On a Semilinear Elliptic Equation in $R^2$ When the Exponent Approaches Infinity

XIAOFENG REN* AND JUNCHENG WEI

School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

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We consider a semilinear elliptic equation in $R^2$ with the nonlinear exponent approaching infinity. In contrast to the blow-up behavior of the corresponding problem in $R^n$ with $n \geq 3$, the $L^\infty(R^2)$ norms of the solutions to the equation in $R^2$ remain bounded from below and above. After a careful study on the decay rates of several quantities, we prove that the normalized solutions will approach the fundamental solution of $-\Delta + 1$ in $R^2$. So as the exponent tends to infinity, the solutions to the problem look more and more like a peak.

1. INTRODUCTION

This paper is devoted to the behavior of the ground state solution to a semilinear elliptic equation with large exponent of the nonlinear term. Considering

$$
\begin{align*}
\Delta u - u + u^p & = 0 \quad \text{in } R^2 \\
u > 0, \quad \lim_{|x| \to \infty} u(x) & = 0,
\end{align*}
$$

\text{(1.1)}

we would like to understand the behavior of the solutions to (1.1) when the exponent $p$ approaches $\infty$.

The corresponding problem in higher dimensions

$$
\begin{align*}
\Delta u - u + u^p & = 0 \quad \text{in } R^n, \text{ with } n \geq 3 \\
u > 0, \quad \lim_{|x| \to \infty} u(x) & = 0,
\end{align*}
$$

\text{(1.2)}

* Current address: Department of Mathematics, Brigham Young University, Provo, Utah 84602.
was studied by X. Pan and X. Wang in [9]. Due to the existence of the critical exponent, namely \((n + 2)/(n - 2)\) there, they were led to study the behavior of the solution to the corresponding equation in higher dimensions when \(p\) approaches \((n + 2)/(n - 2)\). It turns out that solutions to their problem will blow up as \(p \to (n + 2)/(n - 2)\).

Because of the critical exponent for \(n \geq 3\), in higher dimensions we have a profile equation

\[
\begin{aligned}
\Delta u - u + u^{(n+2)/(n-2)} &= 0 & \text{in } \mathbb{R}^n \\
\mu > 0, \quad \lim_{|x| \to \infty} u(x) &= 0.
\end{aligned}
\]  

(1.3)

In fact the nonexistence of solutions to (1.3) implies the blow-up of solutions to (1.2) when \(p \to (n + 2)/(n - 2)\).

In \(\mathbb{R}^2\), however, due to the lack of Eq. (1.3), we will approach the problem in a different way. We will start with a sharp estimate on the growth rate of \(c_p\) where \(c_p\) is the best constant of the embedding

\[W^{1,2}(\mathbb{R}^2) \hookrightarrow L^{p^*}(\mathbb{R}^2).\]

From this estimate and estimates of some other quantities, we will prove

**Theorem 1.1.** Let \(u_p\) be a solution to (1.1). Then for \(p\) large enough, there exists \(C\) independent of \(p\) such that \(1 < \|u_p\|_{L^p(\mathbb{R}^2)} < C\).

**Theorem 1.2.** Let \(u_p\) be a radially symmetric solution to (1.1). Then, as \(p \to \infty\), we have

1. \[\frac{u_p}{\int_{\mathbb{R}^2} u_p} \to G \quad \text{in the sense of distribution},\]
2. \[\frac{u_p}{\int_{\mathbb{R}^2} u_p} \to G \quad \text{in } C^{\alpha}_{loc}(\mathbb{R}^2 \setminus \{0\}) \text{ for all } \alpha \in (0, 1),\]

where \(G\) is the fundamental solution to \(-\Delta + 1\), i.e.,

\[-\Delta G + G = \delta.\]

**Remark 1.3.** \(G\) has an integral representation

\[G(x) = \frac{1}{\Gamma(1/2)} \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}} \int_0^x e^{-t} t^{-1/2} \left(1 + \frac{1}{2} \frac{t}{x}\right)^{-1/2} dt.\]

Moreover, \(G\) decays exponentially at infinity. We refer to [7] for more information.
We see, in contrast to the higher dimensions, the solutions look more and more like a peak as $p$ gets large.

