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On a Semilinear Elliptic Equation in R^2 When the Exponent Approaches Infinity

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We consider a semilinear elliptic equation in R^2 with the nonlinear exponent approaching infinity. In contrast to the blow-up behavior of the corresponding problem in R^n with $n \geq 3$, the $L^q(R^2)$ norms of the solutions to the equation in R^2 remain bounded from below and above. After a careful study on the decay rates of several quantities, we prove that the normalized solutions will approach the fundamental solution of $-\Delta + 1$ in R^2. So as the exponent tends to infinity, the solutions to the problem look more and more like a peak.

1. INTRODUCTION

This paper is devoted to the behavior of the ground state solution to a semilinear elliptic equation with large exponent of the nonlinear term.

Considering

$$\begin{cases} \Delta u - u + u^p = 0 & \text{in } R^2 \\ u > 0, & \lim_{|x| \to \infty} u(x) = 0, \end{cases} \tag{1.1}$$

we would like to understand the behavior of the solutions to (1.1) when the exponent p approaches ∞.

The corresponding problem in higher dimensions

$$\begin{cases} \Delta u - u + u^p = 0 & \text{in } R^n, \text{ with } n \geq 3 \\ u > 0, & \lim_{|x| \to \infty} = 0, \end{cases} \tag{1.2}$$

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was studied by X. Pan and X. Wang in [9]. Due to the existence of the critical exponent, namely $(n+2)/(n-2)$ there, they were led to study the behavior of the solution to the corresponding equation in higher dimensions when p approaches $(n+2)/(n-2)$. It turns out that solutions to their problem will blow up as $p \to (n+2)/(n-2)$.

Because of the critical exponent for $n \geq 3$, in higher dimensions we have a profile equation

$$\begin{cases} \Delta u - u + u^{(n+2)/(n-2)} = 0 & \text{in } R^n \\ u > 0, \qquad \lim_{|x| \to \infty} u(x) = 0. \end{cases} \tag{1.3}$$

In fact the nonexistence of solutions to (1.3) implies the blow-up of solutions to (1.2) when $p \to (n+2)/(n-2)$.

In R^2, however, due to the lack of Eq. (1.3), we will approach the problem in a different way. We will start with a sharp estimate on the growth rate of c_p where c_p is the best constant of the embedding

$$W^{1,2}(R^2) \hookrightarrow L^{p+1}(R^2).$$

From this estimate and estimates of some other quantities, we will prove

THEOREM 1.1. *Let u_p be a solution to* (1.1). *Then for p large enough, there exists C independent of p such that* $1 < \|u_p\|_{L^\infty(R^2)} < C$.

THEOREM 1.2. *Let u_p be a radially symmetric solution to* (1.1). *Then, as $p \to \infty$, we have*

(1) $\dfrac{u_p}{\int_{R^2} u_p} \to G$ *in the sense of distribution,*

(2) $\dfrac{u_p}{\int_{R^2} u_p} \to G$ *in $C^{2,\alpha}_{loc}(R^2 \backslash \{0\})$ for all $\alpha \in (0, 1)$,*

where G is the fundamental solution to $-\Delta + 1$, i.e.,

$$-\Delta G + G = \delta.$$

Remark 1.3. G has an integral representation

$$G(x) = \frac{1}{\Gamma(1/2)} \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}} \int_0^\infty e^{-t} t^{-1/2} \left(1 + \frac{1}{2}\frac{t}{x}\right)^{-1/2} dt.$$

Moreover, G decays exponentially at infinity. We refer to [7] for more information.

We see, in contrast to the higher dimensions, the solutions look more and more like a peak as p gets large.

Our paper is organized as follows. In Section 2, we give necessary background of the solutions to (1.1) together with some integral identities. Then we prove a growth rate estimate of c_p in Section 3. Finally we prove Theorem 1.1 in Section 4 and Theorem 1.2 in Section 5.

2. Preliminaries

There are various ways to obtain solutions to (1.1). We shall adapt the variational approach developed in [2].

