ON SERRIN'S OVERDETERMINED PROBLEM AND A CONJECTURE OF BERESTYCKI, CAFFARELLI AND NIRENBERG

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Abstract. This paper concerns rigidity results to Serrin's overdetermined problem in an epigraph

$$\begin{cases} \Delta u + f(u) = 0, \quad \text{in } \Omega = \{(x', x_n) : x_n > \varphi(x')\}, \\ u > 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \\ |\nabla u| = const., \text{ on } \partial\Omega. \end{cases}$$

We prove that up to isometry the epigraph must be an half space and that the solution u must be one-dimensional, provided that one of the following assumptions are satisfied: either $n = 2$; or φ is globally Lipschitz, or $n \leq 8$ and $\frac{\partial u}{\partial x_n} > 0$ in Ω. In view of the counterexample constructed in [9] in dimensions $n \geq 9$ this result is optimal. This partially answers a conjecture of Berestycki, Caffarelli and Nirenberg [5].

1. Introduction

This paper is concerned with the one dimensional symmetry problem for the Serrin's overdetermined problem in an epigraph. More precisely we consider solutions to the following overdetermined problem:

$$\begin{cases} \Delta u + f(u) = 0, \quad \text{in } \Omega \\ u > 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \\ \dfrac{\partial u}{\partial \nu} = const., \text{ on } \partial\Omega \end{cases} \tag{1.1}$$

where f is a Lipschitz nonlinearity, ν is the outer normal at $\partial\Omega$, and $\frac{\partial u}{\partial \nu}$ is a constant which is not prescribed a priori.

A classical result of Serrin's [28] asserts that if Ω is a bounded and smooth domain for which there is a positive solution to the overdetermined equation (1.1) then Ω is a sphere and u is radially symmetric.

In the analysis of blown up version of free boundary problem, it is natural also to consider Serrin's overdetermined problem in unbounded

domains. (See Berestycki-Caffarelli-Nirenberg [4].) A natural class of unbounded domains to be considered are epigraphs, namely domains Ω of the form

$$\Omega = \{x \in \mathbb{R}^n \ / \ x_N > \varphi(x^{'})\} \tag{1.2}$$

where $x^{'} = (x_1, \ldots, x_{n-1})$ and $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a smooth function. In [5], Berestycki, Caffarelli and Nirenberg proved, under conditions on f that are satisfied for instance the Allen-Cahn nonlinearity below, the following result: If φ is uniformly Lipschitz and *asymptotically flat* at infinity, and Problem (1.1) is solvable, then φ must be a linear function, in other words Ω must be a *half-space*. This result was improved by Farina and Valdinoci [12], by lifting the asymptotic flatness condition and smoothness on f, under the dimension constraint $n \leq 3$ and other assumptions (see remarks below). When the epigraph is coercive (see (1.6) below) they can also consider an arbitrary nonlinearity.

In [5, pp.1110], the following conjecture on Serrin's overdetermined problem in unbounded domains was raised.

Berestycki-Caffarelli-Nirenberg Conjecture: *Assume that Ω is a smooth domain with Ω^c connected and that there is a bounded positive solution of (1.1) for some Lipschitz function f then Ω is either a half-space, or a cylinder $\Omega = B_k \times \mathbb{R}^{n-k}$, where B_k is a k-dimensional Euclidean ball, or the complement of a ball or a cylinder.*

In this paper we are mainly concerned with the BCN conjecture in the epigraph case (1.2), namely the following overdetermined problem

$$\begin{cases} \Delta u + f(u) = 0, \ u > 0 \ \text{ in } \Omega = \{x_n > \varphi(x^{'})\} \\ u = 0, \quad \text{on } \ \{x_n = \varphi(x^{'})\}, \\ \dfrac{\partial u}{\partial \nu} = const., \ \text{on } \quad \{x_n = \varphi(x^{'})\}. \end{cases} \tag{1.3}$$

In this case, the BCN conjecture states that if Serrin's problem (1.3) is solvable, then Ω must be an half-space. In a recent paper, del Pino, Pacard and the second author [9] constructed an epigraph, which is a perturbation of the Bombieri-De Giorgi-Giusti minimal graph, such that problem (1.3) admits a solution. This counterexample requires dimension $n \geq 9$. It remains open if the BCN Conjecture is true in low dimensions $n \leq 8$. In this paper we shall give an affirmative answer to this question.

Before we proceed, we introduce the assumptions on the nonlinearity. Let $W(u) = -\int_0^u f(s)ds$. We assume that W is a standard double well potential, that is, $W \in C^2([0, +\infty))$, satisfying

W1) $W \geq 0$, $W(1) = 0$ *and* $W > 0$ *in* $[0, 1)$*;*

W2) *for some* $\gamma \in (0,1)$, $W' < 0$ *on* $(\gamma, 1)$*;*
W3) *there exists a constant* $\kappa > 0$, $W'' \geq \kappa > 0$ *for all* $x \geq \gamma$*;*
W4) *there exists a constant* $p > 1$, $W'(u) \geq c(u-1)^p$ *for* $u > 1$.

Moreover, we also assume that W satisfies

W5) $W' < 0$ *in* $(0,1)$*, and either* $W'(0) \neq 0$ *or*

$$W'(0) = 0 \quad \textit{and} \quad W''(0) \neq 0.$$

A prototype example is $W(u) = (1-u^2)^2/4$ which gives the Allen-Cahn equation.

Under hypothesis (W1-4), there exists a unique function g satisfying

$$\begin{cases} g'' = W'(g), & \text{on } [0, +\infty), \\ g(0) = 0, & \lim\limits_{t \to +\infty} g(t) = 1. \end{cases} \tag{1.4}$$

Moreover, g has the following first integral:

$$g'(t) = \sqrt{2W(g(t))} > 0, \quad \text{on } [0, +\infty). \tag{1.5}$$

As $t \to +\infty$, $g(t)$ converges to 1 exponentially. Hence the following quantity is finite:

$$\sigma_0 := \int_0^{+\infty} \frac{1}{2}\big|g'(t)\big|^2 + W(g(t))dt < +\infty.$$

From now on we always assume that W satisfies (W1-5).

Our first result proves the conjecture in dimension 2 for any epigraph.

Theorem 1.1. *Let* $n = 2$ *and* W *satisfy (W1-5). If Serrin's overdetermined problem (1.3) has a solution then* $\Omega = \{x_n > \varphi(x')\}$ *must be a half space and up to isometry* $u(x) \equiv g(x \cdot e)$ *for some unit vector* e.

Our second result proves the conjecture in all dimensions for any Lipschitz or coercive graph.

Theorem 1.2. *Assume that* φ *is globally Lipschitz. If Serrin's overdetermined problem* (1.3) *has a solution then* $\Omega = \{x_n > \varphi(x')\}$ *must be a half space and up to isometry* $u(x) \equiv g(x \cdot e)$ *for some unit vector* e.

Theorem 1.3. *Assume that* φ *is coercive, i.e.*

$$\lim_{x' \to \infty} \varphi(x') = +\infty. \tag{1.6}$$

Then there is no solution to Serrin's overdetermined problem (1.3) *in* $\Omega = \{x_n > \varphi(x')\}$.

The last result proves the conjecture in dimensions $n \leq 8$, under an additional assumption.

