the inner product of $\mathcal{W}_\Phi$. Consequently

$$\int_{\mathbb{R}^2} \zeta_K \cdot \zeta^* |\nabla\Phi|^2 = 0. \tag{3.31}$$

Since ζ_K satisfies $\Delta\zeta_K + 2(\nabla\Phi : \nabla\zeta_K)\Phi + |\nabla\Phi|^2\zeta_K = 0$, the above orthogonality is also equivalent to the orthogonality in $\dot{H}^1$.

$$\int_{\mathbb{R}^2} \nabla\zeta_K : \nabla\zeta^* = 0. \tag{3.32}$$

Claim 1. *Assume $\delta < \tilde{\eta}(\Omega)$. There exists $C = C(\Omega)$ such that*

$$\int_{\mathbb{R}^2} |\zeta|^2|\nabla\Phi|^2 \le \int_{\mathbb{R}^2} |\zeta^*|^2|\nabla\Phi|^2 + C\delta^4. \tag{3.33}$$

Indeed, the observation starts from the following identity

$$\int_{\mathbb{R}^2} |\zeta|^2|\nabla\Phi|^2 = \int_{\mathbb{R}^2} (|\zeta_\parallel|^2 + |\zeta_K|^2 + |\zeta^*|^2)|\nabla\Phi|^2 \tag{3.34}$$

which follows from $(\zeta_K + \zeta^*)\cdot\zeta_\parallel = 0$ and (3.31).

First, using the smallness assumption $\delta < \tilde{\eta}(\Omega) \le \eta_5(\Omega_{\varepsilon_1})$ and $\zeta_\parallel = -\frac{1}{2}|\zeta|^2\Phi$, we can apply Lemma 3.5 to get

$$\int_{\mathbb{R}^2} |\zeta_\parallel|^2|\nabla\Phi|^2 = \frac{1}{4}\int_{\mathbb{R}^2} |u-\Phi|^4|\nabla\Phi|^2 \le C\delta^4. \tag{3.35}$$

Second, since $\|u-\Phi\|_{\dot H^1}$ achieves the infimum at Φ, one has $\int_{\mathbb{R}^2}\nabla\zeta:\nabla\zeta_K=0$. Since $\zeta=\zeta_\parallel+\zeta_K+\zeta^*$ and (3.32), this is equivalent to

$$\int_{\mathbb{R}^2}|\nabla\zeta_K|^2=-\int_{\mathbb{R}^2}\nabla\zeta_K:\nabla\zeta_\parallel. \tag{3.36}$$

Since ζ_K is smooth and decay fast (see Remark 2.3), we may do integration by parts for the right hand side and use the facts $\zeta_K\in\ker\mathcal{L}[\Phi]$ and $\zeta_K\cdot\zeta_\parallel=0$ to get

$$\int_{\mathbb{R}^2}|\nabla\zeta_K|^2=-2\int_{\mathbb{R}^2}(\zeta_\parallel\cdot\Phi)(\nabla\Phi:\nabla\zeta_K)=\int_{\mathbb{R}^2}|u-\Phi|^2(\nabla\Phi:\nabla\zeta_K). \tag{3.37}$$

We apply Hölder's inequality to the above equation and (3.35) to get

$$\int_{\mathbb{R}^2}|\nabla\zeta_K|^2\le C\delta^4. \tag{3.38}$$

Now Lemma 3.1 implies that

$$\int_{\mathbb{R}^2}|\zeta_K|^2|\nabla\Phi|^2\le C\delta^4. \tag{3.39}$$

Inserting (3.35) and (3.39) into (3.34), we can prove the Claim 1.

Claim 2. *Assume $\delta<\tilde\eta(\Omega)$. There exists $C=C(\Omega)$ such that*

$$\int_{\mathbb{R}^2}|\nabla\zeta|^2=\int_{\mathbb{R}^2}|\nabla\zeta_\parallel|^2+|\nabla\zeta_K|^2+|\nabla\zeta^*|^2+2\nabla\zeta_\parallel:\nabla(\zeta-\zeta_\parallel) \tag{3.40}$$

and

$$\left|\int_{\mathbb{R}^2}2\zeta_\parallel:\nabla(\zeta-\zeta_\parallel)\right|\le C\delta^3. \tag{3.41}$$

Indeed, the first identity follows from (3.32). From $\zeta_\parallel=-\frac{1}{2}|\zeta|^2\Phi$, we have $\partial_\alpha\zeta_\parallel^l=-\frac{1}{2}|\zeta|^2\partial_\alpha\Phi^l-\frac{1}{2}\Phi^l\partial_\alpha|\zeta|^2$, where $\alpha\in\{x,y\}$ and $l\in\{1,2,3\}$. Using this identity, we have

$$\begin{aligned}\int_{\mathbb{R}^2}2\nabla\zeta_\parallel:\nabla(\zeta-\zeta_\parallel)&=-\int_{\mathbb{R}^2}|\zeta|^2\partial_\alpha\Phi^l\partial^\alpha(\zeta-\zeta_\parallel)_l+\Phi^l\partial_\alpha|\zeta|^2\partial^\alpha(\zeta-\zeta_\parallel)_l\\&=\int_{\mathbb{R}^2}(\zeta-\zeta_\parallel)_l\partial_\alpha|\zeta|^2\partial^\alpha\Phi^l-|\zeta|^2\partial_\alpha\Phi^l\partial^\alpha(\zeta-\zeta_\parallel)_l\end{aligned} \tag{3.42}$$

where we have used $\partial_\alpha[\Phi\cdot(\zeta-\zeta_\parallel)]=0$. Since $|\nabla\Phi|\le C(1+|z|)^{-2}$, we may integrate by parts for the first term to get

$$\int_{\mathbb{R}^2}(\zeta-\zeta_\parallel)_l\partial_\alpha|\zeta|^2\partial^\alpha\Phi^l=-\int_{\mathbb{R}^2}|\zeta|^2\partial_\alpha\Phi^l\partial^\alpha(\zeta-\zeta_\parallel)_l. \tag{3.43}$$

Therefore,

$$\int_{\mathbb{R}^2}2\nabla\zeta_\parallel:\nabla(\zeta-\zeta_\parallel)=-2\int_{\mathbb{R}^2}|\zeta|^2\partial_\alpha\Phi^l\partial^\alpha(\zeta-\zeta_\parallel)_l. \tag{3.44}$$

Since $\partial_\alpha\Phi_l\partial^\alpha\zeta_\parallel^l=-\frac{1}{2}\partial_\alpha\Phi_l\partial^\alpha[|u-\Phi|^2\Phi^l]=-\frac{1}{2}|u-\Phi|^2||\nabla\Phi|^2$,

$$\int_{\mathbb{R}^2}2\nabla\zeta_\parallel:\nabla(\zeta-\zeta_\parallel)=-2\int_{\mathbb{R}^2}|\zeta|^2\nabla\Phi:\nabla\zeta+\int_{\mathbb{R}^2}|u-\Phi|^4|\nabla\Phi|^2. \tag{3.45}$$

Using Hölder's inequality and Lemma 3.5, we obtain

$$\left|\int_{\mathbb{R}^2} 2\nabla\zeta_{\|} : \nabla(\zeta-\zeta_{\|})\right| \le 2\delta\left(\int_{\mathbb{R}^2}|\zeta|^4|\nabla\Phi|^2\right)^{\frac{1}{2}} + C\delta^4 \le C\delta^3. \tag{3.46}$$

Thus Claim 2 is proved.

Now we can use these two claims to prove the theorem. Recall (3.16) and Lemma 3.4.

$$\begin{aligned}\mathcal{E}(u) - 4\pi|\deg(u)| =& \int_{\mathbb{R}^2}|\nabla\zeta|^2 - |\zeta|^2|\nabla\Phi|^2\\ \ge& \int_{\mathbb{R}^2}|\nabla\zeta^*|^2 - |\zeta^*|^2|\nabla\Phi|^2 + \int_{\mathbb{R}^2}|\nabla\zeta_{\|}|^2 + |\nabla\zeta_K|^2 - C\delta^3 - C\delta^4\\ \ge& (1-\mu^{-1})\int_{\mathbb{R}^2}|\nabla\zeta^*|^2 + \int_{\mathbb{R}^2}|\nabla\zeta_{\|}|^2 + |\nabla\zeta_K|^2 - C\delta^3 - C\delta^4\\ \ge& (1-\mu^{-1})\int_{\mathbb{R}^2}|\nabla\zeta|^2 - C\delta^3 - C\delta^4 = (1-\mu^{-1})\delta^2 - C\delta^3 - C\delta^4.\end{aligned}$$

Choosing $\eta(\Omega) = \min\{\tilde{\eta}(\Omega), \frac{1}{4C}(1-\mu^{-1})\}$ where C is obtained from the above line. If $\delta < \eta(\Omega)$, then

$$\mathcal{E}(u) - 4\pi|\deg(u)| \ge \frac{1}{2}(1-\mu^{-1})\delta^2 = \frac{1}{2}(1-\mu^{-1})\int_{\mathbb{R}^2}|u-\Phi|^2. \tag{3.47}$$

The theorem is proved. □

4. Counterexample with degree 2

In this section, we shall construct some example to fulfill (1.8) and thus Theorem 1.5 is established. The process starts with a particular degree 2 harmonic map $\mathcal{S}((z-r-\mathrm{i}\,r)(z+r+\mathrm{i}\,r))$. Here r will be chosen large enough to satisfy various conditions. We shall introduce some notations firstly. Denote $\vec{\alpha} = (\alpha_i) \in \mathbb{R}^{10}$ and

$$\Psi[\vec{\alpha}](z) = \frac{(\alpha_1 + \mathrm{i}\,\alpha_2)z^2 + (\alpha_3 + \mathrm{i}\,\alpha_4)z + (\alpha_5 + \mathrm{i}\,\alpha_6)}{1 - (\alpha_7 + \mathrm{i}\,\alpha_8)z - (\alpha_9 + \mathrm{i}\,\alpha_{10})z^2}. \tag{4.1}$$

We also define $K_i = K_i[\vec{\alpha}](z)$, for $i = 1, \cdots, 10$, as

$$\begin{aligned}K_i[\vec{\alpha}](z) =& r^{\beta_i}\partial_{\alpha_i}\mathcal{S}(\Psi[\vec{\alpha}])(z)\\ =& \frac{r^{\beta_i}}{(1+|\Psi|^2)^2}\left(2(1+|\Psi|^2)\partial_{\alpha_i}\Psi - 2\Psi\partial_{\alpha_i}|\Psi|^2, 2\partial_{\alpha_i}|\Psi|^2\right)[\vec{\alpha}](z)\end{aligned} \tag{4.2}$$

where $\beta_i = 0$ if $i \in \{1,2,5,6\}$, $\beta_i = -1$ if $i \in \{3,4,7,8\}$, $\beta_i = -2$ if $i \in \{9,10\}$. The reason why we divide some K_i by r or r^2 is to make sure that $\mathcal{J}_{ij}$ (see (4.5)) is bounded above by some constant. It will be clear in Lemma 5.5. The non-degeneracy result implies $dim_{\mathbb{R}} \ker \mathcal{L}[\mathcal{S}(\Psi[\vec{\alpha}])] = 10$ when $\Psi[\vec{\alpha}](z)$ is an irreducible rational function. Actually we can prove directly $\{K_i : i = 1, \cdots, 10\}$ are linearly independent, therefore $\ker \mathcal{L}[\mathcal{S}(\Psi[\vec{\alpha}])] = \mathrm{Span}_{\mathbb{R}}\{K_1, \cdots, K_{10}\}$.