Our paper is organized as follows. In Section 2, we give necessary background of the solutions to (1.1) together with some integral identities. Then we prove a growth rate estimate of $c_p$ in Section 3. Finally we prove Theorem 1.1 in Section 4 and Theorem 1.2 in Section 5.

2. Preliminaries

There are various ways to obtain solutions to (1.1). We shall adapt the variational approach developed in [2].

Let

$$\mathcal{A}_p = \left\{ u \in W^{1,2}(\mathbb{R}^2); \int_{\mathbb{R}^2} u^{p+1} = 1 \right\}.$$

Consider the functional $J_p: \mathcal{A}_p \to \mathbb{R}$ defined by

$$J_p(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2).$$

By the theory of Schwartz symmetrization (see, for example, [6]), we can replace any minimizing sequence of the functional by a radially symmetric minimizing sequence. Then it is easy to show, with the aid of a compactness result of radial embedding in [10], that $J_p$ has a radial minimizer in $\mathcal{A}_p$. If we denote this minimizer by $u'_p$, a scalar multiple of $u'_p$ will solve Eq. (1.1). We denote this radial solution by $u_p$. A further $L^\infty$ estimate shows $\lim_{|x| \to \infty} u(x) = 0$. Hence $u_p$ solves (1.1).

According to a result in [3] all solutions of (1.1) up to a parallel translation are radially symmetric. Applying the uniqueness result in [8] or [5], we know that the radial solution is unique. Therefore (1.1) has a unique solution up to a translation. From now on we will denote this unique radial solution by $u_p$.

It is easy to see that $u_p$ is related to the embedding

$$W^{1,2}(\mathbb{R}^2) \hookrightarrow L^{p+1}(\mathbb{R}^2).$$

We refer to [1] for the proof of this embedding theorem.

If we denote $c_p$ as the best constant of the above embedding, i.e., $c_p$ is the least number among all possible constant $C$'s which make the inequality

$$W^{1,2}(\mathbb{R}^2) \hookrightarrow L^{p+1}(\mathbb{R}^2).$$
\[ \|u\|_{L^{\infty}(\mathbb{R}^3)} \leq C\|u\|_{W^{1,2}(\mathbb{R}^3)} \]
for all \( u \in W^{1,2}(\mathbb{R}^3) \), then \( c_p \) is achieved by \( u_p \). Hence we have

**Lemma 2.1.** Let \( u_p \) be the radial solution to (1.1) as above. Then we have

\[ \frac{\|u_p\|_{W^{1,2}(\mathbb{R}^3)}}{\|u_p\|_{L^{\infty}(\mathbb{R}^3)}} = \inf \left\{ \frac{\|u\|_{W^{1,2}(\mathbb{R}^3)}}{\|u\|_{L^{\infty}(\mathbb{R}^3)}} : u \in W^{1,2}(\mathbb{R}^3), u \neq 0 \right\} = \frac{1}{c_p}, \]

where \( c_p \) is the best constant defined above.

Now let us state some integral identities.

**Lemma 2.2.** Let \( u_p \) be a solution to (1.1). We have

1. \( \int_{\mathbb{R}^3} (|\nabla u_p|^2 + |u_p|^2) = \int_{\mathbb{R}^3} u_p^{p+1} \)
2. \( \int_{\mathbb{R}^3} u_p = \int_{\mathbb{R}^3} u_p^p \)
3. \( \frac{1}{2} \int_{\mathbb{R}^3} u_p^2 = (1/(p + 1)) \int_{\mathbb{R}^3} u_p^{p+1} \).

**Proof.** (1) Multiplying (1.1) by \( u_p \) and integrating over \( B(R) \), the ball of radius \( R \) centered at the origin, we get

\[ \int_{B(R)} (|\nabla u_p|^2 + |u_p|^2) = \int_{\partial B_R} u_p \frac{\partial u_p}{\partial n} + \int_{B_R} u_p^{p+1}. \]

Using an exponential decay property of \( u_p \) which says that there exist \( C \), \( \mu \) independent of \( x \) such that

\[ u_p(x) \leq Ce^{-\mu|x|}, \quad |Du_p(x)| \leq Ce^{-\mu|x|} \]

we can let \( R \) approach \( 0 \) yielding 1. The proof of the exponential decay estimate can be found, for example, in [2]. We shall prove a refined result, Lemma 5.1, by the same method in Section 5.