Theorem 1.4. *Let u be a solution of* (1.3) *satisfying the following monotonicity assumption in one direction*

$$\frac{\partial u}{\partial x_n} > 0, \quad \text{in } \Omega. \tag{1.7}$$

If $n \leq 8$ and $0 \in \partial\Omega$, then $u(x) \equiv g(x \cdot e)$ for some unit vector e and Ω is an half space.

We compare the results of this paper with those in the existing literature. Theorem 1.1 was proved by Farina and Valdinoci [12] under the assumption that the epigraph is globally Lipschitz (and for more general f). They also proved Theorem 1.2 and Theorem 1.3 for more general f under the dimension restriction $n = 2$ or 3. (In the case of coercive epigraph it is assumed to be uniformly Lipschitz. See [Theorem 1.6-1.8, [12]].) When $n = 2$, Theorem 1.4 was also proved in [Theorem 1.2, [12]]. In view of the counterexample constructed by del Pino, Pacard and the second author [9], the dimension restriction in Theorem 1.4 is optimal. (We remark that the solutions constructed in [9] also satisfy (1.7).)

The extra condition (1.7) in Theorem 1.4 is a natural one. See [12, 13]. This condition is always satisfied if the epigraph is globally Lipschitz or coercive ([5]). It seems that the monotonicity condition (1.7) should follow from our other assumptions. However, this is not clear at present. It will be an interesting question to remove or prove this condition in general setting.

Theorems 1.1-1.4 have analogues in De Giorgi Conjecture for Allen-Cahn equation

$$\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^n \tag{1.8}$$

which asserts that the only solution which is monotone in one direction must be one-dimensional. Caffarelli-Cordoba [6] proved the one-dimensional symmetry result under the assumption that the level set is globally Lipschitz. (This corresponds to Theorem 1.2.) De Giorgi's conjecture has been proven to be true for $n = 2$ by Ghoussoub and Gui in [16], for $n = 3$ by Ambrosio and Cabre in [3] and for $4 \leq n \leq 8$ by Savin in [27], under the additional assumption that

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1.$$

This conjecture was proven to be false for $n \geq 9$ by del Pino, Kowalczyk and Wei in [10]. (Another proof of Savin's theorem is recently given by the first author [30]. A more general version of De Giorgi's conjecture was proved by Farina-Valdinoci [14].)

Now we explain the main ideas of our proof. The key observation is that under some conditions (i.e. the monotonicity condition (1.7)) we shall prove that solutions to Serrin's overdetermined problem (1.3) are minimizers of the functional

$$\int \frac{1}{2}|\nabla u|^2 + W(u)\chi_{\{u>0\}}. \tag{1.9}$$

(Here we only need W to satisfy hypothesis (W1-4).) The Euler-Lagrange equation corresponding to (1.9) ((1.10) below) is a one phase free boundary problem in which $|\nabla u| = \sqrt{2W(0)}$ on the boundary. To this end, we first establish

Theorem 1.5. *Let u be a solution of* (1.3)*, where W satisfies (W1-5). Then $|\nabla u| = \sqrt{2W(0)}$ on $\partial\Omega$.*

This is mainly because $\{u > 0\}$ is an epigraph, we can touch $\partial\{u > 0\}$ by arbitrarily large balls from both sides. Then we construct suitable comparisons in these balls to determine $|\nabla u|\lfloor_{\partial\Omega}$. (Similar idea has been used in [5].) Theorem 1.5 does not hold for other unbounded domains. See examples of Delaunay type domains in [9].

With hypothesis (W5) and the monotonicity condition (1.7), we further show that a solution to (1.3) is necessarily a minimizer of (1.9).

Hence the proof of Theorem 1.4 is reduced to the study of solutions to the following one phase free boundary problem:

$$\begin{cases} \Delta u = W'(u), \quad \text{in } \Omega = \{u > 0\}, \\ u > 0, \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \\ |\nabla u| = \sqrt{2W(0)}, \text{ on } \partial\Omega. \end{cases} \tag{1.10}$$

In the general case, a solution u to this equation is a stationary critical points of (1.9).

For this one phase free boundary problem, we have

Theorem 1.6. *Let u be a minimizer of* (1.9) *in $\mathbb{R}^n$ with $0 \in \partial\Omega$. If one of the blowing down limit of u is an half space, then $u(x) \equiv g(x \cdot e)$ for some unit vector e.*

This one phase free boundary problem bears many similarities with the Allen-Cahn equation. Hence previous methods used to prove De Giorgi conjecture for Allen-Cahn equations (cf. Savin [27]) can be employed to study the one dimensional symmetry of solutions to (1.10). In this paper, we shall follow the first author's approach in [30], which uses an energy type quantity, the *excess*. To this aim, we also present

the Huichinson-Tonegawa's theory for the convergence of general stationary critical points, see Section 3. (Similar idea has been used in recent preprints [15] and [30].)

Finally we discuss other related progress made at the BCN conjecture. The conjecture, in the case of cylindrical domains, was disproved by Sicbaldi in [29], where he provided a counterexample in the case when $n \geq 3$ and $f(t) = \lambda t$, $\lambda > 0$ by constructing a periodic perturbation of the cylinder $B^{n-1} \times R$ which supports a bounded solution to (1.3). In the two-dimensional case, Hauswirth, Hèlein and Pacard in [17] provided a counterexample in a strip-like domain for the case $f = 0$. Explicitly, Serrin's overdetermined problem is found to be solvable in the domain

$$\Omega = \{x \in \mathbb{R}^2 \ / \ |x_2| < \frac{\pi}{2} + \cosh(x_1)\},$$

where the solution found is unbounded. Necessary geometric and topological conditions on Ω for solvability in the two-dimensional case have been found by Ros and Sicbaldi in [26]. The overdetermined problem in Riemannian manifolds has been considered by Farina, Mari and Valdinoci in [13].

This paper is organized as follows. In Section 2 we collect some basic facts about the one phase free boundary problem, such as Modica inequality and monotonicity formula. In Section 3 we present the Huichinson-Tonegawa theory for the convergence of general stationary critical points. Section 4 is devoted to prove Theorem 1.6, following [30]. Most of these arguments are suitable adaption of previous ones and we only state the results without proof. Only for the integer multiplicity of the limit varifold in the Huichinson-Tonegawa theory (Theorem 3.13), a new proof is given, which follows the line introduced in Lin-Rivière [22] and we think simplifies the existing methods. Section 5 is devoted to proving that Theorem 1.4 can be reduced to Theorem 1.6.

2. One phase free boundary problem

From this section to Section 4, u always denotes a local minimizer of (1.9) in $\mathbb{R}^n$. We also assume that u is nontrivial and $0 \in \partial\{u > 0\}$.

We can show that $0 \leq u \leq 1$ (see Proposition 2.1 below) and it is Lipschitz continuous in $\mathbb{R}^n$ (see [2] and [7]). Hence $\{u > 0\}$ is an open set, which we denote by Ω. Furthermore, by the partial regularity for free boundaries in [2] and [7], $\partial\Omega$ is a smooth hypersurface except a set of Hausdorff dimension at most $n - 3$. The last condition in (1.10)

is understood in the weak sense, see [2]. At the smooth part of $\partial\Omega$, it also holds in the classical sense.