Lemma 4.1. *Suppose $\vec{\alpha}$ satisfies that $\Psi[\vec{\alpha}](z)$ is an irreducible rational function and $\alpha_1 + \mathrm{i}\,\alpha_2 \neq 0$. Then $\{K_i : i = 1, \cdots, 10\}$ is linearly independent.*

Proof. Suppose $\sum_{i=1}^{10} c_i K_i[\vec{\alpha}](z) = 0$ for some constants $c_i \in \mathbb{R}$. Using the expression of K_i in (4.2), it implies that for all such z

$$\sum_{i=1}^{10} c_i \partial_{\alpha_i} |\Psi|^2[\vec{\alpha}](z) = 0, \quad \sum_{i=1}^{10} c_i \partial_{\alpha_i} \Psi[\vec{\alpha}](z) = 0. \tag{4.3}$$

Dividing the second equation by Ψ, we get

$$\sum_{i=1}^{10} c_i \frac{\partial_{\alpha_i}\Psi}{\Psi}[\vec{\alpha}](z) = 0, \quad z \in \mathbb{C} \setminus \{\text{zeros and poles of } \Psi[\vec{\alpha}]\}. \tag{4.4}$$

That is

$$\frac{(c_1 + \mathrm{i}\, c_2)z^2 + (c_3 + \mathrm{i}\, c_4)z + (c_5 + \mathrm{i}\, c_6)}{(\alpha_1 + \mathrm{i}\, \alpha_2)z^2 + (\alpha_3 + \mathrm{i}\, \alpha_4)z + (\alpha_5 + \mathrm{i}\, \alpha_6)} + \frac{(c_7 + \mathrm{i}\, c_8)z + (c_9 + \mathrm{i}\, c_{10})z^2}{1 - (\alpha_7 + \mathrm{i}\, \alpha_8)z - (\alpha_9 + \mathrm{i}\, \alpha_{10})z^2} = 0.$$

Suppose $\{\zeta_1, \zeta_2\}$ are the roots of $(\alpha_1 + \mathrm{i}\, \alpha_2)z^2 + (\alpha_3 + \mathrm{i}\, \alpha_4) + (\alpha_5 + \mathrm{i}\, \alpha_6) = 0$. Consider the roots of $1 - (\alpha_7 + \mathrm{i}\, \alpha_8)z - (\alpha_9 + \mathrm{i}\, \alpha_{10})z^2 = 0$. It possibly has two roots, say $\{\zeta_3, \zeta_4\}$, or one root, or no root. In the first case, since $\Psi[\vec{\alpha}](z)$ is irreducible, then $\{\zeta_1, \zeta_2\} \cap \{\zeta_3, \zeta_4\} = \emptyset$. Let $z \to \zeta_3$ or ζ_4, then we must have $c_7 + \mathrm{i}\, c_8 = c_9 + \mathrm{i}\, c_{10} = 0$. Consequently $c_1 + \mathrm{i}\, c_2 = c_3 + \mathrm{i}\, c_4 = c_5 + \mathrm{i}\, c_6 = 0$. The other two cases are similar to prove. □

Denote $\mathcal{J}[\vec{\alpha}] = (\mathcal{J}_{ij})_{1 \le i,j \le 10}$ where

$$\mathcal{J}_{ij}[\vec{\alpha}] = \int_{\mathbb{R}^2} |\nabla \mathcal{S}(\Psi[\vec{\alpha}])|^2 K_i[\vec{\alpha}] \cdot K_j[\vec{\alpha}]. \tag{4.5}$$

Lemma 4.2. *Suppose $\vec{\alpha}$ satisfies that $\Psi[\vec{\alpha}](z)$ is an irreducible rational function and $\alpha_1 + \mathrm{i}\, \alpha_2 \neq 0$. Then $\mathcal{J}[\vec{\alpha}]$ is a positive definite matrix for any $r > 0$.*

Proof. Assume not, then there exists some $\vec{c} = (c_1, \cdots, c_{10})$ such that

$$0 \ge \sum_{i,j=1}^{10} \mathcal{J}_{ij}[\vec{\alpha}] c_i c_j = \int_{\mathbb{R}^2} |\nabla \mathcal{S}(\Psi[\vec{\alpha}])|^2 |c_i K_i[\vec{\alpha}]|^2. \tag{4.6}$$

This implies $\sum_{i=1}^{10} c_i K_i = 0$. However, this contradicts to the linear independence of K_i. □

Since $\mathcal{L}[\mathcal{S}(\Psi[\vec{\alpha}])](K_i[\vec{\alpha}]) = 0$ and $K_i \cdot \mathcal{S}(\Psi[\vec{\alpha}]) = 0$, one has

$$\int_{\mathbb{R}^2} \nabla K_i[\vec{\alpha}] : \nabla K_j[\vec{\alpha}] = \int_{\mathbb{R}^2} |\nabla \mathcal{S}(\Psi[\vec{\alpha}])|^2 K_i[\vec{\alpha}] \cdot K_j[\vec{\alpha}] = \mathcal{J}_{ij}[\vec{\alpha}]. \tag{4.7}$$

For any $r > 0$, we denote $\vec{\alpha}_r = (1, 0, 0, 0, 0, 2r^2, 0, 0, 0, 0)$. Then

$$\Psi[\vec{\alpha}_r](z) = (z - r - \mathrm{i}\, r)(z + r + \mathrm{i}\, r).$$

We will write $K_i^r = K_i[\vec{\alpha}_r]$ and $\mathcal{J}^r = \mathcal{J}[\vec{\alpha}_r]$ for short. In the following, one will see that $\mathcal{J}^r$ plays an important role in the analysis near the harmonic map $\mathcal{S}(\Psi[\vec{\alpha}_r])$. It is necessary to have a more detailed knowledge of each entries of $\mathcal{J}^r$, at least the leading orders of them as $r \to \infty$.

Proposition 4.3. *After some row and column permutation, we can represent $\mathcal{J}^r$ as a block diagonal matrix.*

$$\mathcal{J}^r \sim \frac{16\pi}{3} diag\{A_1, A_2, A_3, A_4\} \tag{4.8}$$

with

$$A_1 = \begin{pmatrix} 2 & -4 & \alpha_0 \\ -4 & \lambda_3 & \gamma_0 \\ \alpha_0 & \gamma_0 & 4 \end{pmatrix} + O(r^{-6}), \quad A_2 = \begin{pmatrix} 2 & 4 & -\alpha_0 \\ 4 & \lambda_3 & \gamma_0 \\ -\alpha_0 & \gamma_0 & 4 \end{pmatrix} + O(r^{-6}),$$
$$A_3 = \begin{pmatrix} \lambda_1 & \beta_0 - r^{-2} \\ \beta_0 - r^{-2} & \lambda_2 \end{pmatrix} + O(r^{-6}), \quad A_4 = \begin{pmatrix} \lambda_1 & r^{-2} - \beta_0 \\ r^{-2} - \beta_0 & \lambda_2 \end{pmatrix} + O(r^{-6}),$$

where A_1 corresponds to $i,j \in \{1,10,6\}$, A_2 corresponds to $i,j \in \{2,9,5\}$, A_3 corresponds to $i,j \in \{3,8\}$, and A_4 corresponds to $i,j \in \{4,7\}$. Here $\alpha_0 \approx \beta_0 \approx \gamma_0 = O(r^{-\frac{9}{2}})$, $\lambda_1 = 8 + \frac{1}{4}r^{-4}$, $\lambda_2 = 4 + \frac{1}{2}r^{-4}$ and $\lambda_3 = 8 + 4r^{-4}$.

Remark 4.4. *One can compute the determinant of $\mathcal{J}^r$.*

$$\begin{aligned} \det \mathcal{J}^r &= \frac{16^{10}\pi^{10}}{3^{10}} (\det A_1)(\det A_2)(\det A_3)(\det A_4) \\ &= \frac{16^{10}\pi^{10}}{3^{10}} \left(32r^{-4} + O(r^{-6})\right)^2 \left(32 + 4r^{-4} + O(r^{-6})\right)^2 \\ &= \frac{2^{60}\pi^{10}}{3^{10}} r^{-8} + O(r^{-10}). \end{aligned} \tag{4.9}$$

One can see the degenerate tendency of $\mathcal{J}^r$ as $r \to \infty$.

The proof of this lemma needs some involved integration. We need to expand the integrand to the third order to prove the result. Instead of diving into massive computations, we defer the proof of it to the next section and continue the main thread of our construction. Let

$$\mathcal{H} = \{u \in H^1_{loc}(\mathbb{R}^2; \mathbb{R}^3) : \int_{\mathbb{R}^2} |\nabla u|^2 + (1+|z|)^{-4}|u|^2 < \infty\}. \tag{4.10}$$

It is easy to see that $\mathcal{H}$ is a Hilbert space.

Proposition 4.5. *Fix any $r > 0$, there exists a $\varepsilon_2(r)$ and $\eta_6(r)$ such that for any $u : \mathbb{R}^2 \to \mathbb{S}^2$ with $\|u - \mathcal{S}(\Psi[\vec{\alpha}_r])\|_{\mathcal{H}} < \eta_6(r)$, then there exists a unique $\vec{\alpha} = \vec{\alpha}(u)$ satisfying $\|\vec{\alpha} - \vec{\alpha}_r\| < \varepsilon_2(r)$ and*

$$\int_{\mathbb{R}^2} \nabla u : \nabla K = 0, \quad \text{for } K \in \ker \mathcal{L}[\mathcal{S}(\Psi[\vec{\alpha}])]. \tag{4.11}$$

Proof. Define the following map

$$\begin{aligned} &F : \mathcal{H} \times \mathbb{R}^{10} \to \mathbb{R}^{10} \\ &\quad (u, \vec{\alpha}) \mapsto \left(\int_{\mathbb{R}^2} \nabla u : \nabla K_1[\vec{\alpha}], \cdots, \int_{\mathbb{R}^2} \nabla u : \nabla K_{10}[\vec{\alpha}] \right). \end{aligned} \tag{4.12}$$

Such map F is well-defined because $u \in \mathcal{H}$. It is easy to see F is smooth on $\vec{\alpha}$ because K_i depends on $\vec{\alpha}$ smoothly. F is at least C^1 on u.

For any $r > 0$, there exists $\varepsilon_2(r)$ such that if $\|\vec{\alpha} - \vec{\alpha}_r\| < \varepsilon_2(r)$, then α satisfies the assumption of Lemma 4.1. Therefore $\int_{\mathbb{R}^2} |\nabla \mathcal{S}(\Psi[\vec{\alpha}])|^2 = 16\pi$. Differentiating on $\vec{\alpha}$, it infers that

$$\int_{\mathbb{R}^2} \nabla \mathcal{S}(\Psi[\vec{\alpha}]) : \nabla K_i[\vec{\alpha}] = 0, \quad i = 1, \cdots, 10. \tag{4.13}$$

Equivalently, this is $F(\mathcal{S}(\Psi[\vec{\alpha}]), \vec{\alpha}) = 0$ and $F(u, \vec{\alpha}) = F(u - \mathcal{S}(\Psi[\vec{\alpha}]), \vec{\alpha})$. We intend to apply implicit function theorem to F at $(\mathcal{S}(\Psi[\vec{\alpha}_r]), \vec{\alpha}_r)$. The Jacobian matrix with respect to $\vec{\alpha}$ at $(\mathcal{S}(\Psi[\vec{\alpha}_r]), \vec{\alpha}_r)$ is

$$\frac{\partial F}{\partial \vec{\alpha}}((\mathcal{S}(\Psi[\vec{\alpha}_r]), \vec{\alpha}_r)) = \left(\int_{\mathbb{R}^2} \nabla K_i^r : \nabla K_j^r \right)_{1 \le i,j \le 10} = \mathcal{J}^r. \tag{4.14}$$

Here we have used (4.7). Lemma 4.2 says that such Jacobian is non-degenerate. Therefore, using the implicit function theorem, there exist $\eta_6(r)$ and $\varepsilon_2(r)$ small enough such that if $\|u - \mathcal{S}(\Psi[\vec{\alpha}_r])\|_{\mathcal{H}} < \eta_6(r)$, then there exists a unique $\vec{\alpha} = \vec{\alpha}(u)$ such that $|\vec{\alpha} - \vec{\alpha}_r| < \varepsilon_2(r)$ and $F(u, \vec{\alpha}) = 0$. □

Introduce the cut-off function

$$\Theta^r(z) = \begin{cases} 1 & |z| < r^{\frac{1}{2}}, \\ 2 - 2\log(|z|)/\log r & r^{\frac{1}{2}} \le |z| \le r, \\ 0 & |z| > r. \end{cases} \tag{4.15}$$

Define

$$f^r(x, y) = \Theta^r(x + \mathrm{i}\, y - r - \mathrm{i}\, r) - \Theta^r(x + \mathrm{i}\, y + r + \mathrm{i}\, r) \tag{4.16}$$

and

$$\mathcal{K}^r = 2K_2^r - K_9^r. \tag{4.17}$$

See the explicit formulae of K_2^r, K_9^r in section 5.