(2) Similar to (1), but we integrate the equation directly.

(3) This is the \( \mathbb{R}^2 \) version of the well known Pohozaev identity. We refer to [2] for a proof.

We also present some simple radial lemmas which will be used in later sections.

**Lemma 2.3.** Let \( n \geq 2 \). Every radial function \( u \in W^{1,2}(\mathbb{R}^n) \) is almost equal to a function \( U(x) \) which is continuous for \( x \neq 0 \); furthermore
\[ |U(x)| \leq C_n |x|^{(1-n)/2} \|u\|_{W^{1,2}(\mathbb{R}^n)} \]

for \( |x| \geq \alpha_n \) where \( C_n, \alpha_n \) depend on \( n \) only.

The proof of this lemma can be found in [2].

**Lemma 2.4.** Let \( u \in L^2(\mathbb{R}^2) \) be a radial, nonnegative, non-increasing in \( |x| > 0 \), continuous function except at 0. Then we have

\[ u(|x|) \leq \frac{2}{\sqrt{3\pi}} |x|^{-1} \|u\|_{L^2(\mathbb{R}^2)}. \]

**Proof.**

\[ \|u\|^2_{L^2(\mathbb{R}^2)} = 2\pi \int_0^\infty u^2(r) r \, dr \geq 2\pi \int_{|x|/2}^{|x|} u^2(s)s^{-1} \, ds \]

\[ \geq 2\pi \frac{1}{2} u^2(|x|) \left( |x|^2 - \frac{|x|^2}{4} \right) = \frac{3\pi}{4} |x|^2 u^2(|x|). \]

Therefore we have the lemma.

**3. On the Growth Rate of \( c_p \)**

In this section, we establish a sharp estimate for the growth rate of \( c_p \). Notice that according to the Schwartz symmetrization theory, the \( c_p \) is actually obtained in the class of radial functions. We will use \( C \) to denote various constants independent of \( p \).

**Lemma 3.1.** If \( u \in W^{1,2}_0(\Omega) \) where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^2 \), then for every \( t \geq 2 \)

\[ \|u\|_{L^t(\Omega)} \leq C t^{1/2} \|\nabla u\|_{L^2(\Omega)}. \]

**Proof.** Let \( u \in W^{1,2}_0(\Omega) \). We know

\[ \frac{1}{\Gamma(s + 1)} x^s \leq e^x \]

for all \( x \geq 0, s \geq 0 \) where \( \Gamma \) is the Gamma function. From Trudinger's Inequality (see [4, p. 160]), we have

\[ \int_\Omega \exp \left[ C_1 \left( \frac{u}{\|\nabla u\|_{L^2}} \right)^2 \right] \, dx \leq C_2 |\Omega|, \]
where $C_1$, $C_2$ are constants which depend on the dimension of $\Omega$ only and $\| \|$ denotes the Lebesgue measure. Therefore

$$\frac{1}{\Gamma(t/2 + 1)} \int_{\Omega} u'^t \, dx$$

$$= \frac{1}{\Gamma(t/2 + 1)} \int_{\Omega} \left[ C_1 \left( \frac{u}{\| \nabla u \|_{L^2}} \right)^{\frac{t}{2}} \right] \, dx \leq C_1 t^{\frac{t}{2}} \| \nabla u \|_{L^2}$$

$$\leq \int_{\Omega} \exp \left[ C_1 \left( \frac{u}{\| \nabla u \|_{L^2}} \right)^{\frac{t}{2}} \right] \, dx \leq C_2 |\Omega| t^{\frac{t}{2}} \| \nabla u \|_{L^2}.$$