Proposition 2.1. *$u \leq 1$ on $\mathbb{R}^n$.*

Proof. Following an idea of Brezis, first by the Kato inequality we can show that

$$\Delta (u-1)_+ \geq W'(u)\chi_{\{u>1\}} \geq c\,(u-1)_+^p\,.$$

Then the claim follows from the Keller-Osserman theory. □

From this bound and the strong maximum principle, we can further show that $u < 1$ strictly in Ω.

Proposition 2.2 (Modica inequality)**.**

$$\frac{1}{2}|\nabla u|^2 \leq W(u), \quad \text{in } \Omega.$$

Proof. Assume

$$\sup_{\Omega}\left(\frac{1}{2}|\nabla u|^2 - W(u)\right) =: \delta > 0,$$

and $x_i \in \Omega$ approaches this sup bound.

If $\limsup \operatorname{dist}(x_i, \partial\Omega) > 0$, we can argue as in the proof of the usual Modica inequality (e.g. [23]) to get a contradiction.

If $\lim \operatorname{dist}(x_i, \partial\Omega) = 0$, then $u(x_i) \to 0$. Hence for all i large,

$$\frac{1}{2}|\nabla u(x_i)|^2 \geq W(0) + \frac{\delta}{2}.$$

Then we can proceed as in the proof of the gradient estimate for one phase free boundary problem (cf. [2, Corollary 6.5]) to get a contradiction. □

Remark 2.3. *The Modica inequality implies that $\partial\{u > 0\}$ is mean convex (see for example* [7]*).*

By considering domain variations, we can deduce the following stationary condition:

$$\int_\Omega \left(\frac{1}{2}|\nabla u|^2 + W(u)\chi_{\{u>0\}}\right) \operatorname{div} X - DX(\nabla u, \nabla u) = 0. \tag{2.1}$$

As usual, this implies the following Pohozaev identity:

$$\int_{B_R} \frac{n-2}{2}|\nabla u|^2 + nW(u)\chi_{\{u>0\}} = R\int_{\partial B_R} \frac{|\nabla u|^2}{2} - |u_r|^2 + W(u)\chi_{\{u>0\}}. \tag{2.2}$$

Together with the Modica inequality, this gives the following monotonicity formula.

Proposition 2.4 (Monotonicity formula)**.**

$$E(r;u,x) := r^{1-n}\int_{B_r(x)\cap\Omega}\frac{1}{2}|\nabla u|^2 + W(u)\chi_{\{u>0\}}$$

is non-decreasing in $r>0$.

Moreover,

$$\frac{d}{dr}E(r;u,x) = 2r^{1-n}\int_{\partial B_r(x)}\left|\nabla u(y)\cdot\frac{y-x}{|y-x|}\right|^2 + r^{-n}\int_{B_r(x)}\left[W(u)\chi_\Omega - \frac{|\nabla u|^2}{2}\right].$$

Corollary 2.5. *Let* u *be a non-trivial solution of* (1.10)*. Then there exists a constant* $c>0$ *such that, for any* $x\in\partial\Omega$ *and* $R>1$*,*

$$E(R;u,x)\geq cR^{n-1}.$$

Proof. Because $x\in\partial\Omega$, by the non-degeneracy of u near $\partial\Omega$ (see [2, Section 3]), there exists a universal constant c such that $|\Omega\cap B_1(x)|\geq c$. Then because $|\nabla u|\leq C$, $W(u)\geq c$ in $\Omega\cap B_h(x)$ for a universal constant h. This implies that $E(1;u,x)\geq c$ and the claim follows from the monotonicity formula. □

On the other hand, for minimizers we have the following upper bound.

Proposition 2.6. *There exists a universal constant* C *such that, for any* $x\in\mathbb{R}^n$ *and* $R>1$*,*

$$\int_{B_R(x)}\frac{1}{2}|\nabla u|^2 + W(u)\chi_{\{u>0\}}\leq CR^{n-1}.$$

Proof. In $B_R(x)$, construct a comparison function in the following form:

$$w(y) = \begin{cases} 1, & \text{in } B_{R-1}(x),\\ |y-x|-R+1+(R-|y-x|)\,u(y) & \text{in } B_R(x)\setminus B_{R-1}(x).\end{cases}$$

Note that $w>0$ in $B_R(x)$. A direct verification shows that

$$\int_{B_R(x)}\frac{1}{2}|\nabla w|^2 + W(w)\leq CR^{n-1}.$$

The energy bound on u follows from its minimality because $w=u$ on $\partial B_R(x)$. □

3. Huitchinson-Tonegawa theory

In this section we consider the convergence theory for general stationary critical points of the functional

$$\int\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\chi_{\{u_\varepsilon>0\}}. \tag{3.1}$$

Let u_ε be a sequence of stationary solutions in the unit ball B_1, to the singularly perturbed problem

$$\begin{cases} \varepsilon\Delta u_\varepsilon = \dfrac{1}{\varepsilon}W'(u_\varepsilon), \quad \text{in } \{u_\varepsilon > 0\}, \\ u_\varepsilon = 0, \quad \text{on } \partial\{u_\varepsilon > 0\}, \\ |\nabla u_\varepsilon| = \dfrac{1}{\varepsilon}\sqrt{2W(0)}, \text{ on } \partial\{u_\varepsilon > 0\}. \end{cases} \tag{3.2}$$

The stationary condition means that for any vector field $X \in C_0^\infty(B_1, \mathbb{R}^n)$,

$$\int_\Omega \left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\chi_{\{u_\varepsilon>0\}}\right) \mathrm{div}X - \varepsilon DX(\nabla u_\varepsilon, \nabla u_\varepsilon) = 0. \tag{3.3}$$

We also assume that the energy of u_ε is uniformly bounded, that is,

$$\limsup_{\varepsilon\to 0} \int_{B_1} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\chi_{\{u_\varepsilon>0\}} < +\infty. \tag{3.4}$$

Finally, to make the presentation simpler, we assume that $0 \le u_\varepsilon \le 1$ and it satisfies the Modica inequality

$$\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 \le \frac{1}{\varepsilon}W(u_\varepsilon), \quad \text{in } \{u_\varepsilon > 0\}. \tag{3.5}$$

See [19] for the general case, where two weaker conditions (but sufficient for the application below) are derived from (3.2)-(3.4).

Of course, what is used in this paper is the following sequences

$$u_\varepsilon(x) := u(\varepsilon^{-1}x), \quad \varepsilon \to 0,$$

where u is a local minimizer of (1.9) in $\mathbb{R}^n$. By results in the previous section, they satisfy all of the above assumptions.

We can assume that, up to a subsequence of $\varepsilon \to 0$,

$$\varepsilon|\nabla u_\varepsilon|^2 dx \rightharpoonup \mu_1,$$

$$\frac{1}{\varepsilon}W(u_\varepsilon)dx \rightharpoonup \mu_2,$$

weakly as Radon measures, on any compact set of B_1.

A caution on our notation: in the following, unless otherwise stated, $\varepsilon \to 0$ means only a sequence $\varepsilon_i \to 0$.

In the following $\mu = \mu_1/2 + \mu_2$ and $\Sigma = \mathrm{spt}\mu$.