Lemma 4.6. *For $r > 0$ large enough, we have*

$$\int_{\mathbb{R}^2} |\nabla(f^r \mathcal{K}^r)|^2 = \frac{64\pi}{3} r^{-4} + O(|\log r|^{-1} r^{-4}), \tag{4.18}$$

$$\int_{\mathbb{R}^2} |\nabla \mathcal{S}(\Psi[\vec{\alpha}_r])|^2 |f^r \mathcal{K}^r|^2 = \frac{64\pi}{3} r^{-4} + O(r^{-6}). \tag{4.19}$$

Denote

$$p_j = \int_{\mathbb{R}^2} \nabla(f^r \mathcal{K}^r) : \nabla K_j^r \, dxdy. \tag{4.20}$$

Lemma 4.7. *We have*

$$p_1 = -2p_{10} = O(r^{-\frac{13}{2}}), \quad p_2 = O(r^{-6}), \quad p_3 = -p_4 = O(r^{-\frac{9}{2}}), \quad p_5 = O(r^{-\frac{9}{2}}),$$
$$p_6 = O(r^{-\frac{9}{2}}), \quad p_7 = p_8 = -\frac{16\pi}{3} r^{-4} + O(r^{-6}), \quad p_9 = O(r^{-6}).$$

Consider the solution $\vec{c} = (c_1, \cdots, c_{10})^T$ *of* $\mathcal{J}^r \vec{c} = \vec{p}$ *where* $\vec{p} = (p_1, \cdots, p_{10})^T$*,*

$$\begin{aligned} c_1 &= 2c_{10} + O(r^{-\frac{13}{2}}) = O(r^{-\frac{5}{2}}), \quad c_5 = O(r^{-\frac{9}{2}}), \\ c_2 &= -2c_9 + O(r^{-6}) = O(r^{-2}), \quad c_6 = O(r^{-\frac{9}{2}}), \\ c_3 &= -c_4 = O(r^{-\frac{9}{2}}), \quad c_7 = c_8 = -\frac{1}{4} r^{-4} + O(r^{-6}). \end{aligned} \tag{4.21}$$

Lemma 4.6 and Lemma 4.7 are crucial for our construction. Since it requires technical computations, we also postpone the proofs of them to the next section.

Proposition 4.8. *For any* $r > 0$*, there exists* $\varepsilon_3(r)$ *with the following significance. For any* $\varepsilon < \varepsilon_3(r)$*, there exists* $\{h_1, \cdots, h_{10}\}$ *which are* $\dot{H}^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ *functions and depend on* ε, r *continuously such that* $u = \varepsilon h^i K_i^r + \sqrt{1 - \varepsilon^2 |h^i K_i^r|^2} \mathcal{S}(\Psi[\vec{\alpha}_r])$ *satisfies*

$$\int_{\mathbb{R}^2} \nabla u : \nabla K_i^r = 0, \quad i = 1, \cdots, 10. \tag{4.22}$$

Furthermore,

$$\int_{\mathbb{R}^2} |\nabla (h^i K_i^r)|^2 = \frac{64\pi}{3} r^{-4} + O(|\log r|^{-1} r^{-4}) + O(\varepsilon), \tag{4.23}$$

$$\int_{\mathbb{R}^2} |\nabla (h^i K_i^r)|^2 - \int_{\mathbb{R}^2} |\nabla \mathcal{S}(\Psi[\vec{\alpha}_r])|^2 |h^i K_i^r|^2 = O(|\log r|^{-1} r^{-4}) + O(\varepsilon). \tag{4.24}$$

Proof. We can take $\vec{h} = (h_1, \cdots, h_{10})$ where

$$h^i K_i^r = f^r \mathcal{K}^r - c^i K_i^r \tag{4.25}$$

with c_i to be determined. Here we use Einstein summation convention for i. Define a map

$$F : \mathbb{R}_+ \times \mathbb{R}^{10} \to \mathbb{R}^{10} \tag{4.26}$$

$$(\varepsilon, \vec{c}) \to \left(\int_{\mathbb{R}^2} \nabla v : \nabla K_1^r, \cdots, \int_{\mathbb{R}^2} \nabla v : \nabla K_{10}^r \right) \tag{4.27}$$

where

$$v = h^i K_i^r - \frac{\varepsilon |h^i K_i^r|^2}{\sqrt{1 - |\varepsilon h^i K_i^r|^2} + 1} \mathcal{S}(\Psi[\vec{\alpha}_r]). \tag{4.28}$$

The map F is well-defined because K_i^r and $\mathcal{S}(\Psi[\vec{\alpha}_r])$ both belong to $\dot{H}^1(\mathbb{R}^2)$. For ε and $|\vec{c}|$ small, F is a smooth map.

At $\varepsilon = 0$, $F(0, \vec{c}) = 0$ if and only if

$$\mathcal{J}^r \vec{c} = \vec{p} \tag{4.29}$$

where $\vec{p} = (p_1, \cdots, p_{10})$, where p_j is defined in (4.20). Since $\mathcal{J}^r$ is non-degenerate, using Lemma 4.7, the above equation has a unique solution (4.21). We denote it as $\vec{c}_* = (c_1^*, \cdots, c_{10}^*)$. The Jacobian of F at $(0, \vec{c}_*)$ with respect to ε is

$$(\partial_{c_i} F^j)(0, \vec{c}_*) = -\mathcal{J}^r. \tag{4.30}$$

By the implicit function theorem, there exists $\varepsilon_3(r)$ such that for any $0 \le \varepsilon < \varepsilon_3(r)$ there exists $\vec{c} = \vec{c}(\varepsilon) = \vec{c}_* + O(\varepsilon)$ satisfies $F(\varepsilon, \vec{c}) = 0$. Since (4.25) and $f^r \in \dot{H}^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, then $h_i \in \dot{H}^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ for $1 \le i \le 10$. Using the form of v, one readily check $u = \varepsilon h^i K_i^r + \sqrt{1 - \varepsilon^2 |h^i K_i^r|^2} \mathcal{S}(\Psi[\vec{\alpha}_r])$ satisfies (4.22).

To establish (4.23), we compute explicitly

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla(h^i K_i^r)|^2 &= \int_{\mathbb{R}^2} |\nabla(f^r \mathcal{K}^r)|^2 - 2\sum_{i=1}^{10} c_i \int_{\mathbb{R}^2} \nabla(f^r\mathcal{K}^r) : \nabla K_i^r + \sum_{i,j=1}^{10} c_i c_j \mathcal{J}_{ij}^r \\ &= \frac{64\pi}{3} r^{-4} - 2\sum_{i=1}^{10} c_i^* p_i + \sum_{i,j=1}^{10} c_i^* c_j^* \mathcal{J}_{ij}^r + O(|\log r|^{-1} r^{-4}) + O(\varepsilon) \end{aligned} \tag{4.31}$$

where we have used Lemma 4.6 and $\vec{c} = \vec{c}_* + O(\varepsilon)$. By Lemma 4.7, we have

$$2\sum_{i=1}^{10} c_i^* p_i = O(r^{-8}). \tag{4.32}$$

To compute $\sum c_i^* c_j^* \mathcal{J}_{ij}^r$, we shall use Proposition 4.3 to see that $\mathcal{J}^r$ can written as a block diagonal matrix. Combining (4.21), Proposition 4.3, we have

$$\begin{aligned} \sum_{i,j\in\{1,6,10\}} c_i^* c_j^* \mathcal{J}_{ij}^r &= (c_1^*)^2 \mathcal{J}_{11}^r + 2c_1^* c_{10}^* \mathcal{J}_{1,10}^r + (c_{10}^*)^2 \mathcal{J}_{10,10}^r + O(r^{-8}) \\ &= \frac{32\pi}{3}\left((c_1^*)^2 - 2(c_1^*)^2 + (c_1^*)^2\right) + O(r^{-8}) = O(r^{-8}) \end{aligned} \tag{4.33}$$

where we have used $c_1^* = 2c_{10}^* + O(r^{-\frac{13}{2}}) = O(r^{-\frac{5}{2}})$ in the second line. Similarly one can use (4.21), Proposition 4.3 and $c_2^* = -2c_9^* + O(r^{-6}) = O(r^{-2})$ to derive

$$\sum_{i,j\in\{2,9,5\}} c_i^* c_j^* \mathcal{J}_{ij}^r = O(r^{-8}), \qquad \sum_{i,j\in\{3,8\}} c_i^* c_j^* \mathcal{J}_{ij}^r = \sum_{i,j\in\{4,7\}} c_i^* c_j^* \mathcal{J}_{ij}^r = O(r^{-8}). \tag{4.34}$$

Plugging the equations (4.32)-(4.34) to (4.31), we can get (4.23).

To establish (4.24), we shall use $h^i K_i^r = f^r \mathcal{K}^r - c^i K_i^r$ and $K_i^r \in \ker \mathcal{L}[\mathcal{S}(\Psi[\vec{\alpha}_r])]$ to derive

$$\int_{\mathbb{R}^2} |\nabla(h^i K_i^r)|^2 - \int_{\mathbb{R}^2} |\nabla \mathcal{S}(\Psi[\vec{\alpha}_r])|^2 |h^i K_i^r|^2 = \int_{\mathbb{R}^2} |\nabla(f^r\mathcal{K}^r)|^2 - |\nabla\mathcal{S}(\Psi[\vec{\alpha}_r])|^2 |f^r \mathcal{K}^r|^2.$$

Then (4.24) just follows from Lemma 4.6. □

Proposition 4.9. *For any $r > 0$, there exists a $\varepsilon_4(r)$ such that for $\varepsilon < \varepsilon_4(r)$ there exists $\{h_1, \cdots, h_{10}\}$ which are $\dot{H}^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ functions and depend on ε, r continuously such that for $u = \varepsilon h^i K_i^r + \sqrt{1 - \varepsilon^2 |h^i K_i^r|^2} \mathcal{S}(\Psi[\vec{\alpha}_r])$ the following infimum is achieved at $\mathcal{S}(\Psi[\vec{\alpha}_r])$.*

$$\inf_{(p,q)\in\mathcal{A}_2} \|u - \mathcal{S}(p/q)\|_{\dot{H}^1}^2 = \|u - \mathcal{S}(\Psi[\vec{\alpha}_r])\|_{\dot{H}^1}^2 = \varepsilon^2 \|h^i K_i^r\|_{\dot{H}^1}^2 + O_r(\varepsilon^3). \tag{4.35}$$

Here $|O_r(\varepsilon^3)| \le C(r)\varepsilon^3$ as $\varepsilon \to 0$.

Proof. First we choose $\varepsilon_4(r) < \varepsilon_3(r)$. Then it follows from Proposition 4.8 that one can find $\{h_1, \cdots, h_{10}\}$ which are $\dot{H}^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ functions and depend on ε, r continuously such that $u = \varepsilon h^i K_i^r + \sqrt{1-\varepsilon^2 |h^i K_i^r|^2}\mathcal{S}(\Psi[\vec{\alpha}_r])$ satisfies

$$\int_{\mathbb{R}^2} \nabla u : \nabla K_i^r = 0, \quad i = 1, \cdots, 10. \tag{4.36}$$

Taking $\varepsilon_4(r)$ even smaller, we can make

$$\|u - \mathcal{S}(\Psi[\vec{\alpha}_r])\|_{\mathcal{H}} < \min\{\eta_2(r), \eta_6(r), \eta_4(r, \varepsilon_2(r)/100)\}. \tag{4.37}$$

It follows from Lemma 2.5 that the infimum (4.35) can be achieved , say $\tilde{\Phi} = \mathcal{S}(\tilde{p}/\tilde{q})$ for some canonical $(\tilde{p}, \tilde{q}) \in \mathcal{A}_2$. Corollary 2.7 implies that all minimizers are near $\mathcal{S}(\Psi[\vec{\alpha}_r])$. More precisely

$$|\tilde{p} - (z - r - \mathrm{i}\, r)(z + r + \mathrm{i}\, r)|_\infty + |\tilde{q} - 1|_\infty \le \frac{\varepsilon_2(r)}{100}. \tag{4.38}$$

Thus we can assume $\tilde{\Phi} = \mathcal{S}(\Psi[\vec{\alpha}])$ for some $\vec{\alpha}$ satisfies $\|\vec{\alpha} - \vec{\alpha}_r\| < \varepsilon_2(r)$.