Hence

$$\left( \int_{\Omega} u'^t \, dx \right)^{\frac{1}{t}} \leq \left( \Gamma \left( \frac{t}{2} + 1 \right) \right)^{\frac{1}{t}} C_1^{\frac{1}{2}} t^{\frac{1}{2}} \| \nabla u \|_{L^2(\Omega)}.$$

Notice that, according to Stirling's formula,

$$\left( \Gamma \left( \frac{t}{2} + 1 \right) \right)^{\frac{1}{t}} \sim \left( \left( \frac{t}{e} \right)^{\frac{t}{2}} \sqrt{2\pi t} \right)^{\frac{1}{t}} \sim Ct^{\frac{1}{2}},$$

where $0 < \theta_t < 1/12$. This completes our lemma.

**Lemma 3.2.** For each $u \in W^{1,2}(R^3)$, we have

$$\| u \|_{L^t(R^3)} \leq C t^{1/2} \| u \|_{W^{1,2}(R^3)}$$

for every $t \geq 4$.

**Proof.** As we mentioned earlier, by the Schwartz symmetrization theory, we need only to prove the inequality for radial functions. So in the proof all functions are assumed to be radial. Let $u := u_1 + u_2 := u \chi_{[r < \alpha_2]} + u \chi_{[r \geq \alpha_2]}$ where $\alpha_2$ is defined in Lemma 2.3.

Then

$$\| u \|_{L^t(R^3)} \leq \| u_1 \|_{L^t} + \| u_2 \|_{L^t}.$$

And by Lemma 2.3

$$\| u_2 \|_{L^t} \leq \left( \int_{R^3 \setminus B_{\alpha_2}} u'^t \right)^{\frac{1}{t}} \leq \left( \frac{2\pi}{\alpha_2^{t-2}} \right)^{\frac{1}{t}} \left( \frac{2}{t-1} \right)^{\frac{1}{t}} \| u \|_{W^{1,2}(R^3)}$$

$$\leq C \| u \|_{W^{1,2}(R^3)} \left( \frac{2}{t-1} \right)^{\frac{1}{t}} \left( \alpha_2^{t-2} \right)^{\frac{1}{t}} \leq C \| u \|_{W^{1,2}(R^3)}.$$
Hence combining this with Lemma 3.1, we obtain
\[ \|u\|_{L^1} \leq C t^{1/2} \|u\|_{W^{1,2}(R^2)} + C \|u_2\|_{W^{1,2}(R^2)} \leq C t^{1/2} \|u\|_{W^{1,2}(R^2)}. \]

**Lemma 3.3.** For \( p \) large enough, there exist \( C_1, C_2 \) independent of \( p \) such that
\[ C_1 p^{1/2} \leq c_p \leq C_2 p^{1/2}. \]

**Proof.** The second inequality follows from Lemma 3.2. To prove the first inequality, we consider the so-called Moser's function
\[
\phi(r) = \begin{cases} 
(p + 1)^{-1/2} \log \frac{1}{r}, & 0 \leq r \leq e^{-(p+1)} \\
(p + 1)^{-1/2} \log e^{p+1}, & e^{-(p+1)} \leq r \leq 1 \\
(p + 1)^{1/2}, & 0 \leq r \leq e^{-(p+1)}.
\end{cases}
\]

So \( \phi \in W^{1,2}_0(B_1) \subset W^{1,2}(R^2) \) where the trivial extension is used to interpret the second inclusion.

On the one hand
\[ \|\nabla \phi\|_{L^2}^2 = 2\pi \int_0^1 \phi^2 r \, dr = 2\pi \int_{e^{-(p+1)}}^1 (p + 1)^{-1} \frac{1}{r} \, dr = 2\pi. \]

Therefore by Poincaré's Inequality on \( B_1 \),
\[ \|\phi\|_{W^{1,2}(R^2)} = \|\phi\|_{W^{1,2}(B_1)} \leq C \|\nabla \phi\|_{L^2(B_1)} = C, \quad (3.1) \]
where \( C \) is a constant depending on the first eigenvalue of \( \Delta \) on the unit disk with Dirichlet boundary condition only.