Let

$$\mathcal{A}_p = \left\{ u \in W^{1,2}(R^2) \colon \int_{R^2} u^{p+1} = 1 \right\}.$$

Consider the functional $J_p \colon \mathcal{A}_p \to R$ defined by

$$J_p(u) = \frac{1}{2} \int_{R^2} [|\nabla u|^2 + u^2].$$

By the theory of Schwartz symmetrization (see, for example, [6]), we can replace any minimizing sequence of the functional by a radially symmetric minimizing sequence. Then it is easy to show, with the aid of a compactness result of radial embedding in [10], that J_p has a radial minimizer in $\mathcal{A}_p$. If we denote this minimizer by u_p', a scalar multiple of u_p' will solve Eq. (1.1). We denote this radial solution by u_p. A further L^∞ estimate shows $\lim_{|x|\to\infty} u(x) = 0$. Hence u_p solves (1.1).

According to a result in [3] all solutions of (1.1) up to a parallel translation are radially symmetric. Applying the uniqueness result in [8] or [5], we know that the radial solution is unique. Therefore (1.1) has a unique solution up to a translation. From now on we will denote this unique radial solution by u_p.

It is easy to see that u_p is related to the embedding

$$W^{1,2}(R^2) \hookrightarrow L^{p+1}(R^2).$$

We refer to [1] for the proof of this embedding theorem.

If we denote c_p as the best constant of the above embedding, i.e., c_p is the least number among all possible constant C's which make the inequality

$$\|u\|_{L^{p+1}(R^2)} \leq C\|u\|_{W^{1,2}(R^2)}$$

for all $u \in W^{1,2}(R^2)$, then c_p is achieved by u_p. Hence we have

LEMMA 2.1. *Let u_p be the radial solution to* (1.1) *as above. Then we have*

$$\frac{\|u_p\|_{W^{1,2}(R^2)}}{\|u_p\|_{L^{p+1}(R^2)}} = \inf\left\{\frac{\|u\|_{W^{1,2}(R^2)}}{\|u\|_{L^{p+1}(R^2)}} : u \in W^{1,2}(R^2),\ u \neq 0\right\} = \frac{1}{c_p},$$

where c_p is the best constant defined above.

Now let us state some integral identities.

LEMMA 2.2. *Let u_p be a solution to* (1.1). *We have*

(1) $\int_{R^2} [|\nabla u_p|^2 + |u_p|^2] = \int_{R^2} u_p^{p+1}$

(2) $\int_{R^2} u_p = \int_{R^2} u_p^p$

(3) $\frac{1}{2}\int_{R^2} u_p^2 = (1/(p+1))\int_{R^2} u_p^{p+1}$.

Proof. (1) Multiplying (1.1) by u_p and integrating over $B(R)$, the ball of radius R centered at the origin, we get

$$\int_{B(R)} [|\nabla u_p|^2 + |u_p|^2] = \int_{\partial B_R} u_p \frac{\partial u_p}{\partial n} + \int_{B_R} u_p^{p+1}.$$

Using an exponential decay property of u_p which says that there exist C, μ independent of x such that

$$u_p(x) \leq Ce^{-\mu|x|}, \qquad |Du_p(x)| \leq Ce^{-\mu|x|}$$

we can let R approach ∞ yielding 1. The proof of the exponential decay estimate can be found, for example, in [2]. We shall prove a refined result, Lemma 5.1, by the same method in Section 5.

(2) Similar to (1), but we integrate the equation directly.

(3) This is the R^2 version of the well known Pohozaev identity. We refer to [2] for a proof. ∎

We also present some simple radial lemmas which will be used in later sections.

LEMMA 2.3. *Let $n \geq 2$. Every radial function $u \in W^{1,2}(R^n)$ is almost equal to a function $U(x)$ which is continuous for $x \neq 0$; furthermore*

$$|U(x)| \le C_n |x|^{(1-n)/2} \|u\|_{W^{1,2}(R^n)}$$

for $|x| \ge \alpha_n$ *where* C_n, α_n *depend on* n *only.*

The proof of this lemma can be found in [2].

LEMMA 2.4. *Let* $u \in L^2(R^2)$ *be a radial, nonnegative, non-increasing in* $|x| > 0$, *continuous function except at* 0. *Then we have*

$$u(|x|) \le \frac{2}{\sqrt{3\pi}} |x|^{-1} \|u\|_{L^2(R^2)}.$$

Proof.

$$\|u\|^2_{L^2(R^2)} = 2\pi \int_0^\infty u^2(r) r \, dr \ge 2\pi \int_{|x|/2}^{|x|} u^2(s) s^{-1} \, ds$$

$$\ge 2\pi \frac{1}{2} u^2(|x|)\left(|x|^2 - \frac{|x|^2}{4}\right) \ge \frac{3\pi}{4} |x|^2 u^2(|x|).$$

Therefore we have the lemma. ∎

3. On the Growth Rate of c_p

In this section, we establish a sharp estimate for the growth rate of c_p. Notice that according to the Schwartz symmetrization theory, the c_p is actually obtained in the class of radial functions. We will use C to denote various constants independent of p.