Since the infimum achieved at $\mathcal{S}(\Psi[\vec{\alpha}])$, then one has the orthogonality condition

$$\int_{\mathbb{R}^2} \nabla u : \nabla K = 0, \quad \forall\, K \in \ker \mathcal{L}[\mathcal{S}(\Psi[\vec{\alpha}])]. \tag{4.39}$$

However, Proposition 4.5 says that such $\vec{\alpha}$ is unique if $\|\vec{\alpha} - \vec{\alpha}_r\| < \varepsilon_2(r)$. Our choice of $\vec{h}$ from Proposition 4.8 makes sure that $\vec{\alpha}$ has to be $\vec{\alpha}_r$.

Finally, since $h^i K_i^r$ is smooth and bounded on $\mathbb{R}^2$, we can compute explicitly

$$\|u - \mathcal{S}(\Psi[\vec{\alpha}_r])\|_{\dot{H}^1}^2 = \|\varepsilon h^i K_i^r + O(\varepsilon^2 |\vec{h}|^2)\|_{\dot{H}^1}^2 = \varepsilon^2 \|h^i K_i^r\|_{\dot{H}^1}^2 + O_r(\varepsilon^3). \tag{4.40}$$

□

Finally we can prove the main theorem of this section.

Proof of Theorem 1.5. We shall take u from Proposition 4.9. Since $h^i K_i \in \dot{H}^1(\mathbb{R}^2; \mathbb{R}^3) \cap L^\infty(\mathbb{R}^2; \mathbb{R}^3)$, we shall apply Lemma 2.4 to get

$$\begin{aligned} \mathcal{E}(u) - 4\pi|\deg(u)| &= \frac{1}{2}\varepsilon^2 \int_{\mathbb{R}^2} |\nabla(h^i K_i^r)|^2 - |\nabla \mathcal{S}(\Psi[\vec{\alpha}_r])|^2 |h^i K_i^r|^2 + O_r(\varepsilon^3) \\ &= \varepsilon^2 O(|\log r|^{-1} r^{-4}) + O_r(\varepsilon^3) \end{aligned} \tag{4.41}$$

where we have used (4.24) in the last step.

On the other hand, it follows from Proposition 4.9 and (4.23) that

$$\inf_{(p,q)\in\mathcal{A}_2} \|u - \mathcal{S}(p/q)\|_{\dot{H}^1}^2 = \frac{64\pi}{3} r^{-4}\varepsilon^2 + O_r(\varepsilon^3) + O(\varepsilon^2 r^{-6}). \tag{4.42}$$

Now for any $M > 0$, we choose r large such that $64\pi/(3M) > O(|\log r|^{-1})$. **Fixing** such r, we can choose ε small such that

$$\inf_{\vec{\alpha}\in\mathbb{R}^{10}} \|u - \mathcal{S}(\Psi[\vec{\alpha}])\|_{\dot{H}^1}^2 > M(\mathcal{E}(u) - 4\pi|\deg u|). \tag{4.43}$$

Thus the theorem is proved. □

5. Explicit computation of the Jacobian

In this section, we shall compute the Jacobian $\mathcal{J}^r$ explicitly and give proofs of Proposition 4.3, Lemma 4.6, and Lemma 4.7.

Now consider $\Psi[\vec{\alpha}]$ defined in (4.1). Identifying $\mathbb{C}^5 = \{(a,b,c,d,e)\}$ and $\mathbb{R}^{10} = \{\vec{\alpha}\}$ in the way of $a = \alpha_1 + \mathrm{i}\,\alpha_2$, $b = \alpha_3 + \mathrm{i}\,\alpha_4$, etc. In this notation, we rewrite it as

$$\Psi[\vec{\alpha}](z) = \frac{az^2 + bz + c}{1 - dz - ez^2}.$$

Also we rewrite $\vec{\alpha}_r = (1, 0, -(r+\mathrm{i}\,r)^2, 0, 0) \in \mathbb{C}^5$, and denote $\Psi[\vec{\alpha}_r] = \psi_r$ for short

$$\psi_r(z) = (z - r - \mathrm{i}\,r)(z + r + \mathrm{i}\,r) = x^2 - y^2 + \mathrm{i}\,(2xy - 2r^2).$$

We always assume that r is large enough. We want to compute the vector fields K_i defined in (4.2) at $\vec{\alpha}_r$ explicitly. One needs to differentiate $\mathcal{S}(\Psi[\vec{\alpha}])$ with respect to real and imaginary part of a, b, c, d, e at $\vec{\alpha}_r$. Therefore we need to compute $\partial\Psi[\vec{\alpha}]$ and $\partial|\Psi[\vec{\alpha}]|^2$ at $\vec{\alpha}_r$. It is easy to see

$$\partial_{a_1}\Psi[\vec{\alpha}_r] = \psi_r, \quad \partial_{a_2}\Psi[\vec{\alpha}_r] = \mathrm{i}\,\psi_r, \quad \partial_{b_1}\Psi[\vec{\alpha}_r] = z, \quad \partial_{b_2}\Psi[\vec{\alpha}_r] = \mathrm{i}\,z, \quad \partial_{c_1}\Psi[\vec{\alpha}_r] = 1,$$
$$\partial_{c_2}\Psi[\vec{\alpha}_r] = \mathrm{i}\,,\ \partial_{d_1}\Psi[\vec{\alpha}_r] = z\psi_r,\ \partial_{d_2}\Psi[\vec{\alpha}_r] = \mathrm{i}\,z\psi_r,\ \partial_{e_1}\Psi[\vec{\alpha}_r] = z^2\psi_r,\ \partial_{e_2}\Psi[\vec{\alpha}_r] = \mathrm{i}\,z^2\psi_r.$$

Note that $\partial|\psi|^2 = \partial(\psi\bar{\psi}) = 2\mathrm{Re}\,(\psi\partial\bar{\psi}) = 2\mathrm{Re}\,(\psi\overline{\partial\psi})$. Then

$$\begin{aligned}
&\partial_{a_1}|\Psi[\vec{\alpha}_r]|^2 = 2|\psi_r|^2, \quad \partial_{a_2}|\Psi[\vec{\alpha}_r]|^2 = 0,\\
&\partial_{b_1}|\Psi[\vec{\alpha}_r]|^2 = 2(x^3 - 2r^2y + xy^2), \quad \partial_{b_2}|\Psi[\vec{\alpha}_r]|^2 = 2(y^3 - 2r^2x + x^2y),\\
&\partial_{c_1}|\Psi[\vec{\alpha}_r]|^2 = 2x^2 - 2y^2, \quad \partial_{c_2}|\Psi[\vec{\alpha}_r]|^2 = 4xy - 4r^2,\\
&\partial_{d_1}|\Psi[\vec{\alpha}_r]|^2 = 2x|\psi_r|^2, \quad \partial_{d_2}|\Psi[\vec{\alpha}_r]|^2 = -2y|\psi_r|^2,\\
&\partial_{e_1}|\Psi[\vec{\alpha}_r]|^2 = 2(x^2 - y^2)|\psi_r|^2, \quad \partial_{e_2}|\Psi[\vec{\alpha}_r]|^2 = -4xy|\psi_r|^2.
\end{aligned}$$

Then we have

$$\begin{aligned}
K_1^r &:= \partial_{a_1}u = \zeta\left((1 - |\psi_r|^2)\psi_r, 2|\psi_r|^2\right),\\
K_2^r &:= \partial_{a_2}u = \zeta\left((1 + |\psi_r|^2)\mathrm{i}\,\psi_r, 0\right),\\
K_3^r &:= r^{-1}\partial_{b_1}u = \zeta r^{-1}\left((1 + |\psi_r|^2)z - 4(x^3 - 2r^2y + xy^2)\psi_r, 4(x^3 - 2r^2y + xy^2)\right),\\
K_4^r &:= r^{-1}\partial_{b_2}u = \zeta r^{-1}\left((1 + |\psi_r|^2)\mathrm{i}\,z - 4(y^3 - 2r^2x + x^2y)\psi_r, 4(y^3 - 2r^2x + x^2y)\right),\\
K_5^r &:= \partial_{c_1}u = \zeta\left((1 + |\psi_r|^2) - 4(x^2 - y^2)\psi_r, 4(x^2 - y^2)\right),\\
K_6^r &:= \partial_{c_2}u = \zeta\left((1 + |\psi_r|^2)\mathrm{i}\, - 8(xy - r^2)\psi_r, 8(xy - r^2)\right),\\
K_7^r &:= r^{-1}\partial_{d_1}u = \zeta r^{-1}\left((1 + |\psi_r|^2)z\psi_r - 2x|\psi_r|^2\psi_r, 2x|\psi_r|^2\right),\\
K_8^r &:= r^{-1}\partial_{d_2}u = \zeta r^{-1}\left((1 + |\psi_r|^2)\mathrm{i}\,z\psi_r + 2y|\psi_r|^2\psi_r, -2y|\psi_r|^2\right),\\
K_9^r &:= r^{-2}\partial_{e_1}u = \zeta r^{-2}\left((1 + |\psi_r|^2)z^2\psi_r - 2(x^2 - y^2)|\psi_r|^2\psi_r, 2(x^2 - y^2)|\psi_r|^2\right),\\
K_{10}^r &:= r^{-2}\partial_{e_2}u = \zeta r^{-2}\left((1 + |\psi_r|^2)\mathrm{i}\,z^2\psi_r + 4xy|\psi_r|^2\psi_r, -4xy|\psi_r|^2\right).
\end{aligned}$$

where $\zeta = 2(1+|\psi_r|^2)^{-2}$. We will introduce the notation $\mathcal{I}^r_{ij} = \frac{1}{4}(1+|\psi_r|^2)^2 K^r_i \cdot K^r_j$ to denote the inner product of K^r_i and K^r_j. We have

$$\begin{aligned}
&\mathcal{I}^r_{11} = |\psi_r|^2 = \mathcal{I}^r_{22}, \quad \mathcal{I}^r_{12} = 0, \quad \mathcal{I}^r_{13} = r^{-1}(x^3 - 2r^2y + xy^2),\\
&\mathcal{I}^r_{14} = r^{-1}(y^3 - 2r^2x + x^2y), \quad \mathcal{I}^r_{15} = (x^2 - y^2), \quad \mathcal{I}^r_{16} = 2(xy - r^2), \quad \mathcal{I}^r_{17} = r^{-1}x|\psi_r|^2,\\
&\mathcal{I}^r_{18} = -r^{-1}y|\psi_r|^2, \quad \mathcal{I}^r_{19} = r^{-2}(x^2-y^2)|\psi_r|^2, \quad \mathcal{I}^r_{1,10} = -2r^{-2}xy|\psi_r|^2,\\
&\mathcal{I}^r_{23} = r^{-1}[2r^2x - x^2y - y^3)], \quad \mathcal{I}^r_{24} = r^{-1}[(x^3 - 2r^2y + xy^2)], \quad \mathcal{I}^r_{25} = 2(r^2 - xy),\\
&\mathcal{I}^r_{26} = (x^2 - y^2), \quad \mathcal{I}^r_{27} = r^{-1}y|\psi_r|^2, \quad \mathcal{I}^r_{28} = r^{-1}x|\psi_r|^2,\\
&\mathcal{I}^r_{29} = 2r^{-2}xy|\psi_r|^2, \quad \mathcal{I}^r_{2,10} = r^{-2}(x^2-y^2)|\psi_r|^2.
\end{aligned}$$

$$\begin{aligned}
&\mathcal{I}^r_{33} = r^{-2}(x^2+y^2) = \mathcal{I}^r_{44}, \quad \mathcal{I}^r_{34} = 0, \quad \mathcal{I}^r_{35} = r^{-1}x,\\
&\mathcal{I}^r_{36} = r^{-1}y, \quad \mathcal{I}^r_{37} = r^{-2}(x^4 - y^4), \quad \mathcal{I}^r_{38} = 2r^{-2}(r^2 - xy)(x^2+y^2),\\
&\mathcal{I}^r_{39} = r^{-3}(x^2+y^2)(x^3 + 2r^2y - 3xy^2), \quad \mathcal{I}^r_{3,10} = -r^{-3}(x^2+y^2)(2r^2x - 3x^2y + y^3),\\
&\mathcal{I}^r_{45} = -r^{-1}y, \quad \mathcal{I}^r_{46} = r^{-1}x, \quad \mathcal{I}^r_{47} = -r^{-2}(r^2 - xy)(x^2+y^2), \quad \mathcal{I}^r_{48} = r^{-2}(x^4 - y^4),\\
&\mathcal{I}^r_{49} = -r^{-3}(x^2+y^2)(2r^2x - 3x^2y + y^3), \quad \mathcal{I}^r_{4,10} = r^{-3}(x^2+y^2)(x^3 + 2r^2y - 3xy^2).
\end{aligned}$$