We can also assume the matrix valued measures

$$\varepsilon\nabla u_\varepsilon \otimes \nabla u_\varepsilon dx \rightharpoonup [\tau_{\alpha\beta}]\mu_1,$$

where $[\tau_{\alpha\beta}]$, $1 \le \alpha, \beta \le n$, is measurable with respect to μ_1. Moreover, τ is nonnegative definite μ_1-almost everywhere and

$$\sum_{\alpha=1}^{n} \tau_{\alpha\alpha} = 1, \quad \mu_1 - a.e.$$

First, we need the following simple clearing out result, which is a direct consequence of Corollary 2.5.

Proposition 3.1. *There exists a universal constant η small so that the following holds. For any $r > 0$, if*

$$r^{1-n} \int_{B_r(x)} \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \le \eta,$$

then either $u_\varepsilon \equiv 0$ in $B_{r/2}(x)$ or $u_\varepsilon \ge 1 - \gamma$.

In the latter case of the previous lemma, we can improve the decay estimate to an exponential one.

Lemma 3.2. *If $u_\varepsilon \ge 1 - \gamma$ in $B_r(x)$, then*

$$u_\varepsilon \ge 1 - Ce^{-\frac{r}{C\varepsilon}} \quad \text{in } B_{r/2}(x).$$

Proof. By (W3),

$$\Delta (1 - u_\varepsilon) = -\frac{1}{\varepsilon^2} W'(u_\varepsilon) \ge \frac{c}{\varepsilon^2} (1 - u_\varepsilon).$$

From this the decay estimate can be deduced, e.g. by a comparison with an upper solution. □

Combining the monotonicity formula (Proposition 2.4) and Proposition 3.1, we get

Lemma 3.3. *For any $x \in \Sigma$,*

$$\frac{1}{C} r^{n-1} \le \mu(B_r(x)) \le C r^{n-1},$$

for some universal constant C.

Another consequence of Proposition 3.1 is:

Lemma 3.4. *On any connected compact set of $B_1 \setminus \Sigma$, either $u_\varepsilon \to 1$ uniformly or $u_\varepsilon \equiv 0$ for all ε small.*

This is because for every x not in Σ, there exists an $r > 0$ such that $\mu(B_r(x)) \le \eta r^{n-1}/2$. Hence for all ε small,

$$\int_{B_r(x)} \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + W(u_\varepsilon) \chi_{\{u_\varepsilon > 0\}} \le \eta r^{n-1},$$

and Proposition 3.1 applies.

Similar to [19], by the Modica inequality (Proposition 2.2) and the monotonicity formula (Proposition 2.4), we can show that

Lemma 3.5. *In* $L^1_{loc}(B_1)$,

$$\frac{1}{\varepsilon}W(u_\varepsilon)\chi_{\{u_\varepsilon>0\}} - \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 \to 0.$$

As a consequence, we have the following energy partition relation.

Corollary 3.6. $\mu_1/2 = \mu_2$.

By passing to the limit in the monotonicity formula, we obtain the corresponding monotonicity formula for the limit measure μ.

Lemma 3.7. *For any* $x \in B_1$,

$$r^{1-n}\mu(B_r(x))$$

is non-decreasing in $r > 0$. *Moreover, for any* $0 < r_1 < r_2 < +\infty$,

$$r_2^{1-n}\mu(B_{r_2}(x)) - r_1^{1-n}\mu(B_{r_1}(x)) = 2\int_{\Sigma\cap(B_{r_2}\setminus B_{r_1})} \frac{\sum_{\alpha,\beta=1}^n \tau_{\alpha\beta}(y)(y-x)_\alpha(y-x)_\beta}{|x-y|^{n+1}} d\mu.$$

By this lemma, we can define

$$\Theta(x) := \lim_{r\to 0}\frac{\mu(B_r(x))}{r^{n-1}}.$$

This is an upper semi-continuous function. By Lemma 3.3, $1/C \le \Theta(x) \le C$ everywhere on Σ.

Combining Proposition 3.1, Lemma 3.2 and Lemma 3.4, we have the following characterization of Σ.

Corollary 3.8. $x \in \Sigma \Longleftrightarrow \Theta(x) > 0 \Longleftrightarrow \Theta(x) \ge 1/C$.

By the Preiss theorem [25] (or by following the direct proof in [21]), we can show that

Lemma 3.9. Σ *is countably* $(n-1)$*-rectifiable.*

By differentiation of Radon measures, the measure μ has the following representation.

Corollary 3.10. $\mu = \Theta\mathcal{H}^{n-1}\lfloor_\Sigma$.

Next we show that

Lemma 3.11. $I - \tau = T_x\Sigma$, $\mathcal{H}^{n-1}$*-a.e. on* Σ.

This can be proved as in [19]. However, here we would like to give a new proof, which uses several ideas from [22].

As in [22], to prove this lemma, we only need to consider the special case where $\Sigma = \mathbb{R}^{n-1}$.

Notation: $\mathcal{C}_1 := B_1^{n-1} \times (-1, 1)$.

Proposition 3.12. *If $\Sigma = \mathbb{R}^{n-1}$, then*

$$\lim_{\varepsilon\to 0}\int_{\mathcal{C}_1}\varepsilon\sum_{\alpha=1}^{n-1}\left(\frac{\partial u_\varepsilon}{\partial x_\alpha}\right)^2 = 0.$$

This clearly implies Lemma 3.11 in this special case. This proposition can be proved as in [30, Lemma 4.6]. This proof is by choosing $X = \varphi\psi x_n e_n$ in the stationary condition (3.3), where $\varphi \in C_0^\infty(B_1^{n-1})$ and $\psi \in C_0^\infty((-1,1))$. For another proof using the monotonicity formula, see the derivation of [21, Eq. (2.11)].

With the help of Proposition 3.12, we can get the following quantization result for $\Theta(x)$.

Theorem 3.13. *$\Theta(x)/\sigma_0$ equals positive integer $\mathcal{H}^{n-1}$-a.e. on Σ.*

To prove this theorem, we need a lemma.

Lemma 3.14. *For any $\delta > 0$, there exists a $b \in (0,1)$ such that, for all ε small,*

$$\int_{\mathcal{C}_1\cap\{u_\varepsilon>1-b\}}\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) \le \delta.$$

The proof uses the strict convexity of W near 1, in particular,

$$\Delta\left[\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\right] \ge \frac{\kappa}{\varepsilon^2}\left[\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\right], \quad \text{in } \{u_\varepsilon > 1-b_1\}, \tag{3.6}$$

where $b_1 > 0$ is small. For more details, see [30, Corollary 6.4].

Proof of Theorem 3.13. We still need only to consider the special case where $\Sigma = \mathbb{R}^{n-1}$ and $\mu = \Theta\mathcal{H}^{n-1}\lfloor_{\mathbb{R}^{n-1}}$, with Θ a constant. We want to prove that Θ/σ_0 is a positive integer.

For $x' \in B_1^{n-1}$, let

$$f_\varepsilon(x') := \int_{-1}^{1}\frac{\varepsilon}{2}|\nabla u_\varepsilon(x',x_n)|^2 + \frac{1}{\varepsilon}W(u_\varepsilon(x',x_n))\chi_{\{u_\varepsilon>0\}}dx_n.$$

By (3.4), f_ε are uniformly bounded in $L^1(B_1^{n-1})$. By the convergence of $\varepsilon|\nabla u_\varepsilon|^2dx$ etc., f_ε converges to Θ weakly in $L^1(B_1^{n-1})$.