On the other hand
\[
\int_{B_1} \phi^{p+1} = 2\pi \int_0^1 \phi^{p+1} r \, dr \geq 2\pi \int_{e^{-(p+1)/2}}^1 (p + 1)^{-(p+1)/2} \left( \log \frac{1}{r} \right)^{p+1} \frac{1}{r} \, dr \\
= 2\pi (p + 1)^{-(p+1)/2} \int_0^{p+1/2} t^{p+1} e^{-t} \, dt \\
\geq 2\pi (p + 1)^{-(p+1)/2} \int_{(p+1)/2}^{p+1/2} t^{p+1} e^{-t} \, dt \\
\geq 2\pi (p + 1)^{-(p+1)/2} \left( \frac{p + 1}{2} \right)^{p+1} e^{-(p+1)} \frac{p + 1}{2} \\
\geq 2\pi 2^{-(p+2)} (p + 1)^{(p+3)/2} e^{-(p+1)}.
\]
Hence
\[ \|\phi\|_{L^{p+1}} \geq (2\pi)^{1(p+1)/2} e^{-1(p+1)(p+1)/2} \geq C(p + 1)^{1/2}. \]  
(3.2)

Combining (3.1) and (3.2), we have the lemma. □

4. ON THE $L^\infty(R^2)$ NORM OF $u_p$

We are now in the position to study the $L^\infty(R^2)$ norms of $u_p$. Because our solutions $u_p$ are radially symmetric, they satisfy the following ordinary differential equation
\[ \begin{cases} u'' + \frac{1}{r} u' - u + u^p &= 0 \\ u'(0) &= 0, \quad \lim_{r \to \infty} u(r) = 0. \end{cases} \]  
(4.1)

From [8] or [5], we know $u_p'(r) < 0$ for all $r > 0$ where $u_p$ is a solution to (1.1) with the exponent in the equation being $p$. We define
\[ \gamma_p := u_p(0) = \|u_p\|_{L^\infty}. \]

**Lemma 4.1.** $\|u_p\|_{W^{1,2}(R^2)} \leq C/p^{1/2}$ when $p$ is large.

**Proof.** From the integral identity (1) in Lemma 2.2, we deduce
\[ \|u_p\|_{W^{1,2}(R^2)}^2 = \|u_p\|_{L^{p+1}(R^2)}^{p+1}. \]
Combining this with Lemma 2.1 and Lemma 3.2, we get
\[ \|u_p\|_{W^{1,2}(R^2)} = C(p+1)(p-1)p^{-1/2} \leq C p^{-1/2} \]
when $p$ is large. □

We now prove Theorem 1.1.

**Proof of Theorem 1.1.** The uniform lower bound 1 follows from the maximum principle. To find a uniform upper bound let
\[ E_p(r) = -ru_p'(r) - \frac{1}{2} r^2(u_p^p - u). \]
Then $E_p(0) = 0$, and
\[
\begin{align*}
E'_p(r) &= -u'_p - ru''_p - r(u''_p - u_p) - \frac{1}{2} r^2 (pu''_p - 1)u'_p \\
&= - \frac{1}{2} r^2 (pu''_p - 1)u'_p.
\end{align*}
\]
Let $r_0 > 0$ be such that $u_p(r_0) = (1/p)^{1/(p-1)}$. Clearly such $r_0$ exists, and it depends on $p$. Notice $E'_p(r) \geq 0$ for $r \in (0, r_0)$. Therefore $E_p \geq 0$ on $(0, r_0)$. From the definition of $E$, we have
\[
-u'_p(r) - \frac{1}{2} r(u''_p - u_p) \geq 0
\]
on $(0, r_0)$. Combining this with Eq. (4.1) of $u_p$, we get
\[
u''_p \geq \frac{1}{2} (u_p - u''_p), \quad u''_p \geq \frac{1}{2} (\gamma_p - \gamma''_p) \tag{4.2}
\]
for all $r$ in $(0, r_0)$.

Integrating both sides of (4.2) twice with respect to $r$ on $(0, r)$, we obtain
\[
\gamma_p - \frac{1}{2} u_p(r) \leq \frac{1}{4} r^2 (\gamma''_p - \gamma_p) \tag{4.3}
\]
for $r \in (0, r_0)$.