$$\begin{aligned}
&\mathcal{I}^r_{55} = 1 = \mathcal{I}^r_{66},\ \mathcal{I}^r_{56} = 0,\ \mathcal{I}^r_{57} = r^{-1}(x^3 + 2r^2y - 3xy^2),\ \mathcal{I}^r_{58} = r^{-1}(2r^2x - 3x^2y + y^3),\\
&\mathcal{I}^r_{59} = r^{-2}(x^4 + 4r^2xy - 6x^2y^2 + y^4),\ \mathcal{I}^r_{5,10} = 2r^{-2}(r^2 - 2xy)(x^2 - y^2),\\
&\mathcal{I}^r_{67} = r^{-1}(2r^2x - 3x^2y + y^3), \quad \mathcal{I}^r_{68} = r^{-1}(x^3 + 2r^2y - 3xy^2),\\
&\mathcal{I}^r_{69} = 2r^{-2}(r^2 - 2xy)(y^2 - x^2), \quad \mathcal{I}^r_{6,10} = r^{-2}(x^4 + 4r^2xy - 6x^2y^2 + y^4).
\end{aligned}$$

$$\begin{aligned}
&\mathcal{I}^r_{77} = r^{-2}(x^2+y^2)|\psi_r|^2, \quad \mathcal{I}^r_{78} = 0,\\
&\mathcal{I}^r_{79} = r^{-3}x(x^2+y^2)|\psi_r|^2, \quad \mathcal{I}^r_{7,10} = -r^{-3}y(x^2+y^2)|\psi_r|^2,\\
&\mathcal{I}^r_{88} = r^{-2}(x^2+y^2)|\psi_r|^2, \quad \mathcal{I}^r_{89} = r^{-3}y(x^2+y^2)|\psi_r|^2, \quad \mathcal{I}^r_{8,10} = r^{-3}x(x^2+y^2)|\psi_r|^2,\\
&\mathcal{I}^r_{99} = r^{-4}(x^2+y^2)^2|\psi_r|^2, \quad \mathcal{I}^r_{9,10} = 0, \quad \mathcal{I}^r_{10,10} = r^{-4}(x^2+y^2)^2|\psi_r|^2.
\end{aligned}$$

The calculations here and later are a little bit tedious but still manageable by bare hands. To make the life easier, we perform them with the help of Mathematica[1].

Using (2.7), we get

$$|\nabla \mathcal{S}(\Psi[\vec{\alpha}_r])|^2 = \frac{4|\partial_x \psi_r|^2 + 4|\partial_y \psi_r|^2}{(1+|\psi_r|^2)^2} = \frac{32(x^2+y^2)}{(1+|\psi_r|^2)^2}. \tag{5.1}$$

Thus (4.5) implies that

$$\mathcal{J}^r_{ij} = \int_{\mathbb{R}^2} \frac{128(x^2+y^2)}{(1+|\psi_r|^2)^4} \mathcal{I}^r_{ij}\, dxdy. \tag{5.2}$$

[1]https://www.wolframcloud.com/obj/bingomat/Published/deg2-HM-final.nb
A link for source code.

Lemma 5.1. *The following entries of $\mathcal{J}^r$ are all equal to* 0.

$$\mathcal{J}^r_{12}, \mathcal{J}^r_{13}, \mathcal{J}^r_{14}, \mathcal{J}^r_{15}, \mathcal{J}^r_{17}, \mathcal{J}^r_{18}, \mathcal{J}^r_{19}, \mathcal{J}^r_{23}, \mathcal{J}^r_{24}, \mathcal{J}^r_{26}, \mathcal{J}^r_{27}, \mathcal{J}^r_{28}, \mathcal{J}^r_{2,10},$$
$$\mathcal{J}^r_{34}, \mathcal{J}^r_{35}, \mathcal{J}^r_{36}, \mathcal{J}^r_{37}, \mathcal{J}^r_{39}, \mathcal{J}^r_{3,10}, \mathcal{J}^r_{45}, \mathcal{J}^r_{46}, \mathcal{J}^r_{48}, \mathcal{J}^r_{49}, \mathcal{J}^r_{4,10},$$
$$\mathcal{J}^r_{56}, \mathcal{J}^r_{57}, \mathcal{J}^r_{58}, \mathcal{J}^r_{5,10}, \mathcal{J}^r_{67}, \mathcal{J}^r_{68}, \mathcal{J}^r_{69}, \mathcal{J}^r_{78}, \mathcal{J}^r_{79}, \mathcal{J}^r_{7,10}, \mathcal{J}^r_{89}, \mathcal{J}^r_{8,10}, \mathcal{J}^r_{9,10}.$$

Proof. These facts follow from the symmetries. The integrand of $\mathcal{J}^r_{13}$ is odd with respect to the operation $(x, y) \to (-x, -y)$ while the integration domain is even. Thus it is equal to 0. The same symmetry holds for

$$\mathcal{J}^r_{14}, \mathcal{J}^r_{17}, \mathcal{J}^r_{18}, \mathcal{J}^r_{23}, \mathcal{J}^r_{24}, \mathcal{J}^r_{27}, \mathcal{J}^r_{28}, \mathcal{J}^r_{35}, \mathcal{J}^r_{36}, \mathcal{J}^r_{39}, \mathcal{J}^r_{3,10},$$
$$\mathcal{J}^r_{45}, \mathcal{J}^r_{46}, \mathcal{J}^r_{49}, \mathcal{J}^r_{4,10}, \mathcal{J}^r_{57}, \mathcal{J}^r_{58}, \mathcal{J}^r_{67}, \mathcal{J}^r_{68}, \mathcal{J}^r_{79}, \mathcal{J}^r_{7,10}, \mathcal{J}^r_{89}, \mathcal{J}^r_{8,10}.$$

On the other hand, if we switch x and y, then we get $\mathcal{J}^r_{15} = -\mathcal{J}^r_{15}$. The same symmetry holds for

$$\mathcal{J}^r_{19}, \mathcal{J}^r_{26}, \mathcal{J}^r_{2,10}, \mathcal{J}^r_{37}, \mathcal{J}^r_{48}, \mathcal{J}^r_{5,10}, \mathcal{J}^r_{69}.$$

It is easy to see $\mathcal{J}^r_{12} = \mathcal{J}^r_{34} = \mathcal{J}^r_{56} = \mathcal{J}^r_{78} = \mathcal{J}^r_{9,10} = 0$ since $\mathcal{I}^r_{12} = \mathcal{I}^r_{34} = \mathcal{I}^r_{56} = \mathcal{I}^r_{78} = \mathcal{I}^r_{9,10} = 0$. □

To compute other terms of $\mathcal{J}^r_{ij}$, we need the following lemma.

Lemma 5.2. *Suppose $p(x, y, r)$ is a homogeneous polynomial on x, y, r with degree $k \geq 0$. Assume $l \geq 3$ and $l > \frac{k}{4} + \frac{1}{2}$. For r large, one has*

$$\begin{aligned}\int_{\mathbb{R}^2} \frac{p(x,y,r)}{(1+|\psi_r|^2)^l} =& \frac{\pi}{8(l-1)}[p(\mathbf{1}) + p(-\mathbf{1})]r^{k-2} + \frac{\pi[\Delta_{x,y}p(\mathbf{1}) + \Delta_{x,y}p(-\mathbf{1})]}{256(l-1)(l-2)}r^{k-6} \\ &+ \frac{\pi[-p_x(\mathbf{1}) - p_y(\mathbf{1}) + p_x(-\mathbf{1}) + p_y(-\mathbf{1}) + p(\mathbf{1}) + p(-\mathbf{1})]}{128(l-1)(l-2)}r^{k-6} \\ &+ O(r^{k-8})\end{aligned}$$

where $\mathbf{1} = (1, 1, 1)$ and $-\mathbf{1} = (-1, -1, 1)$.

Proof. Let

$$\begin{aligned}\Omega_1 &= \{(x, y) : (x-r)^2 + (y-r)^2 < r\}, \\ \Omega_2 &= \{(x, y) : (x+r)^2 + (y+r)^2 < r\}, \\ \Omega_3 &= \{(x, y) : (x-r)^2 + (y-r)^2 \geq r, |y| + |x| \leq 3r, x + y \geq 0\}, \\ \Omega_4 &= \{(x, y) : (x+r)^2 + (y+r)^2 \geq r, |y| + |x| \leq 3r, x + y \leq 0\},\end{aligned}$$

and $\Omega^{out} = \mathbb{R}^2 - \Omega_1 - \Omega_2 - \Omega_3 - \Omega_4 \subset \{(x, y) : |x| \geq r, |y| \geq r\}$. We shall reserve the notations of Ω_i for the rest of paper. See Figure 1 for illustration. Denote

$$Int(\Omega_i) = \int_{\Omega_i} \frac{p(x,y,r)}{(1+|\psi_r|^2)^l} dxdy.$$

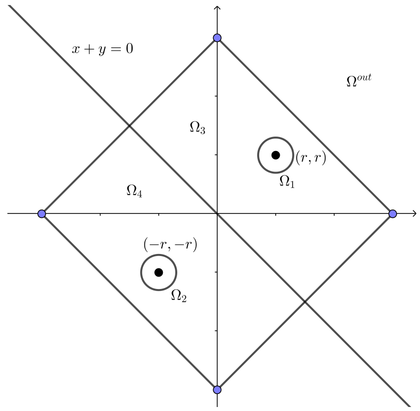

FIGURE 1. Integration domains Ω_i.

On Ω_1, making a change of variable $x = r + r^{-1}\tilde{x}$, $y = r + r^{-1}\tilde{y}$, we have $\Omega_1 = \{(\tilde{x}, \tilde{y}) : \tilde{x}^2 + \tilde{y}^2 < r^3\}$. Since $|\psi_r|^2 = [(x-r)^2 + (y-r)^2][(x+r)^2 + (y+r)^2]$, we rewrite it as

$$\begin{aligned} |\psi_r|^2 =& (\tilde{x}^2 + \tilde{y}^2)[(2 + r^{-2}\tilde{x})^2 + (2 + r^{-2}\tilde{y})^2] \\ =& (\tilde{x}^2 + \tilde{y}^2)[8 + 4r^{-2}(\tilde{x} + \tilde{y}) + r^{-4}(\tilde{x}^2 + \tilde{y}^2)]. \end{aligned} \tag{5.3}$$

Denote $\tilde{x}^2 + \tilde{y}^2 = \rho^2$ and $A = 1 + 8(\tilde{x}^2 + \tilde{y}^2) = 1 + 8\rho^2$. Since $r^{-2}|\tilde{x}| \le r^{-1/2}$ and $r^{-2}|\tilde{y}| \le r^{-1/2}$ in Ω_1, then $4r^{-2}(\tilde{x} + \tilde{y}) + r^{-4}(\tilde{x}^2 + \tilde{y}^2) \ll 8$. Then we can make the following expansion.

$$\begin{aligned} & (1 + |\psi_r|^2)^{-l} = \left(A + 4r^{-2}\rho^2(\tilde{x} + \tilde{y}) + r^{-4}\rho^4\right)^{-l} \\ =& A^{-l} - lA^{-l-1}[4r^{-2}\rho^2(\tilde{x} + \tilde{y}) + r^{-4}\rho^4] + \frac{l(l+1)}{2}A^{-l-2}[4r^{-2}\rho^2(\tilde{x} + \tilde{y}) + r^{-4}\rho^4]^2 \\ & + O(A^{-l-3}r^{-6}\rho^6(|\tilde{x}|^3 + |\tilde{y}|^3)). \end{aligned} \tag{5.4}$$

Since p is a homogeneous polynomial and $r^{-2}|\tilde{x}| \le r^{-1/2}$ and $r^{-2}|\tilde{y}| \le r^{-1/2}$, then

$$\begin{aligned} & p(x, y, r) = r^k p(1 + r^{-2}\tilde{x}, 1 + r^{-2}\tilde{y}, 1) \\ =& r^k p(\mathbf{1}) + r^{k-2}[p_x(\mathbf{1})\tilde{x} + p_y(\mathbf{1})\tilde{y}] + \frac{1}{2}r^{k-4}[p_{xx}(\mathbf{1})\tilde{x}^2 + p_{xy}(\mathbf{1})\tilde{x}\tilde{y} + p_{yy}(\mathbf{1})\tilde{y}^2] \\ & + O(r^{k-6}(|\tilde{x}|^3 + |\tilde{y}|^3)). \end{aligned} \tag{5.5}$$