Fix a $\psi \in C_0^\infty((-1,1))$ such that $\psi \equiv 1$ in $(-1/2, 1/2)$. Let

$$\tilde{f}_\varepsilon(x') := \int_{-1}^{1}\left[\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\chi_{\{u_\varepsilon>0\}}\right]\psi(x_n)dx_n.$$

By Lemma 3.2 and Lemma 3.4,

$$\int_{B_1^{n-1}}\left|f_\varepsilon - \tilde{f}_\varepsilon\right| \le Ce^{-\frac{1}{C\varepsilon}}. \tag{3.7}$$

By substituting $X = \varphi\psi e_i$ with $\varphi \in C_0^\infty(B_1^{n-1})$, we see

$$\frac{\partial \tilde{f}_\varepsilon}{\partial x_i} = \sum_{j=1}^n \frac{\partial}{\partial x_j} A_{ij}^\varepsilon + g_\varepsilon^i, \quad \forall 1 \le i \le n-1,$$

where

$$A_{ij}^\varepsilon := \int_{-1}^1 \varepsilon \frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_j}\psi(x_n)dx_n,$$
$$g_\varepsilon^i = \int_{-1}^1 \varepsilon \frac{\partial u_\varepsilon}{\partial x_i}\frac{\partial u_\varepsilon}{\partial x_n}\psi'(x_n)dx_n.$$

By Proposition 3.12 and Cauchy inequality, for all $1 \le i \le n-1$ and $1 \le j \le n$, A_{ij}^ε and g_ε^i converges to 0 in $L^1_{loc}(B_1^{n-1})$. Then by Allard's strong constancy lemma [1], $\tilde{f}_\varepsilon$ converges to Θ in $L^1_{loc}(B_1^{n-1})$, which also holds for f_ε by (3.7).

By Lemma 3.12 and the weak L^1 estimate for Hardy-Littlewood maximal function, there exists a set E_ε^1 with $|B_{1/2}^{n-1} \setminus E_\varepsilon^1| < |B_{1/2}^{n-1}|/4$, such that,

$$\lim_{\varepsilon\to 0} \sup_{r\in(0,1/2)} r^{1-n} \int_{\mathcal{C}_r(x')} \varepsilon \sum_{\alpha=1}^{n-1} \left(\frac{\partial u_\varepsilon}{\partial x_\alpha}\right)^2 = 0, \quad \forall x' \in E_\varepsilon^1. \tag{3.8}$$

By Lemma 3.14, for any $\delta > 0$, there exists a $b \in (0,1)$ and a set E_ε^2 with $|B_{1/2}^{n-1} \setminus E_\varepsilon^2| < |B_{1/2}^{n-1}|/4$, such that

$$\limsup_{\varepsilon\to 0} \sup_{r\in(0,1/2)} r^{1-n} \int_{\mathcal{C}_r(x')\cap\{u_\varepsilon > 1-b\}} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + W(u_\varepsilon) \le C\delta, \quad \forall x' \in E_\varepsilon^2. \tag{3.9}$$

By applying the weak L^1 estimate for Hardy-Littlewood maximal function to $|f_\varepsilon - \Theta|$, we get a set E_ε^3 with $|B_{1/2}^{n-1} \setminus E_\varepsilon^3| < |B_{1/2}^{n-1}|/4$, such that

$$\lim_{\varepsilon\to 0} \sup_{r\in(0,1/2)} r^{1-n} \int_{B_r(x')} |f_\varepsilon(x') - \Theta| = 0, \quad \forall x'_\varepsilon \in E_\varepsilon^3. \tag{3.10}$$

Now choose $x'_\varepsilon \in E_\varepsilon^1 \cap E_\varepsilon^2 \cap E_\varepsilon^3$. For any $x_\varepsilon := (x'_\varepsilon, x_\varepsilon^n) \in \partial\{u_\varepsilon > 0\}$,

$$v^\varepsilon(x) := u_\varepsilon(x_\varepsilon + \varepsilon x)$$

converges to a limit v^∞ in $C_{loc}(\mathbb{R}^n) \cap H^1_{loc}(\mathbb{R}^n)$ (by the a priori estimates in [2]). By (3.8), v^∞ depends only on the x_n variable, hence equals the one dimensional profile g. Thus for all $\varepsilon > 0$ small, $v^\varepsilon(0, x_n) > 0$ in $(0, g^{-1}(b))$ and

$$\lim_{\varepsilon\to 0} \int_0^{g^{-1}(b)} \frac{1}{2}\left(\frac{\partial v^\varepsilon}{\partial x_n}\right)^2 + W(v^\varepsilon) = \int_0^{g^{-1}(b)} \frac{1}{2}(g')^2 + W(g) = \sigma_0 + o_b(1). \tag{3.11}$$

Assume that $\Pi^{-1}(x'_\varepsilon)\cap\partial\{u_\varepsilon>0\}$ consists N_ε points, t^i_ε, $1\le i\le N_\varepsilon$. By the analysis above, for all ε small, $u_\varepsilon>1-b$ or $u_\varepsilon=0$ outside

$$G_\varepsilon := B^{n-1}_\varepsilon(x'_\varepsilon)\times\cup_{1\le i\le N_\varepsilon}(t^i_\varepsilon-M\varepsilon,t^i_\varepsilon+M\varepsilon),$$

where M is a constant depending only on b.

Then

$$\begin{aligned}
&\varepsilon^{1-n}\int_{C_\varepsilon(x'_\varepsilon)} e_\varepsilon(u_\varepsilon)\\
=\ &\varepsilon^{1-n}\int_{G_\varepsilon} e_\varepsilon(u_\varepsilon)+\varepsilon^{1-n}\int_{C_\varepsilon(x'_\varepsilon)\setminus G_\varepsilon} e_\varepsilon(u_\varepsilon)\\
=\ &\sum_{i=1}^{N_\varepsilon}\varepsilon^{1-n}\int_{B^{n-1}_\varepsilon(x'_\varepsilon)}\int_{-M}^{M}\left(\frac{1}{2}|\nabla v^\varepsilon|^2+W(v^\varepsilon)\chi_{\{v^\varepsilon>0\}}\right)+O(\delta)\\
=\ &N_\varepsilon\left(\sigma_0+o_\varepsilon(1)+o_b(1)\right)+O(\delta).\quad \text{(by (3.11))}
\end{aligned}$$

On the other hand, by (3.10),

$$\lim_{\varepsilon\to0}\varepsilon^{1-n}\int_{C_\varepsilon(x'_\varepsilon)} e_\varepsilon(u_\varepsilon)=\Theta.$$

Hence

$$\lim_{\varepsilon\to0}N_\varepsilon=\frac{\Theta}{\sigma_0}+o_b(1)+O(\delta).$$

The last two terms can be made arbitrarily small. Then because N_ε is a positive integer, it must be constant for all ε small, which also equals Θ/σ_0. □

Define the varifold V by

$$<V,\varphi>:=\int_\Sigma\varphi(x,T_x\Sigma)\Theta(x)d\mathcal{H}^{n-1},$$

for $\varphi\in C^\infty_0(B_1\times\mathbb{RP}^n)$. (We view the space of hyperplanes of $\mathbb{R}^n$ as the projective space $\mathbb{RP}^n$.) By passing to the limit in the stationary condition for u_ε and noting Lemma 3.11, we obtain

Lemma 3.15. *V is stationary.*

Finally, we would like to compare this convergence theory with the Γ-convergence theory. Let

$$w_\varepsilon(x):=\Phi(u_\varepsilon(x))=\int_0^{u_\varepsilon(x)}\sqrt{2W(t)}dt.$$

Then

$$\int_{B_1}|\nabla w_\varepsilon| \ = \ \int_{B_1}\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|$$

$$\leq \int_{B_1} \frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\chi_{\{u_\varepsilon>0\}} \leq C.$$

Since $0 \leq w_\varepsilon \leq \int_0^1 \sqrt{2W(t)}dt$, it is uniformly bounded in $BV_{loc}(B_1)$. Then up to a subsequence w_ε converges in $L^1_{loc}(B_1)$ to a function $w_\infty \in BV_{loc}(B_1)$.