LEMMA 3.1. *If* $u \in W_0^{1,2}(\Omega)$ *where* Ω *is a bounded smooth domain in* R^2, *then for every* $t \ge 2$

$$\|u\|_{L^t(\Omega)} \le C t^{1/2} |\Omega|^{1/t} \|\nabla u\|_{L^2(\Omega)}.$$

Proof. Let $u \in W_0^{1,2}(\Omega)$. We know

$$\frac{1}{\Gamma(s+1)} x^s \le e^x$$

for all $x \ge 0$, $s \ge 0$ where Γ is the Γ function. From Trudinger's Inequality (see [4, p. 160]), we have

$$\int_\Omega \exp\left[C_1 \left(\frac{u}{\|\nabla u\|_{L^2}}\right)^2\right] dx \le C_2 |\Omega|,$$

where C_1, C_2 are constants which depend on the dimension of Ω only and $|\ |$ denotes the Lebesgue measure. Therefore

$$\frac{1}{\Gamma(t/2+1)}\int_\Omega u^t\,dx$$
$$= \frac{1}{\Gamma(t/2+1)}\int_\Omega \left[C_1\left(\frac{u}{\|\nabla u\|_{L^2}}\right)^2\right]^{t/2} dx C_1^{-t/2}\|\nabla u\|_{L^2}^t$$
$$\leq \int_\Omega \exp\left[C_1\left(\frac{u}{\|\nabla u\|_{L^2}}\right)^2\right] dx C_1^{-t/2}\|\nabla u\|_{L^2}^t \leq C_2|\Omega|C_1^{-t/2}\|\nabla u\|_{L^2}^t.$$

Hence

$$\left(\int_\Omega u^t\,dx\right)^{1/t} \leq \left(\Gamma\left(\frac{t}{2}+1\right)\right)^{1/t} C_2^{1/t}C_1^{-1/2}|\Omega|^{1/t}\|\nabla u\|_{L^2(\Omega)}.$$

Notice that, according to Stirling's formula,

$$\left(\Gamma\left(\frac{t}{2}+1\right)\right)^{1/t} \sim \left(\left(\frac{t/2}{e}\right)^{t/2}\sqrt{te}e^{\theta_t}\right)^{1/t} \sim Ct^{1/2},$$

where $0<\theta_t<1/12$. This completes our lemma. ∎

LEMMA 3.2. *For each $u \in W^{1,2}(R^2)$, we have*

$$\|u\|_{L^t(R^2)} \leq Ct^{1/2}\|u\|_{W^{1,2}(R^2)}$$

for every $t \geq 4$.

Proof. As we mentioned earlier, by the Schwartz symmetrization theory, we need only to prove the inequality for radial functions. So in the proof all functions are assumed to be radial. Let $u := u_1 + u_2 := u\chi_{\{r\leq\alpha_2\}} + u\chi_{\{r\geq\alpha_2\}}$ where α_2 is defined in Lemma 2.3.

Then

$$\|u\|_{L^t(R^2)} \leq \|u_1\|_{L^t} + \|u_2\|_{L^t}.$$

And by Lemma 2.3

$$\|u_2\|_{L^t} \leq \left(\int_{R^2\setminus B_{\alpha_2}} u^t\right)^{1/t} \leq \left(2\pi\int_{r>\alpha_2} r^{-t/2+1}(C\|u\|_{W^{1,2}(R^2)})^t\right)^{1/t}$$
$$\leq C\|u\|_{W^{1,2}(R^2)}\left(\frac{2}{|t-1|}\right)^{1/t}(\alpha_2^{-t/2+2})^{1/t} \leq C\|u\|_{W^{1,2}(R^2)}.$$

Hence combining this with Lemma 3.1, we obtain

$$\|u\|_{L^t} \le Ct^{1/2}\|u\|_{W^{1,2}(R^2)} + C\|u_2\|_{W^{1,2}(R^2)} \le Ct^{1/2}\|u\|_{W^{1,2}(R^2)}. \quad \blacksquare$$