Here we write $(1, 1, 1)$ as $\mathbf{1}$ for short. Now we combine (5.4) and (5.5) to get

$$\frac{p(x, y, r)}{(1 + |\psi_r|^2)^l} = r^k \frac{p(\mathbf{1})}{[1 + 8\rho^2]^l} + r^{k-2}B_1 + r^{k-4}B_2 + O(r^{k-6}A^{-l}(|\tilde{x}|^3 + |\tilde{y}|^3)), \tag{5.6}$$

where

$$B_1 = \frac{p_x(\mathbf{1})\tilde{x} + p_y(\mathbf{1})\tilde{y}}{[1 + 8\rho^2]^l} - \frac{4p(\mathbf{1})l\rho^2(\tilde{x} + \tilde{y})}{[1 + 8\rho^2]^{l+1}} \tag{5.7}$$

and

$$\begin{aligned} B_2 = & \frac{[p_{xx}(\mathbf{1})\tilde{x}^2 + p_{xy}(\mathbf{1})\tilde{x}\tilde{y} + p_{yy}(\mathbf{1})\tilde{y}^2]}{2[1+8\rho^2]^l} - \frac{4\rho^2 l(\tilde{x}+\tilde{y})[p_x(\mathbf{1})\tilde{x} + p_y(\mathbf{1})\tilde{y}]}{[1+8\rho^2]^{l+1}} \\ & - \frac{p(\mathbf{1}) l \rho^4}{[1+8\rho^2]^{l+1}} + \frac{8p(\mathbf{1}) l(l+1)\rho^4(\tilde{x}+\tilde{y})^2}{[1+8\rho^2]^{l+2}}. \end{aligned} \tag{5.8}$$

Making a change of variable $\tilde{x} = \rho\cos\theta$, $\tilde{y} = \rho\sin\theta$, we have

$$\begin{aligned} Int(\Omega_1) &= 2\pi r^{k-2}\int_0^{r^{\frac{3}{2}}} \frac{p(\mathbf{1})\rho}{[1+8\rho^2]^l} d\rho + r^{k-6}\int_{\Omega_1} B_2 d\tilde{x}d\tilde{y} + O(r^{k-8})\int_0^{r^{\frac{3}{2}}} \frac{\rho^3}{[1+8\rho^2]^l} d\rho \\ &= 2\pi p(\mathbf{1}) r^{k-2}\int_0^{r^{\frac{3}{2}}} \frac{\rho}{(1+8\rho^2)^l} d\rho + r^{k-6}\int_{\Omega_1} B_2 d\tilde{x}d\tilde{y} + O(r^{k-8}). \end{aligned}$$

Here we have used the assumption $l \geq 3$ and $\int_{\Omega_1} B_1 d\tilde{x}d\tilde{y} = 0$. One can compute

$$\begin{aligned} \int_{\Omega_1} B_2 d\tilde{x}d\tilde{y} = & \frac{\pi}{2}\Delta_{x,y} p(\mathbf{1})\int_0^{r^{\frac{3}{2}}} \frac{\rho^3}{[1+8\rho^2]^l} d\rho - 4\pi[p_x(\mathbf{1}) + p_y(\mathbf{1})] l \int_0^{r^{\frac{3}{2}}} \frac{\rho^5}{[1+8\rho^2]^{l+1}} d\rho \\ & - 2\pi p(\mathbf{1}) l \int_0^{r^{\frac{3}{2}}} \frac{\rho^5}{[1+8\rho^2]^{l+1}} d\rho + 16\pi p(\mathbf{1}) l(l+1)\int_0^{r^{\frac{3}{2}}} \frac{\rho^7}{[1+8\rho^2]^{l+2}} d\rho. \end{aligned}$$

Now some elementary integration shows

$$\begin{aligned} \int_0^{r^{\frac{3}{2}}} \frac{\rho}{[1+8\rho^2]^l} d\rho &= \frac{1}{16(l-1)} + O(r^{-6}), \\ \int_0^{r^{\frac{3}{2}}} \frac{\rho^3}{[1+8\rho^2]^l} d\rho &= \frac{1}{128(l-1)(l-2)} + O(r^{-3}), \\ \int_0^{r^{\frac{3}{2}}} \frac{\rho^5}{[1+8\rho^2]^{l+1}} d\rho &= \frac{1}{512 l(l-1)(l-2)} + O(r^{-3}), \\ \int_0^{r^{\frac{3}{2}}} \frac{\rho^7}{[1+8\rho^2]^{l+2}} d\rho &= \frac{3}{4096(l+1)l(l-1)(l-2)} + O(r^{-3}). \end{aligned}$$

Plugging in these identities to $\int_{\Omega_1} B_2 d\tilde{x}d\tilde{y}$ to get

$$\begin{aligned} & \int_{\Omega_1} B_2 d\tilde{x}d\tilde{y} \\ =& \frac{\pi\Delta_{x,y} p(\mathbf{1})}{256(l-1)(l-2)} - \frac{2\pi[p_x(\mathbf{1}) + p_y(\mathbf{1})] + \pi p(\mathbf{1})}{256(l-1)(l-2)} + \frac{3\pi p(\mathbf{1})}{256(l-1)(l-2)} + O(r^{-3}) \\ =& \frac{\pi[\Delta_{x,y} p(\mathbf{1}) - 2p_x(\mathbf{1}) - 2p_y(\mathbf{1}) + 2p(\mathbf{1})]}{256(l-1)(l-2)} + O(r^{-3}). \end{aligned}$$

Therefore

$$Int(\Omega_1) = \frac{\pi p(\mathbf{1})}{8(l-1)} r^{k-2} + \frac{\pi[\Delta_{x,y} p(\mathbf{1}) - 2p_x(\mathbf{1}) - 2p_y(\mathbf{1}) + 2p(\mathbf{1})]}{256(l-1)(l-2)} r^{k-6} + O(r^{k-8}). \tag{5.9}$$

By symmetry, we have a corresponding equality for $Int(\Omega_2)$,

$$Int(\Omega_2) = \frac{\pi p(-\mathbf{1})}{8(l-1)} r^{k-2} + \frac{\pi[\Delta_{x,y} p(-\mathbf{1}) + 2p_x(-\mathbf{1}) + 2p_y(-\mathbf{1}) + 2p(-\mathbf{1})]}{256(l-1)(l-2)} r^{k-6} + O(r^{k-8}) \tag{5.10}$$

where $p(-\mathbf{1}) = (-1,-1,1)$.

Now in Ω_3, we have $r^{-2}|\tilde{x}| + r^{-2}|\tilde{y}| \le C$ and $|\psi_r|^2 \ge C^{-1} r^2[(x-r)^2 + (y-r)^2]$ for some uniform constant C. Let $x = r + r^{-1}\tilde{x}$ and $y = r + r^{-1}\tilde{y}$, one gets

$$\begin{aligned} |Int(\Omega_3)| &\le C r^k \int_{\Omega_3} |\psi_r|^{-2l} dx dy \le C r^{k-2} \int_{r^{\frac{3}{2}}}^{\infty} [\tilde{x}^2 + \tilde{y}^2]^{-l} d\tilde{x} d\tilde{y} \\ &\le C r^{k-2} \int_{r^{\frac{3}{2}}}^{+\infty} \rho^{-2l+1} d\rho \le C r^{k-2} r^{-3l+3} = O(r^{k-8}). \end{aligned} \tag{5.11}$$

The same estimate holds for $Int(\Omega_4)$. In Ω^{out}, we make a change of variable $x = r\tilde{x}$, $y = r\tilde{y}$,

$$\begin{aligned} |Int(\Omega^{out})| &\le \int_{\Omega^{out}} |p(x,y,r)| |\psi_r|^{-2l} dx dy \\ &\le \int_{|\tilde{x}|\ge 1, |\tilde{y}| \ge 1} \frac{|p(\tilde{x},\tilde{y},1)| r^{k-4l+2}}{[((\tilde{x}-1)^2 + (\tilde{y}-1)^2)((\tilde{x}-1)^2 + (\tilde{y}-1)^2)]^l} d\tilde{x} d\tilde{y} \\ &\le r^{k-4l+2} \int_{|\tilde{x}|\ge 1, |\tilde{y}|\ge 1} \frac{|\tilde{x}|^k + |\tilde{y}|^k + C}{[\tilde{x}^2 + \tilde{y}^2]^{2l}} d\tilde{x} d\tilde{y} \le C r^{k-4l+2} = O(r^{k-10}) \end{aligned} \tag{5.12}$$

provided $4l > k+2$.

Collecting the results of $Int(\Omega_i)$, $i = 1,2,3,4$ and $Int(\Omega^{out})$, we get

$$\begin{aligned} Int(\mathbb{R}^2) =& \frac{\pi}{8(l-1)} [p(\mathbf{1}) + p(-\mathbf{1})] r^{k-2} + \frac{\pi[\Delta_{x,y} p(\mathbf{1}) + \Delta_{x,y} p(-\mathbf{1})]}{256(l-1)(l-2)} r^{k-6} \\ &+ \frac{\pi[-p_x(\mathbf{1}) - p_y(\mathbf{1}) + p_x(-\mathbf{1}) + p_y(-\mathbf{1}) + p(\mathbf{1}) + p(-\mathbf{1})]}{128(l-1)(l-2)} r^{k-6} \\ &+ O(r^{k-8}). \end{aligned} \tag{5.13}$$

This proves our conclusion. □

Corollary 5.3. *Suppose $p(x,y,r)$ is a homogeneous polynomial on x,y,r with degree $k\geq 0$. Assume $l\geq 4$ and $l>\frac{k}{4}+\frac{3}{2}$. For r large, one has*

$$\begin{aligned}&\int_{\mathbb{R}^2}\frac{p(x,y,r)|\psi_r|^2}{(1+|\psi_r|^2)^l}\\=&\frac{\pi}{8(l-1)(l-2)}[p(\mathbf{1})+p(-\mathbf{1})]r^{k-2}+\frac{\pi[\Delta_{x,y}p(\mathbf{1})+\Delta_{x,y}p(-\mathbf{1})]}{128(l-1)(l-2)(l-3)}r^{k-6}\\&+\frac{\pi[-p_x(\mathbf{1})-p_y(\mathbf{1})+p_x(-\mathbf{1})+p_y(-\mathbf{1})+p(\mathbf{1})+p(-\mathbf{1})]}{64(l-1)(l-2)(l-3)}r^{k-6}+O(r^{k-8}).\end{aligned}\tag{5.14}$$

Proof. Notice the following equality

$$\int_{\mathbb{R}^2}\frac{p(x,y,r)|\psi_r|^2}{[1+|\psi_r|^2]^l}=\int_{\mathbb{R}^2}\frac{p(x,y,r)}{[1+|\psi_r|^2]^{l-1}}-\int_{\mathbb{R}^2}\frac{p(x,y,r)}{[1+|\psi_r|^2]^l},\tag{5.15}$$

one can apply Lemma 5.2. □

Next we shall prove that if $p(1,1,1)=p(-1,-1,1)=0$ then the remainder of term in (5.13) could be improved.