By extending Φ suitably to (-1,1), there exists a continuous inverse of it. Then $u_\varepsilon = \Phi^{-1}(w_\varepsilon)$ converges to $\Phi^{-1}(w_\infty)$ in $L^1_{loc}(\mathbb{R}^n)$. Since

$$\int_{B_1} W(u_\varepsilon)\chi_{\{u_\varepsilon>0\}} \leq C\varepsilon,$$

$u_\varepsilon \to 0$ or 1 a.e. in B_1. Hence there exists a measurable set Ω_∞ such that

$$u_\varepsilon \to \chi_{\Omega_\infty}, \quad \text{in } L^1_{loc}(\mathbb{R}^n).$$

Because $w_\infty = (\int_0^1 \sqrt{2W(t)}dt)\chi_{\Omega_\infty}$, $\chi_{\Omega_\infty} \in BV_{loc}(\mathbb{R}^n)$.

For minimizers, combining the above two approaches gives

Proposition 3.16. *If u_ε are minimizers, then $\Sigma = \partial\Omega_\infty$ and $\mu = \sigma_0\mathcal{H}^{n-1}\lfloor_{\partial\Omega_\infty}$. Moreover, Ω_∞ is a set with minimal perimeter.*

The first claim can be proved by the method of cut and paste, i.e. constructing suitable comparison functions. The second one follows from the standard Γ-convergence theory (see [24]).

4. Improvement of flatness

Now we return to the study of entire solutions of (1.10). Let u be a local minimizer of the functional (1.9) in $\mathbb{R}^n$. For $\varepsilon \to 0$, we can apply results in the previous section to study the convergence of the blowing down sequence

$$u_\varepsilon(x) = u(\varepsilon^{-1}x).$$

In this section we assume the blowing gown limit $\Omega_\infty = \{x_n > 0\}$ for a subsequence $\varepsilon_i \to 0$. (However, at this stage we do not know whether this limit depends on the subsequence of $\varepsilon \to 0$.) Note that this is always true if $n \leq 7$, by Bernstein theorem.

The following quantity will play an important role in our analysis.

Definition 4.1 (Excess). *Let P be an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ and e one of its unit normal vector, $B_r^{n-1}(x) \subset P$ an open ball and $\mathcal{C}_r(x) = B_r^{n-1}(x) \times (-1,1)$ the cylinder over $B_r(x)$. The excess of u_ε in $\mathcal{C}_r(x)$ with respect to P is*

$$E(r; x, u_\varepsilon, P) := r^{1-n} \int_{\mathcal{C}_r(x)} \left[1 - (\nu_\varepsilon \cdot e)^2\right] \varepsilon|\nabla u_\varepsilon|^2. \tag{4.1}$$

Here $\nu_\varepsilon = \nabla u_\varepsilon / |\nabla u_\varepsilon|$ when $|\nabla u_\varepsilon| \neq 0$, otherwise we take it to be an arbitrary unit vector.

In Proposition 3.12, we have shown that if the blowing down limit (of u_ε) is a hyperplane, then the excess with respect to this hyperplane converges to 0.

Our main tool to prove Theorem 1.6 is the following decay estimate. As in [30], we state this theorem for a general stationary critical point of (3.1), not necessarily a minimizer.

Theorem 4.2 (**Tilt-excess decay**). *For any given constant $b \in (0,1)$, there exist five universal constants $\delta_0, \tau_0, \varepsilon_0 > 0$, $\theta \in (0, 1/4)$ and K_0 large so that the following holds. Let u_ε be a solution of* (1.10) *with $\varepsilon \leq \varepsilon_0$ in B_4, satisfying the Modica inequality, $0 \in \partial\{u_\varepsilon > 0\}$, and*

$$4^{-n} \int_{\mathcal{B}_4} \frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \chi_{\{u_\varepsilon > 0\}} \leq (1 + \tau_0)\, \sigma_0 \omega_n. \tag{4.2}$$

Suppose the excess with respect to $\mathbb{R}^{n-1}$

$$\delta_\varepsilon^2 := E(2; 0, u_\varepsilon, \mathbb{R}^{n-1}) \leq \delta_0^2, \tag{4.3}$$

where $\delta_\varepsilon \geq K_0 \varepsilon$. Then there exists another plane P, such that

$$E(\theta; 0, u_\varepsilon, P) \leq \frac{\theta}{2} E(2; 0, u_\varepsilon, \mathbb{R}^n). \tag{4.4}$$

Moreover, there exists a universal constant C such that

$$\|e - e_{n+1}\| \leq C E(2; 0, u_\varepsilon, \mathbb{R}^n)^{1/2}, \tag{4.5}$$

where e is the unit normal vector of P pointing to the above.

The proof of this theorem is similar to the one in [30]. It is mainly divided into four steps:

Step 1. $\partial\{u_\varepsilon > 0\}$ and $\{u_\varepsilon = t\}$ (for $t \in (0, 1-b)$ with $b > 0$ fixed) can be represented by Lipschitz graphs over $\mathbb{R}^{n-1}$, $x_n = h_\varepsilon^t(x')$, except a bad set of small measure (controlled by $E(2; 0, u_\varepsilon, \mathbb{R}^{n-1})$). This can be achieved by the weak L^1 estimate for Hardy-Littlewood maximal functions, as in the proof of Theorem 3.13.

Step 2. By writing the excess using the (x', t) coordinates (t as in Step 1), $h_\varepsilon^t / \delta_\varepsilon$ are uniformly bounded in $W^{1,2}_{loc}(B_1^{n-1})$. Then we can assume that they converge weakly to a limit h_∞. Here we need the assumption $\delta_\varepsilon \gg \varepsilon$ to guarantee the limit is independent of t.

Step 3. By choosing $X = \varphi \psi e_n$ in the stationary condition (3.3), where $\varphi \in C_0^\infty(B_1^{n-1})$ and $\psi \in C_0^\infty((-1,1))$, and then passing to the limit, it is shown that h_∞ is harmonic in B_1^{n-1}.

Step 4. By choosing $X = \varphi\psi x_n e_n$ in the stationary condition (3.3) and then passing to the limit, it is shown that (roughly speaking) $h^t_\varepsilon/\delta_\varepsilon$ converges strongly in $W^{1,2}_{loc}(B^{n-1}_1)$. The tilt-excess decay estimate then follows from some basic estimates on harmonic functions.

As in [30], using this theorem we can prove the following estimate.