LEMMA 3.3. *For p large enough, there exist C_1, C_2 independent of p such that*

$$C_1 p^{1/2} \le c_p \le C_2 p^{1/2}.$$

Proof. The second inequality follows from Lemma 3.2. To prove the first inequality, we consider the so-called Moser's function

$$\Phi(r) = \begin{cases} (p+1)^{-1/2} \log \dfrac{1}{r}, & e^{-(p+1)} \le r \le 1 \\ (p+1)^{-1/2} \log e^{p+1}, & 0 \le r \le e^{-(p+1)} \end{cases}$$

$$= \begin{cases} (p+1)^{-1/2} \log \dfrac{1}{r}, & e^{-(p+1)} \le r \le 1 \\ (p+1)^{1/2}, & 0 \le r \le e^{-(p+1)}. \end{cases}$$

So $\phi \in W_0^{1,2}(B_1) \subset W^{1,2}(R^2)$ where the trivial extension is used to interpret the second inclusion.

On the one hand

$$\|\nabla\phi\|_{L^2}^2 = 2\pi \int_0^1 \phi_r^2 r\, dr = 2\pi \int_{e^{-(p+1)}}^1 (p+1)^{-1} \frac{1}{r}\, dr = 2\pi.$$

Therefore by Poincaré's Inequality on B_1,

$$\|\phi\|_{W^{1,2}(R^2)} = \|\phi\|_{W^{1,2}(B_1)} \le C\|\nabla\phi\|_{L^2(B_1)} = C, \tag{3.1}$$

where C is a constant depending on the first eigenvalue of Δ on the unit disk with Dirichlet boundary condition only.

On the other hand

$$\begin{aligned} \int_{B_1} \phi^{p+1} &= 2\pi \int_0^1 \phi^{p+1} r\, dr \ge 2\pi \int_{e^{-(p+1)}}^1 (p+1)^{-(p+1)/2} \left(\log \frac{1}{r}\right)^{p+1} r\, dr \\ &= 2\pi(p+1)^{-(p+1)/2} \int_0^{p+1} t^{p+1} e^{-t}\, dt \\ &\ge 2\pi(p+1)^{-(p+1)/2} \int_{(p+1)/2}^{p+1} t^{p+1} e^{-t}\, dt \\ &\ge 2\pi(p+1)^{-(p+1)/2} \left(\frac{p+1}{2}\right)^{p+1} e^{-(p+1)} \frac{p+1}{2} \\ &\ge 2\pi 2^{-(p+2)} (p+1)^{(p+3)/2} e^{-(p+1)}. \end{aligned}$$

Hence

$$\|\phi\|_{L^{p+1}} \geq (2\pi)^{1/(p+1)} 2^{-(p+2)/(p+1)} (p+1)^{(1/2)((p+3)/(p+1))} e^{-1} \geq C(p+1)^{1/2}. \tag{3.2}$$

Combining (3.1) and (3.2), we have the lemma. ∎

4. On the $L^\infty(R^2)$ Norm of u_p

We are now in the position to study the $L^\infty(R^2)$ norms of u_p. Because our solutions u_p are radially symmetric, they satisfy the following ordinary differential equation

$$\begin{cases} u'' + \dfrac{1}{r} u' - u + u^p = 0 \\ u'(0) = 0, \qquad \lim_{r\to\infty} u(r) = 0. \end{cases} \tag{4.1}$$

From [8] or [5], we know $u_p'(r) < 0$ for all $r > 0$ where u_p is a solution to (1.1) with the exponent in the equation being p. We define

$$\gamma_p := u_p(0) = \|u_p\|_{L^\infty}.$$

LEMMA 4.1. $\|u_p\|_{W^{1,2}(R^2)} \leq C/p^{1/2}$ *when p is large.*

Proof. From the integral identity (1) in Lemma 2.2, we deduce

$$\|u_p\|^2_{W^{1,2}(R^2)} = \|u_p\|^{p+1}_{L^{p+1}(R^2)}.$$

Combining this with Lemma 2.1 and Lemma 3.2, we get

$$\|u_p\|_{W^{1,2}(R^2)} = C^{(p+1)/(p-1)} p^{-(1/2)((p+1)/(p-1))} \leq Cp^{-1/2}$$

when p is large. ∎

We now prove Theorem 1.1.