Lemma 5.4. *Suppose that $p(x,y,r)$ is a homogeneous polynomial on x,y,r with degree $k\geq 0$. Assume $p(1,1,1)=p(-1,-1,1)=0$, $l\geq 3$, $l>\frac{k}{4}+\frac{1}{2}$, then*

$$\begin{aligned}&\int_{\mathbb{R}^2}\frac{p(x,y,r)}{(1+|\psi_r|^2)^l}dxdy\\=&\frac{\pi[\Delta_{x,y}p(\mathbf{1})+\Delta_{x,y}p(-\mathbf{1})-2p_x(\mathbf{1})-2p_y(\mathbf{1})+2p_x(-\mathbf{1})+2p_y(-\mathbf{1})]r^{k-6}}{256(l-1)(l-2)}+O(r^{k-\frac{17}{2}}).\end{aligned}$$

Proof. We shall use the notations in the proof of Lemma 5.2 and refine the proof there. In Ω_1,

$$\begin{aligned}&(1+|\psi_r|^2)^{-l}=\left(A+4r^{-2}\rho^2(\tilde{x}+\tilde{y})+r^{-4}\rho^4\right)^{-l}\\=&A^{-l}-lA^{-l-1}[4r^{-2}\rho^2(\tilde{x}+\tilde{y})+r^{-4}\rho^4]+\frac{l(l+1)}{2}A^{-l-2}[4r^{-2}\rho^2(\tilde{x}+\tilde{y})+r^{-4}\rho^4]^2\\&-\frac{l(l+1)(l+2)}{6}A^{-l-3}[4r^{-2}\rho^2(\tilde{x}+\tilde{y})+r^{-4}\rho^4]^3+O(A^{-l-4}r^{-8}\rho^8(|\tilde{x}|^4+|\tilde{y}|^4)|).\end{aligned}\tag{5.16}$$

Using our assumption, one has

$$\begin{aligned}&p(x,y,r)=r^kp(1+r^{-2}\tilde{x},1+r^{-2}\tilde{y},1)\\=&r^{k-2}[p_x(\mathbf{1})\tilde{x}+p_y(\mathbf{1})\tilde{y}]+\frac{1}{2}r^{k-4}[p_{xx}(\mathbf{1})\tilde{x}^2+p_{xy}(\mathbf{1})\tilde{x}\tilde{y}+p_{yy}(\mathbf{1})\tilde{y}^2]\\&+\frac{1}{6}r^{k-6}Q+O(r^{k-8}(|\tilde{x}|^4+|\tilde{y}|^4))\end{aligned}\tag{5.17}$$

where

$$Q=p_{xxx}(\mathbf{1})\tilde{x}^3+3p_{xxy}(\mathbf{1})\tilde{x}^2\tilde{y}+3p_{xyy}(\mathbf{1})\tilde{x}\tilde{y}^2+p_{yyy}(\mathbf{1})\tilde{y}^3.\tag{5.18}$$

Now we combine (5.16) and (5.17) to get

$$\frac{p(x,y,r)}{(1+|\psi_r|^2)^l} = r^{k-2}B_1 + r^{k-4}B_2 + r^{k-6}B_3 + O(r^{k-8}A^{-l}(|\tilde{x}|^4 + |\tilde{y}|^4)) \tag{5.19}$$

where

$$B_1 = \frac{p_x(\mathbf{1})\tilde{x} + p_y(\mathbf{1})\tilde{y}}{[1+8\rho^2]^l},$$

$$B_2 = \frac{[p_{xx}(\mathbf{1})\tilde{x}^2 + p_{xy}(\mathbf{1})\tilde{x}\tilde{y} + p_{yy}(\mathbf{1})\tilde{y}^2]}{2[1+8\rho^2]^l} - \frac{4\rho^2 l(\tilde{x}+\tilde{y})[p_x(\mathbf{1})\tilde{x} + p_y(\mathbf{1})\tilde{y}]}{[1+8\rho^2]^{l+1}},$$

$$B_3 = \frac{Q}{6[1+8\rho^2]^l} - \frac{l\rho^4[p_x(\mathbf{1})\tilde{x} + p_y(\mathbf{1})\tilde{y}] + 2l\rho^2(\tilde{x}+\tilde{y})[p_{xx}(\mathbf{1})\tilde{x}^2 + p_{xy}(\mathbf{1})\tilde{x}\tilde{y} + p_{yy}(\mathbf{1})\tilde{y}^2]}{[1+8\rho^2]^{l+1}}.$$

Using $\int_{\Omega_1} B_3 d\tilde{x}d\tilde{y} = 0 = \int_{\Omega_1} B_1 d\tilde{x}d\tilde{y}$, we can get the following estimate

$$Int(\Omega_1) = \frac{\pi[\Delta_{x,y}p(\mathbf{1}) - 2p_x(\mathbf{1}) - 2p_y(\mathbf{1})]}{256(l-1)(l-2)} r^{k-6} + O(r^{k-10}). \tag{5.20}$$

Similar equality hold for $Int(\Omega_2)$. In Ω_3, using our assumption,

$$\begin{aligned}|Int(\Omega_3)| &\le Cr^{k-2}\int_{\Omega_3}(|\tilde{x}|+|\tilde{y}|)|\psi_r|^{-2l}dxdy \le Cr^{k-4}\int_{r^{\frac{3}{2}}}^{\infty}[\tilde{x}+\tilde{y}][\tilde{x}^2+\tilde{y}^2]^{-l}d\tilde{x}d\tilde{y}\\ &\le Cr^{k-4}\int_{r^{\frac{3}{2}}}^{+\infty}\rho^{-2l+2}d\rho \le Cr^{k-4}r^{-3l+\frac{9}{2}} = O(r^{k-\frac{17}{2}}).\end{aligned}$$

The same estimate holds for $Int(\Omega_4)$. For Ω^{out}, (5.12) still holds. Collecting the results of $Int(\Omega_i)$, $i = 1,2,3,4$ and $Int(\Omega^{out})$, we get the conclusion. □

Lemma 5.5. *We have*

$$\begin{aligned}
&\mathcal{J}^r_{11} = \mathcal{J}^r_{22} = \frac{32\pi}{3} + O(r^{-6}), \quad \mathcal{J}^r_{1,10} = -\frac{64\pi}{3} + O(r^{-6}),\\
&\mathcal{J}^r_{16} = -\mathcal{J}^r_{25} = O(r^{-\frac{9}{2}}), \quad \mathcal{J}^r_{29} = \frac{64\pi}{3} + O(r^{-6}),\\
&\mathcal{J}^r_{33} = \mathcal{J}^r_{44} = \frac{128}{3}\pi + \frac{4\pi}{3}r^{-4} + O(r^{-6}), \quad \mathcal{J}^r_{38} = -\mathcal{J}^r_{47} = -\frac{16\pi}{3}r^{-2} + O(r^{-\frac{9}{2}}),\\
&\mathcal{J}^r_{55} = \mathcal{J}^r_{66} = \frac{64\pi}{3} + O(r^{-6}), \quad \mathcal{J}^r_{59} = \mathcal{J}^r_{6,10} = O(r^{-\frac{9}{2}}),\\
&\mathcal{J}^r_{77} = \mathcal{J}^r_{88} = \frac{64\pi}{3} + \frac{8\pi}{3}r^{-4} + O(r^{-6}),\\
&\mathcal{J}^r_{99} = \mathcal{J}^r_{10,10} = \frac{128\pi}{3} + \frac{64\pi}{3}r^{-4} + O(r^{-6}).
\end{aligned}$$

Proof. Recall (5.2). Applying Corollary 5.3 with $k = 2, l = 4$ and $p(x,y,r) = 128(x^2+y^2)$, we have

$$\mathcal{J}^r_{11} = \int_{\mathbb{R}^2}\frac{128(x^2+y^2)|\psi_r|^2}{(1+|\psi_r|^2)^4}dxdy = \frac{32\pi}{3} + O(r^{-6}). \tag{5.21}$$

In the same way, one can compute $\mathcal{J}^r_{1,10}, \mathcal{J}^r_{22}, \mathcal{J}^r_{29}, \mathcal{J}^r_{33}, \mathcal{J}^r_{44}, \mathcal{J}^r_{55}, \mathcal{J}^r_{66}, \mathcal{J}^r_{77}, \mathcal{J}^r_{88}, \mathcal{J}^r_{99}, \mathcal{J}^r_{10,10}$.

Applying Lemma 5.4 with $k = 4, l = 4$ and $p(x, y, r) = 256(x^2 + y^2)(xy - r^2)$, we have

$$\mathcal{J}^r_{16} = \int_{\mathbb{R}^2} \frac{256(x^2 + y^2)(xy - r^2)}{(1 + |\psi_r|^2)^4} dxdy = O(r^{-\frac{9}{2}}). \tag{5.22}$$

In the same way, one can compute $\mathcal{J}^r_{25}, \mathcal{J}^r_{38}, \mathcal{J}^r_{47}, \mathcal{J}^r_{59}, \mathcal{J}^r_{6,10}$. □

Next, we give proofs of Proposition 4.3 and two key lemmas required in section 4.

Proof of Proposition 4.3. Combining Lemma 5.1 and Lemma 5.5, we know that $\mathcal{J}^r$ has the specific form. □

Proof of Lemma 4.7. It is easy to see that

$$\int_{\mathbb{R}^2} \nabla(f^r K^r_i) : \nabla K^r_j dxdy = \int_{\mathbb{R}^2} |\nabla \mathcal{S}(\Psi[\vec{\alpha}_r])|^2 f^r K^r_i \cdot K^r_j dxdy, \quad 1 \le i, j \le 10. \tag{5.23}$$

We shall adopt the notation

$$Int_{ij}(\Omega) = \int_{\Omega} |\nabla \mathcal{S}(\Psi[\vec{\alpha}_r])|^2 f^r(x, y) K^r_i \cdot K^r_j dxdy. \tag{5.24}$$

Recall the definition of p_j in (4.20). Then

$$p_j = 2\left(\sum_{k=1}^{4} Int_{2j}(\Omega_k) + Int_{2j}\left(\Omega^{out}\right)\right) - \left(\sum_{k=1}^{4} Int_{9j}(\Omega_k) + Int_{9j}\left(\Omega^{out}\right)\right)$$

where $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ and Ω^{out} defined in the proof of Lemma 5.2.

It follows from symmetry that $p_1 = -2p_{10}$, $p_3 = -p_4$, and $p_7 = p_8$. Indeed, using the expression of $\mathcal{I}^r_{ij}$, we have $\mathcal{K}^r \cdot K^r_1(x, y) = -2\mathcal{K}^r \cdot K^r_{10}(x, y)$, $\mathcal{K}^r \cdot K^r_3(x, y) = -\mathcal{K}^r \cdot K^r_4(y, x)$ and $\mathcal{K}^r \cdot K^r_7(x, y) = \mathcal{K}^r \cdot K^r_8(y, x)$. Since Θ^r is a radial function, then $f^r(x, y) = f^r(y, x)$. It follows from (5.2) that $p_1 = -2p_{10}$, $p_3 = -p_4$, and $p_7 = p_8$.

It is easy to see that $Int_{21}(\mathbb{R}^2) = 0$ since $\mathcal{I}^r_{21} = 0$. We have

$$\begin{aligned} Int_{91}\left(\mathbb{R}^2\right) &= \frac{1}{r^2} \int_{\mathbb{R}^2} \frac{128(x^4 - y^4)|\psi_r|^2 f^r(x, y)}{(1 + |\psi_r|^2)^4} dxdy \\ &= \frac{1}{r^2} \int_{\mathbb{R}^2} \left(\frac{128(x^4 - y^4)}{(1 + |\psi_r|^2)^3} - \frac{128(x^4 - y^4)}{(1 + |\psi_r|^2)^4}\right) f^r(x, y) dxdy. \end{aligned}$$

Now let $p(x, y, r) = 128(x^4 - y^4)$, it holds that $p(1, 1, 1) = p(-1, -1, 1) = 0$. Then we shall refine the proof of Lemma 5.4 to compute $Int_{91}(\mathbb{R}^2)$. Since $f^r = 1$ on Ω_1, similar to (5.20) with $k = 4, l = 3, 4$ and $p(x, y, r) = 128(x^4 - y^4)$, we have

$$Int_{91}(\Omega_1) = \frac{\pi[\Delta_{x,y} p(\mathbf{1}) - 2p_x(\mathbf{1}) - 2p_y(\mathbf{1})]}{128 \cdot 3 \cdot 2 \cdot 1} r^{-4} + O(r^{-8}) = O(r^{-8}). \tag{5.25}$$

Note that $f^r = -1$ on Ω_2, we also have

$$Int_{91}(\Omega_2) = -\frac{\pi[\Delta_{x,y} p(-\mathbf{1}) + 2p_x(-\mathbf{1}) + 2p_y(-\mathbf{1})]}{128 \cdot 3 \cdot 2 \cdot 1} r^{-4} + O(r^{-8}) = O(r^{-8}). \tag{5.26}$$

Note that $|f^r| \leq 1$, similar to (5.11) and (5.12), we have

$$|Int_{91}(\Omega_3)| + |Int_{91}(\Omega_4)| + \left|Int_{91}\left(\Omega^{out}\right)\right| = O(r^{-\frac{13}{2}}). \tag{5.27}$$