Lemma 4.3. *There exists a unit vector e_∞ and a universal constant $C(n)$ such that*

$$\int_{\mathcal{B}_R(x)} \left[1-(\nu\cdot e_\infty)^2\right]|\nabla u|^2 \le C(n)R^{n-2}, \quad \forall\, x\in\mathbb{R}^n,\ R>1. \tag{4.6}$$

Note that here the exponent $n-2<n-1$, which is the energy growth order of u (see Corollary 2.5 and Proposition 2.6). The blowing down analysis in Section 3 only gives

$$\int_{\mathcal{B}_R(x)} \left[1-(\nu\cdot e_{R,x})^2\right]|\nabla u|^2 = o(R^{n-1}),$$

where the unit vector $e_{R,x}$ may also depend on x and R. However, by iterating Theorem 4.2 on balls of the form $B_{\theta^{-i}}(x)$, we not only get the decay of the excess on these balls, but also get a control on the variation of $e_{x,\theta^{-i}}$ (through the estimate (4.5)).

Still as in [30], (4.6) implies the uniqueness of the blowing down limit constructed in the previous section.

Next consider the distance type function

$$\Psi(x) := g^{-1}\circ u.$$

It satisfies

$$-\Delta\Psi = f(\Psi)(1-|\nabla\Psi|^2), \quad \text{in } \{\Psi>0\}=\{u>0\},$$

where $f(t) := \frac{W'(g(t))}{\sqrt{2W(g(t))}}$.

By the vanishing viscosity method, as $\varepsilon\to 0$,

$$\Psi_\varepsilon(x) := \varepsilon\Psi(\varepsilon^{-1}x)$$

converges to a limit Ψ_∞ uniformly on any compact set of $\mathbb{R}^n$, and in C^1 on any compact set of $\{\Psi_\infty>0\}$. Moreover, in $\{\Psi_\infty>0\}$, Ψ_∞ is a viscosity solution to the eikonal equation

$$|\nabla\Phi_0|^2-1=0.$$

By definition, we can show that $\{\Psi_\infty>0\}=\Omega_\infty$. Using the estimate (4.6) we know Ψ_∞ depends only on the e_∞ direction. Hence after suitable rotation $\Psi_\infty = x_n^+$.

The C^1 convergence of Ψ_ε then implies that $\nabla\Psi$ is arbitrarily close to e_n, as far as u is close enough to 1, in a uniform manner. This then enables us to apply the sliding method to deduce that u depends only the x_n variable, hence finish the proof of Theorem 1.6.

5. Serrin's overdetermined problem

In this section we assume that u is a solution of (1.3) satisfying the monotonicity condition (1.7), where W is a double well potential satisfying the hypothesis *(W1-5)*.

We first need a technical lemma for the application below.

Lemma 5.1. *Let u be a C^2 solution of*

$$\Delta u = W'(u), \quad \text{in } \mathbb{R}^n,$$

satisfying $0 < u \leq 1$. Then $u \equiv 1$.

For a proof see [20, Section 4.1].

Lemma 5.2. *For any $x' \in \mathbb{R}^{n-1}$,*

$$\lim_{x_n \to +\infty} u(x', x_n) = 1,$$

and $u(x', -x_n) = 0$ for all $x_n > 0$ large.

Proof. By a contradiction argument, we can show that

$$\lim_{t \to \pm\infty} \operatorname{dist}((x', te_n), \partial\Omega) = +\infty.$$

Thus for any $R > 0$ and $t > 0$ large, $B_R(x', -te_n) \subset \Omega^c$. In particular, $u(x', -te_n) = 0$ for all t large.

By the same reasoning and standard elliptic estimates, as $t \to +\infty$,

$$u^t(\cdot) = u((x', te_n) + \cdot)$$

converges in $C^2_{loc}(\mathbb{R}^n)$ to a limit function u_∞, which is a positive solution of

$$\Delta u_\infty = W'(u_\infty)$$

in $\mathbb{R}^n$. Since $0 < u_\infty \leq 1$, by the previous lemma, $u_\infty \equiv 1$. □

Lemma 5.3. *On $\partial\Omega$, $|\nabla u| \geq \sqrt{2W(0)}$.*

Proof. By the same proof as in the previous lemma, for any $R > 0$ there exists a $t_0 > 0$ such that, for all $t \geq t_0$, the ball $B_R(0, t) \subset \Omega$.

Let v^R be the unique radial solution of

$$\begin{cases} \Delta v^R = W'(v^R), & \text{in } B_R, \\ v^R > 0, & \text{in } B_R, \\ v^R = 0, & \text{on } \partial B_R. \end{cases}$$

For any x and $R > 0$, denote $v_x^R := v^R(\cdot - x)$.

Since $\sup_{B_R} v^R < 1$, if t is large enough, $v_{te_n}^R < u$ in $B_R(te_n)$. Let

$$t^* := \inf\{t : B_R(te_n) \subset \Omega\}.$$

Then $B_R(t^*e_n)$ is tangent to $\partial\Omega$ at some point x_0.

By [5, Lemma 3.1], for all $t \geq t^*$, $u > v_{te_n}^R$ in $B_R(te_n)$. The Hopf lemma implies that

$$|\nabla u(x_0)| = \frac{\partial u}{\partial \nu}(x_0) \geq \frac{\partial v_{t^*e_n}^R}{\partial \nu}(x_0). \tag{5.1}$$

Here ν is the upward unit normal vector of $\partial\Omega$. Because $B_R(t^*e_n)$ is tangent to $\partial\Omega$ at x_0, we have

$$\frac{\partial v_{t^*e_n}^R}{\partial \nu}(x_0) = -\frac{\partial v^R}{\partial r}(Re_n) = |\nabla v(Re_n)|. \tag{5.2}$$

On the other hand, as $R \to +\infty$, $v_R(Re_n + \cdot)$ converges to a positive solution of

$$\begin{cases} \Delta v^\infty = W'(v^\infty), & \text{in } \mathbb{R}^n_+, \\ v^\infty > 0, & \text{in } \mathbb{R}^n_+, \\ v^\infty = 0, & \text{on } \partial\mathbb{R}^n_+. \end{cases}$$

Because v^R is radial, v^∞ depends only on the x_n variable. (In fact, to deduce this we do not need the radial symmetry of v^R, see [5] and references therein.) Hence it satisfies the ODE version of (1.3) and the conservation relation (1.5). In particular,

$$\sqrt{2W(0)} = |\nabla v^\infty(0,0)| = \lim_{R\to+\infty} |\nabla v_R(Re_n)|.$$

Combining this with (5.1) and (5.2) we finish the proof. □

Lemma 5.4. *On $\partial\Omega$, $|\nabla u| \leq \sqrt{2W(0)}$.*

Proof. As in the previous lemma, for any $R > 0$ we find a ball $B_R(0, -t^*e_n) \subset \Omega^c$ tangent to $\partial\Omega$ at a point x_0.