Now let $T_0$ be such that $u_p(T_0) = \frac{1}{2} \gamma_p$. $T_0$ depends on $p$. From (4.3), we have
\[
\gamma_p - \frac{1}{2} \gamma_p \leq \frac{1}{4} T_0^2 (\gamma''_p - \gamma_p), \quad \gamma_p \leq \frac{1}{2} T_0^2 \gamma''_p. \tag{4.4}
\]

Applying Lemma 3.2 with $t = 2p$ and Lemma 4.1, we get
\[
\|u_p\|_{L_t^{t}(\mathbb{R}^1)} \leq C(2p)^{1/2}\|u_p\|_{W^{1,2}(\mathbb{R}^2)} \leq C(2p)^{1/2}Cp^{-1/2} := M,
\]
where $M$ is independent of $p$ for large $p$. Hence
\[
\int_{\mathbb{R}^2} u_p^{2p} \, dx \leq M^{2p}.
\]
\[ \int_{\mathbb{R}^2} u^{2p} \, dx = 2\pi \int_0^\infty u^{2p}(r) r \, dr \geq \int_0^{T_0} u^{2p}_\rho r \, dr \geq \left( \frac{\gamma_p}{2} \right)^{2p} \frac{1}{2} T_0^2. \]

Therefore
\[ \frac{1}{2} T_0^2 \left( \frac{\gamma_p}{2} \right)^{2p} \leq M^{2p}, \quad T_0^2 \leq 2 \left( \frac{2M}{\gamma_p} \right)^{2p}. \tag{4.5} \]

Combining (4.4) with (4.5), we obtain
\[ l \leq \left( \frac{2M}{\gamma_p} \right)^{2p} (\gamma_p^{p-1}) \leq \left( \frac{2M}{\gamma_p} \right)^{2p} \gamma_p^p, \quad \gamma_p \leq (2M)^2. \]

We now derive the decay rates for some quantities.

**Corollary 4.2.** There exist \( C_1, C_2 \) independent of \( p \) such that for large \( p \) we have

1. \( C_1 \frac{1}{p} \leq \int_{\mathbb{R}^2} |\nabla u_\rho|^2 \, dx \leq C_2 \frac{1}{p} \)

2. \( C_1 \frac{1}{p} \leq \int_{\mathbb{R}^2} u_\rho^{p+1} \, dx \leq C_2 \frac{1}{p} \)

3. \( C_1 \frac{1}{p^2} \leq \int_{\mathbb{R}^2} u_\rho^2 \, dx \leq C_2 \frac{1}{p^2} \)

4. \( C_1 \frac{1}{p} \leq \int_{\mathbb{R}^2} u_\rho \, dx = \int_{\mathbb{R}^2} u_\rho^p \, dx \leq C_2 \frac{1}{p} \).

**Proof.** Combining Lemma 2.1, Lemma 2.2, and Lemma 3.3 we have (1)–(3). To prove (4), we see from Theorem 1.1 and (2)

\[ \int_{\mathbb{R}^2} u_\rho^p \geq \frac{1}{\gamma_p} \int_{\mathbb{R}^2} u_\rho^{p+1} \geq \frac{1}{C} \int_{\mathbb{R}^2} u_\rho^{p+1} \geq \frac{C_1}{p}. \]

On the other hand, an interpolation argument shows
\[ \|u_\rho\|_{L^p} \leq \|u_\rho\|_{L^2}^{\frac{p-2}{p}} \|u_\rho\|_{L^p} \|u_\rho\|_{L^{p+1}} \leq \left( \frac{2M}{\gamma_p} \right)^{p+1} \|u_\rho\|_{L^{p+1}} \left( \frac{2M}{\gamma_p} \right)^{(p+1)\gamma_p(p+1-2)}. \]

Therefore (2) and (3) imply
\[ \int_{\mathbb{R}^2} u_\rho^p \leq \frac{C_2}{p}. \]
for $C_2$ independent of $p$ when $p$ is large. Therefore with the aid of Lemma 2.2(2) we have (4).