From (5.25), (5.26) and (5.27), we get $p_1 = -2p_{10} = O(r^{-\frac{13}{2}})$. Using the same method, we have

$$p_3 = -p_4 = O(r^{-\frac{9}{2}}), \quad p_5 = O(r^{-\frac{9}{2}}), \quad p_6 = O(r^{-\frac{9}{2}}). \tag{5.28}$$

We have

$$\begin{aligned} Int_{22}(\mathbb{R}^2) &= \int_{\mathbb{R}^2} \frac{128(x^2+y^2)|\psi_r|^2 f^r(x,y)}{(1+|\psi_r|^2)^4} dxdy \\ &= \int_{\mathbb{R}^2} \left(\frac{128(x^2+y^2)}{(1+|\psi_r|^2)^3} - \frac{128(x^2+y^2)}{(1+|\psi_r|^2)^4} \right) f^r(x,y) dxdy. \end{aligned}$$

We shall refine the proof of Lemma 5.4 to compute $Int_{22}(\mathbb{R}^2)$. As before, similar to (5.9) with $k = 2, l = 3, 4$ and $p(x, y, r) = 128(x^2 + y^2)$, we have

$$\begin{aligned} Int_{22}(\Omega_1) &= \frac{\pi p(\mathbf{1})}{8 \cdot 3 \cdot 2 \cdot 1} + \frac{\pi[\Delta_{x,y} p(\mathbf{1}) - 2p_x(\mathbf{1}) - 2p_y(\mathbf{1}) + 2p(\mathbf{1})]}{128 \cdot 3 \cdot 2 \cdot 1} r^{-4} + O(r^{-6}) \\ &= \frac{16\pi}{3} + O(r^{-6}). \end{aligned} \tag{5.29}$$

Note that $f^r = -1$ on Ω_2, similar to (5.10), we have

$$\begin{aligned} Int_{22}(\Omega_2) &= -\frac{\pi p(-\mathbf{1})}{8 \cdot 3 \cdot 2 \cdot 1} - \frac{\pi[\Delta_{x,y} p(-\mathbf{1}) + 2p_x(-\mathbf{1}) + 2p_y(-\mathbf{1}) + 2p(-\mathbf{1})]}{128 \cdot 3 \cdot 2 \cdot 1} r^{-4} + O(r^{-6}) \\ &= -\frac{16\pi}{3} + O(r^{-6}). \end{aligned} \tag{5.30}$$

Similar to (5.11) and (5.12), we have

$$|Int_{22}(\Omega_3)| + |Int_{22}(\Omega_4)| + \left|Int_{22}\left(\Omega^{out}\right)\right| = O(r^{-6}). \tag{5.31}$$

From (5.29),(5.30) and (5.31), we have $Int_{22}(\mathbb{R}^2) = O(r^{-6})$. Similarly, we can also get $Int_{92}(\mathbb{R}^2) = O(r^{-6})$. Then $p_2 = O(r^{-6})$. Using the same method, we have

$$p_7 = p_8 = -\frac{16\pi}{3} r^{-4} + O(r^{-6}), \quad p_9 = O(r^{-6}).$$

Next, we want to solve the linear system $\mathcal{J}^r \vec{c} = \vec{p}$. To that end, we shall use the expression of $\mathcal{J}^r$ in (4.8) after some row and column switching. It is reduced to solve each block independently. For instance, we solve $A_1(c_1, c_{10}, c_6)^T = (p_1, p_{10}, p_6)^T$. Using Cramer's rule and (5.28),

$$c_1 = \frac{1}{\det A_1} \begin{vmatrix} p_1 & \mathcal{J}^r_{1,10} & \mathcal{J}^r_{16} \\ p_{10} & \mathcal{J}^r_{10,10} & \mathcal{J}^r_{10,6} \\ p_6 & \mathcal{J}^r_{6,10} & \mathcal{J}^r_{6,6} \end{vmatrix} \approx r^4 \begin{vmatrix} O(r^{-\frac{13}{2}}) & 1 & O(r^{-\frac{9}{2}}) \\ O(r^{-\frac{13}{2}}) & 1 & O(r^{-\frac{9}{2}}) \\ O(r^{-\frac{9}{2}}) & O(r^{-\frac{9}{2}}) & 1 \end{vmatrix} = O(r^{-\frac{5}{2}}). \tag{5.32}$$

Using the same method, we have $c_{10} = O(r^{-\frac{5}{2}})$ and $c_6 = O(r^{-\frac{9}{2}})$. Moreover, one can verify that

$$c_1 - 2c_{10} = \frac{1}{\det A_1}\begin{vmatrix} p_1 & \mathcal{J}^r_{1,10} + 2\mathcal{J}^r_{11} & \mathcal{J}^r_{16} \\ p_{10} & \mathcal{J}^r_{10,10} + 2\mathcal{J}^r_{1,10} & \mathcal{J}^r_{10,6} \\ p_6 & \mathcal{J}^r_{6,10} + 2\mathcal{J}^r_{16} & \mathcal{J}^r_{6,6} \end{vmatrix} \approx r^4 \begin{vmatrix} O(r^{-\frac{13}{2}}) & O(r^{-6}) & O(r^{-\frac{9}{2}}) \\ O(r^{-\frac{13}{2}}) & r^{-4} & O(r^{-\frac{9}{2}}) \\ O(r^{-\frac{9}{2}}) & O(r^{-\frac{9}{2}}) & 1 \end{vmatrix} = O(r^{-\frac{13}{2}}). \tag{5.33}$$

To solve $A_2(c_2, c_9, c_5)^T = (p_2, p_9, p_5)^T$, we use Cramer's rule to obtain

$$c_2 = \frac{1}{\det A_2}\begin{vmatrix} p_2 & \mathcal{J}^r_{29} & \mathcal{J}^r_{25} \\ p_9 & \mathcal{J}^r_{99} & \mathcal{J}^r_{95} \\ p_5 & \mathcal{J}^r_{59} & \mathcal{J}^r_{55} \end{vmatrix} \approx r^4 \begin{vmatrix} O(r^{-6}) & 1 & O(r^{-\frac{9}{2}}) \\ O(r^{-6}) & 1 & O(r^{-\frac{9}{2}}) \\ O(-\frac{9}{2}) & O(r^{-\frac{9}{2}}) & 1 \end{vmatrix} = O(r^{-2}) \tag{5.34}$$

and $c_9 = O(r^{-2})$, $c_5 = O(r^{-\frac{9}{2}})$. Furthermore, similar to the approach of (5.33), we can derive $c_2 + 2c_9 = O(r^{-6})$.

To solve $A_3(c_3, c_8)^T = (p_3, p_8)^T$,

$$\begin{aligned} c_3 &= \frac{1}{\det A_3}\begin{vmatrix} p_3 & \mathcal{J}^r_{38} \\ p_8 & \mathcal{J}^r_{88} \end{vmatrix} \approx \begin{vmatrix} O(r^{-\frac{9}{2}}) & r^{-2} \\ r^{-4} & 1 \end{vmatrix} = O(r^{-\frac{9}{2}}), \\ c_8 &= \frac{1}{\det A_3}\begin{vmatrix} \mathcal{J}^r_{33} & p_3 \\ \mathcal{J}^r_{83} & p_8 \end{vmatrix} = -\frac{1}{4}r^{-4} + O(r^{-6}). \end{aligned} \tag{5.35}$$

Using $p_3 = -p_4$, $p_7 = p_8$ and $\mathcal{J}^r_{38} = -\mathcal{J}^r_{47}$, we obtain $c_4 = -c_3$ and $c_7 = c_8$. □

Proof of Lemma 4.6. Making the following expansion

$$\int_{\mathbb{R}^2} |\nabla(f^r\mathcal{K}^r)|^2 = \int_{\mathbb{R}^2} |\nabla f^r||\mathcal{K}^r|^2 + (f^r)^2|\nabla\mathcal{K}^r|^2 + 2f^r\partial_\alpha f^r\mathcal{K}^r_i\cdot\partial^\alpha\mathcal{K}^r_i. \tag{5.36}$$

Integration by parts

$$\int_{\mathbb{R}^2} 2f^r\partial_\alpha f^r\mathcal{K}^r_i\cdot\partial^\alpha\mathcal{K}^r_i = \int_{\mathbb{R}^2}\partial_\alpha(f^r)^2\mathcal{K}^r_i\cdot\partial^\alpha\mathcal{K}^r_i = -\int_{\mathbb{R}^2}(f^r)^2[\mathcal{K}^r\cdot\Delta\mathcal{K}^r + |\nabla\mathcal{K}^r|^2].$$

Since $\mathcal{K}^r \in \mathcal{L}[\mathcal{S}(\Psi[\vec{\alpha}_r])]$, we have

$$\int_{\mathbb{R}^2}(f^r)^2\mathcal{K}^r\cdot\Delta\mathcal{K}^r = -\int_{\mathbb{R}^2}|\nabla\mathcal{S}(\Psi[\vec{\alpha}_r])|^2|f^r\mathcal{K}^r|^2. \tag{5.37}$$

Inserting the above two equations back to (5.36), we get

$$\int_{\mathbb{R}^2}|\nabla(f^r\mathcal{K}^r)|^2 - |\nabla\mathcal{S}(\Psi[\vec{\alpha}_r])|^2|f^r\mathcal{K}^r|^2 = -\int_{\mathbb{R}^2}|\nabla f^r|^2|\mathcal{K}^r|^2. \tag{5.38}$$

Next we want to compute the right hand side. Recall that $\mathcal{K}^r = 2K^r_2 - K^r_9$. Then

$$\begin{aligned} |\mathcal{K}^r|^2 = 4|K^r_2|^2 - 4K^r_2\cdot K^r_9 + |K^r_9|^2 &= \frac{4|\psi_r|^2}{(1+|\psi_r|^2)^2}\left[4 - \frac{8xy}{r^2} + \frac{(x^2+y^2)^2}{r^4}\right] \\ &= \frac{4|\psi_r|^4}{r^4(1+|\psi_r|^2)^2}. \end{aligned} \tag{5.39}$$

Introduce the notation $A_1 = \{(x,y) : r \le (x-r)^2 + (y-r)^2 < r^2\}$ and $A_2 = \{(x,y) : r < (x+r)^2 + (y+r)^2 < r^2\}$. Recall that f^r is defined in (4.16). Then

$$|\nabla f^r|^2 = \frac{4}{|\log r|^2}\frac{1}{(x-r)^2+(y-r)^2}\chi_{A_1} + \frac{4}{|\log r|^2}\frac{1}{(x+r)^2+(y+r)^2}\chi_{A_2} \tag{5.40}$$

where χ_{A_1} is the characteristic function of set A_1. Then

$$\begin{aligned}\int_{A_1} |\nabla f^r|^2|\mathcal{K}^r|^2 &\le \frac{16}{|\log r|^2 r^4}\int_{A_1}\frac{1}{(x-r)^2+(y-r)^2}\frac{|\psi_r|^4}{(1+|\psi_r|^2)^2}\\ &= \frac{16}{|\log r|^2 r^4}\int_{A_1}\frac{[(x+r)^2+(y+r)^2]|\psi_r|^2}{(1+|\psi_r|^2)^2}\\ &\le \frac{16}{|\log r|^2 r^4}C|\log r| = \frac{C}{|\log r| r^4},\end{aligned} \tag{5.41}$$

for some uniform constant C. Similar estimate holds on A_2. Therefore

$$\int_{\mathbb{R}^2}|\nabla f^r|^2|\mathcal{K}^r|^2 = \int_{A_1}|\nabla f^r|^2|\mathcal{K}^r|^2 + \int_{A_2}|\nabla f^r|^2|\mathcal{K}^r|^2 = O(\frac{1}{|\log r| r^4}). \tag{5.42}$$

On the other hand, applying Corollary 5.3, one gets

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla\mathcal{S}(\Psi[\vec{\alpha}_r])|^2|f^r\mathcal{K}^r|^2 &= \int_{\Omega_1\cup\Omega_2}|\nabla\mathcal{S}(\Psi[\vec{\alpha}_r])|^2|\mathcal{K}^r|^2 + \int_{\Omega_3\cup\Omega_4}|\nabla\mathcal{S}(\Psi[\vec{\alpha}_r])|^2|f^r\mathcal{K}^r|^2\\ &= 4\mathcal{J}^r_{22} - 4\mathcal{J}^r_{29} + \mathcal{J}^r_{99} + O(r^{-6}) = \frac{64\pi}{3}r^{-4} + O(r^{-6}).\end{aligned}$$

□

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