In $B_{2R}(0, -t^*e_n) \setminus B_R(0, -t^*e_n)$, by the Kato inequality,

$$\Delta u \geq W'(u).$$

Clearly the constant function 1 is a sup solution of this equation in $B_{2R}(0, -t^*e_n) \setminus B_R(0, -t^*e_n)$. Because $1 > u$, by the standard sup-sub solution method, there exists a solution $u^R > u$ in $B_{2R}(0, -t^*e_n) \setminus B_R(0, -t^*e_n)$ satisfying $u^R = 0$ on $\partial B_R(0, -t^*e_n)$ and $u^R = 1$ on $\partial B_{2R}(0, -t^*e_n)$. Then the Hopf lemma implies that

$$|\nabla u(x_0)| = \frac{\partial u}{\partial \nu}(x_0) \leq \frac{\partial u^R}{\partial \nu}(x_0) = |\nabla u^R(x_0)| \leq \sqrt{2W(0)} + o_R(1).$$

Here the last identity follows from the same argument as in the previous lemma. □

Combining Lemma (5.3) and Lemma (5.4) we obtain the proof of Theorem 1.5. Note that up to now we have not used the monotonicity condition (1.7). However, this condition is crucial for the following result.

Lemma 5.5. *u is a local minimizer of the functional* (1.9) *in* $\mathbb{R}^n$.

Proof. Assume by the contrary, there exists a ball $B_R(x_0)$ such that u is not a minimizer of the functional (1.9) in this ball (under the same boundary condition as u). Let v be such a minimizer.

For any $t \in \mathbb{R}$, consider

$$u^t(x) := u(x + te_n).$$

By Lemma 5.2, for all t large, $u^t > 0$ and $u^t > v$ in $B_R(x_0)$. Let

$$t_+ := \inf\{t : u^s \geq v \text{ on } \overline{B_R(x_0)},\ \forall s > t\}.$$

We claim that $t_+ = 0$.

Assume by the contrary, $t_+ > 0$. By definition and continuity, $u^{t_+} \geq v$ on $\overline{B_R(x_0)}$. Moreover, by the monotonicity of u, $u^{t_+} \neq v$ on $\partial B_R(x_0)$. Then by the strong maximum principle and Hopf lemma, $\{v > 0\} \cap B_R(x_0)$ is strictly contained in $\{u^{t_+} > 0\} \cap B_R(x_0)$ and $u^{t_+} > v$ strictly on $\overline{\{v > 0\} \cap B_R(x_0)}$.

By continuity, there exists an $\epsilon > 0$ such that, for all $t \in (t_+ - \epsilon, t_+]$, $u^t \geq v$ on $\overline{B_R(x_0)}$. This contradicts the definition of t_+. Hence we must have $t_+ = 0$, which implies that $u \geq v$ on $\overline{B_R(x_0)}$.

Because for all $t > 0$ large, $u^{-t} \equiv 0$ on $\overline{B_R(x_0)}$, we can also slide from below. This gives $u \leq v$ on $\overline{B_R(x_0)}$. Hence $u \equiv v$ is the unique minimizer of the energy functional (1.9). □

With this lemma in hand, we can perform the blowing down analysis as in the one phase free boundary problem. By the proof of [27, Theorem 2.4], we can show (using the notations in Section 3)

Lemma 5.6. *If $n \leq 8$, Ω_∞ is an half space.*

With this lemma in hand, we can use the method in the previous section to show that u is one dimensional, thus completing the proof of Theorem 1.4.

Finally we prove Theorem 1.2, 1.3 and 1.1.

Proof of Theorem 1.2 and Theorem 1.3. By [5], u satisfies the monotonicity condition (1.7) in Ω. As in the previous proof, $|\nabla u| = \sqrt{2W(0)}$

on $\partial\Omega$ and u is a local minimizer for the functional (1.9). Then we can perform the blowing down analysis as before.

If φ is globally Lipschitz, the blowing down limit Ω_∞ is still the epigraph associated to a Lipschitz function φ_∞ defined on $\mathbb{R}^{n-1}$. Since Ω_∞ has minimal perimeter, φ_∞ satisfies the minimal surface equation. By [18, Theorem 17.5], φ_∞ must be an affine function. In other words, Ω_∞ is an half space. Then we deduce that Ω is an half space and φ is an affine function. However, this contradicts the coerciveness of φ. $\square$

Proof of Theorem 1.1. In $\mathbb{R}^2$, by Remark 2.3, $\{u = 0\}$ is a convex set. Hence the function φ is concave.

First,

$$\begin{aligned} -\int_{B_R(0)\cap\{u>0\}} W'(u) &= -\int_{B_R(0)\cap\{u>0\}} \Delta u \\ &= -\int_{\partial B_R(0)\cap\{u>0\}} \frac{\partial u}{\partial r} + \int_{B_R(0)\cap\partial\{u>0\}} |\nabla u| \\ &\leq CR. \quad \text{(by the Lipschitz bound on } u) \end{aligned}$$

By hypothesis on W, $-W' \geq cW$ on $(\gamma, 1)$. Thus we obtain

$$\int_{B_R(0)\cap\{u>\gamma\}} W(u) \leq CR. \tag{5.3}$$

Next, as in the proof of [5, Theorem 1.2, (b)], there exists an $M > 0$ so that

$$\{u < \gamma\} \subset \{x : dist(x, \partial\{u > 0\}) < M\}. \tag{5.4}$$

By the convexity of $\partial\{u > 0\}$,

$$\left|\{x : dist(x, \partial\{u > 0\}) < M\} \cap B_R(0)\right| \leq CR.$$

Thus

$$\int_{B_R(0)\cap\{0<u<\gamma\}} W(u) \leq CR. \tag{5.5}$$

Combining (5.3), (5.5) and the Modica inequality we see

$$\int_{B_R(0)} \frac{1}{2}|\nabla u|^2 + W(u)\chi_{\{u>0\}} \leq CR.$$

With this bound in hand, we can perform the blowing down analysis using results in Section 3. In particular, we obtain a stationary integer 1-rectifiable varifold V from the sequence

$$u_\varepsilon(x) := u(\varepsilon^{-1}x).$$

V has the following form: there are finitely many positive integers n_i and unit vectors e_i with the corresponding rays $L_i := \{te_i : t > 0\}$, such that

$$V = \sum_i n_i [L_i], \tag{5.6}$$

where $[L_i]$ is the standard varifold associated to L_i.

Because V is stationary, we have the following balancing formula:

$$\sum_i n_i e_i = 0. \tag{5.7}$$

On the other hand, from the convexity of $\{u = 0\}$ it is clear that as $\varepsilon \to 0$, $\varepsilon\{u > 0\}$ converges to a limit Ω_∞ in the Hausdorff distance, with $\mathbb{R}^2 \setminus \Omega_\infty$ convex. Moreover, by assuming $0 \in \partial\{u = 0\}$, $\mathbb{R}^2 \setminus \Omega_\infty \subset \{u = 0\}$. Hence for all $\varepsilon > 0$, $u_\varepsilon = 0$ on $\mathbb{R}^2 \setminus \Omega_\infty$. By (5.4), $u_\varepsilon \to 1$ a.e. in Ω_∞. This then implies that the support of V lies in $\partial\Omega_\infty$. Combining (5.7) and the convexity of $\mathbb{R}^2 \setminus \Omega_\infty$, Ω_∞ must be an half plane.

What we have proved says that, the limit cone (at infinity) of the concave curve $\{x_2 = \varphi(x_1)\}$ is a line. By convexity, this implies that $\{x_2 = \varphi(x_1)\}$ itself is a line.

Finally, there are many ways to deduce that u is one dimensional, see for example [5] again. □

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