5. **On the Normalized $u_p$**

Before we turn to the proof of Theorem 1.2, we first give a uniform exponential decay estimate in $p$ of $u_p$.

**Lemma 5.1.** There exists a constant $C$ independent of $p$, such that for large $p$ and $r \geq 1$

$$u_p(r) \leq \frac{C}{p} e^{-r^2/2}.$$  

**Proof.** Let $v_p = r^{1/2}u_p$. Then $v_p$ satisfies

$$v''_p = \left[ q(r) - \frac{1}{4r^2} \right] v_p,$$

where $q(r) = 1 - u_p^{\rho-1}(r)$. Note for large $p$, by Lemma 2.4 and Corollary 4.2

$$q(1) - \frac{1}{41^2} \geq 1 - \left[ \frac{2}{\sqrt{3\pi}} \|u_p\|_{L^2} \right]^{\rho-1} - \frac{1}{4} \geq \frac{3}{4} - \left( \frac{C}{p} \right)^{\rho-1} \to \frac{3}{4}$$

as $p \to \infty$. Hence for large $p$ we have

$$q(r) - \frac{1}{4r^2} \geq \frac{1}{2}$$

for $r \geq 1$.

Let $w_p = v^2_p$; then $w_p$ satisfies

$$\frac{1}{2} w''_p = (v_p')^2 + \left[ q(r) - \frac{1}{4r^2} \right] w_p.$$  

Thus for $r \geq 1$, one has $w''_p \geq w_p$ and $w_p \geq 0$.

Let $z_p = e^{-r}(w'_p + w_p)$. We have $z'_p = e^{-r}(w''_p - w_p) \geq 0$; hence $z_p$ is a nondecreasing function on $(1, \infty)$. If there exists $r_1 > 1$ such that $z_p(r_1) > 0$, then $z_p(r) \geq z_p(r_1) > 0$ for $r > r_1$. This implies that

$$w'_p + w_p \geq (z_p(r_1)) e^r$$
whence \( w'_p + w_p \) is not integrable on \((r_1, \infty)\). But \( v^2_p \) and \( v_p v'_p \) are integrable near \( \infty \) for \( u_p \in W^{1,2}(R^2) \), so that \( w'_p \) and \( w_p \) are also integrable, a contradiction.

Hence \( z_p(r) \leq 0 \) for \( r \geq 1 \). This implies that

\[
(e^{r}w_p)' = e^{2r}z_p \leq 0
\]

for \( r \geq 1 \). Hence

\[
w_p \leq e^1 w_p(1)e^{-r}, \quad u_p \leq e^{1/2} u_p(1)e^{-r/2} r^{-1/2}.
\]

But \( u_p(1) \) can be estimated by Lemma 2.4 and Corollary 4.2 as

\[
u_p(1) \leq \frac{2}{\sqrt{3\pi}} \|u_p\|_{L^\infty(R^2)} \leq \frac{C}{p}.
\]

Therefore

\[
u_p(r) \leq \frac{C}{p} e^{-r/2} r^{-1/2}
\]

for \( r \geq 1 \).

\[\]  

**Proof of Theorem 1.2.** Let

\[
A_\varepsilon = \left\{ x \in R^2 : \varepsilon \leq |x| \leq \frac{1}{\varepsilon} \right\}.
\]

Then by Lemma 2.4 and Corollary 4.2 we have that for large \( p \)

\[
\left\| \frac{u_p}{\int u_p} \right\|_{L^\infty(A_\varepsilon)} \leq C_p \|u_p\|_{L^2} \leq C_\varepsilon.
\]

Hence \( \|u_p/\int u_p\|_{L^\infty(A_\varepsilon)} \) is bounded uniformly for large \( p \).

On the other hand, by Lemma 2.4 and Corollary 4.2

\[
\left\| \frac{u_p^p}{\int u_p} \right\|_{L^\infty(A_\varepsilon)} \leq C_p \left( \frac{C_2}{p} \right)^p \varepsilon^{-p}
\]

which is bounded uniformly for large \( p \). So the elliptic \( L^p \)-estimate shows that \( u_p/\int u_p \) is bounded in \( C^\alpha(A_\varepsilon) \) for all \( \alpha \in (0, 1) \). Using the Schauder estimate, we have that \( u_p/\int u_p \) is bounded uniformly in \( C^{2,\alpha}(A_\varepsilon) \) for all \( \alpha \in \)
(0, 1). Therefore a subsequence of \( \{ u_p / \| u_p \| \} \), still denoted by \( \{ u_p / \| u_p \| \} \), will approach a function, say \( G' \), in \( C^2_{\text{lin}}(\mathbb{R}^2 \setminus \{0\}) \). So we almost have (1) except that we need to show \( G' = G \) where \( G \) is the fundamental solution given in Remark 1.3.

We now prove

\[
\frac{u_p}{\| u_p \|} \to G'
\]

in the sense of distribution and

\[-\Delta G' + G' = \delta.\]

Let \( \phi(x) \in C^\infty_0(\mathbb{R}^2) \), \( \varepsilon > 0 \). A standard kernel argument shows that (note \( \int u_p^p = \int u_p \) by Lemma 2.2), with the aid of Lemma 2.4

\[
\left| \int_{\mathbb{R}^2} \phi(x) \frac{u_p^p}{\| u_p \|} \ dx - \phi(0) \right| \leq \int_{\mathbb{R}^2} |\phi(x) - \phi(0)| \frac{u_p^p}{\| u_p \|}
\]

\[
\leq \int_{B_\varepsilon} |\phi(x) - \phi(0)| \frac{u_p^p}{\| u_p \|} + 2 \max_{x \in \mathbb{R}^2} (\phi(x)) \int_{\mathbb{R}^2 \setminus B_\varepsilon} \frac{u_p^p}{\| u_p \|}
\]

\[
\leq \max_{B_\varepsilon} (\phi(x) - \phi(0)) + 2 \max_{\mathbb{R}^2} [\phi(x)] \ C_p \int_{B_\varepsilon} \frac{1}{r^p} \left( \frac{C}{p} \right)^x
\]

\[
\leq \varepsilon + \frac{\varepsilon}{2} = \varepsilon
\]

if we choose \( \delta \) small enough first, then choose \( p \) large enough.

Hence,

\[
\lim_{p \to \infty} \int \phi(x) \frac{u_p^p}{\| u_p \|} = \phi(0).
\]

But

\[
\int \phi(x) \frac{u_p^p}{\| u_p \|} = \int [(-\Delta + 1)\phi(x)] \frac{u_p^p}{\| u_p \|}.
\]

(5.1)

To pass the limit in Eq. (5.1), we need a dominating function.

Let us first consider \( u_p / \| u_p \| \) in \( B_1 \), the unit disc. Lemma 2.4 and Corol-
lary 4.2 imply that

$$\frac{u_p}{\int u_p} \leq \frac{2C}{|x|}. \quad (5.2)$$

So we take the latter to be the dominating function in $B_1$.

Then we use Lemma 5.1 to get the following decay estimate on $R^2 \setminus B_1$

$$\frac{u_p(x)}{\int u_p} \leq Ce^{-|x|/2}|x|^{-|x|/2}, \quad (5.3)$$

where the latter gives a dominating function on $R^2 \setminus B_1$.

Combining (5.2) with (5.3), we can pass the limit in (5.1); hence

$$\int \left[ (-\Delta + 1)\phi(x) \right] G'(x) = \lim_{p \to \infty} \int \left[ (-\Delta + 1)\phi(x) \right] \frac{u_p(x)}{\int u_p}$$

$$= \lim_{p \to \infty} \int \phi(x) \frac{u_p(x)}{\int u_p} = \phi(0).$$

Therefore $G'$ satisfies

$$-\Delta G' + G' = \delta;$$

hence

$$-\Delta (G - G') + (G - G') = 0,$$

where $G$ is the fundamental solution given in Remark 1.3. Because $G - G'$ decays, actually decays exponentially by Remark 1.3 and (5.3), at infinity, the maximum principle implies $G = G'$; hence we get (1) and (2) simultaneously.

References