Proof of Theorem 1.1. The uniform lower bound 1 follows from the maximum principle. To find a uniform upper bound let

$$E_p(r) = -ru_p'(r) - \frac{1}{2} r^2(u_p^p - u).$$

Then $E_p(0) = 0$, and

$$E_p'(r) = -u_p' - ru_p'' - r(u_p^p - u_p) - \frac{1}{2} r^2(pu_p^{p-1} - 1)u_p'$$
$$= -\frac{1}{2} r^2(pu_p^{p-1} - 1)u_p'.$$

Let $r_0 > 0$ be such that $u_p(r_0) = (1/p)^{1/(p-1)}$. Clearly such r_0 exists, and it depends on p. Notice $E_p'(r) \geq 0$ for $r \in (0, r_0)$. Therefore $E_p \geq 0$ on $(0, r_0)$. From the definition of E, we have

$$-u_p'(r) - \frac{1}{2} r(u_p^p - u_p) \geq 0$$

on $(0, r_0)$. Combining this with Eq. (4.1) of u_p, we get

$$u_p'' \geq \frac{1}{2}(u_p - u_p^p), \qquad u_p'' \geq \frac{1}{2}(\gamma_p - \gamma_p^p) \tag{4.2}$$

for all r in $(0, r_0)$.

Integrating both sides of (4.2) twice with respect to r on $(0, r)$, we obtain

$$\gamma_p - \frac{1}{2} u_p(r) \leq \frac{1}{4} r^2(\gamma_p^p - \gamma_p) \tag{4.3}$$

for $r \in (0, r_0)$.

Now let T_0 be such that $u_p(T_0) = \frac{1}{2}\gamma_p$. T_0 depends on p. From (4.3), we have

$$\gamma_p - \frac{1}{2}\gamma_p \leq \frac{1}{4} T_0^2(\gamma_p^p - \gamma_p), \qquad \gamma_p \leq \frac{1}{2} T_0^2\gamma_p^p. \tag{4.4}$$

Applying Lemma 3.2 with $t = 2p$ and Lemma 4.1, we get

$$\|u_p\|_{L^{2p}(R^2)} \leq C(2p)^{1/2}\|u_p\|_{W^{1,2}(R^2)} \leq C(2p)^{1/2}C'p^{-1/2} := M,$$

where M is independent of p for large p. Hence

$$\int_{R^2} u_p^{2p}\, dx \leq M^{2p}.$$

But

$$\int_{R^2} u^{2p}\,dx = 2\pi \int_0^\infty u_p^{2p}(r)r\,dr \geq \int_0^{T_0} u_p^{2p} r\,dr \geq \left(\frac{\gamma_p}{2}\right)^{2p} \frac{1}{2} T_0^2.$$

Therefore

$$\frac{1}{2} T_0^2 \left(\frac{\gamma_p}{2}\right)^{2p} \leq M^{2p}, \qquad T_0^2 \leq 2\left(\frac{2M}{\gamma_p}\right)^{2p}. \tag{4.5}$$

Combining (4.4) with (4.5), we obtain

$$1 \leq \left(\frac{2M}{\gamma_p}\right)^{2p} (\gamma_p^{p-1}) \leq \left(\frac{2M}{\gamma_p}\right)^{2p} \gamma_p^p, \qquad \gamma_p \leq (2M)^2. \quad \blacksquare$$

We now derive the decay rates for some quantities.

COROLLARY 4.2. *There exist C_1, C_2 independent of p such that for large p we have*

(1) $C_1 \dfrac{1}{p} \leq \int_{R^2} |\nabla u_p|^2\,dx \leq C_2 \dfrac{1}{p}$

(2) $C_1 \dfrac{1}{p} \leq \int_{R^2} u_p^{p+1}\,dx \leq C_2 \dfrac{1}{p}$

(3) $C_1 \dfrac{1}{p^2} \leq \int_{R^2} u_p^2\,dx \leq C_2 \dfrac{1}{p^2}$

(4) $C_1 \dfrac{1}{p} \leq \int_{R^2} u_p\,dx = \int_{R^2} u_p^p\,dx \leq C_2 \dfrac{1}{p}.$

Proof. Combining Lemma 2.1, Lemma 2.2, and Lemma 3.3 we have (1)–(3). To prove (4), we see from Theorem 1.1 and (2)

$$\int_{R^2} u_p^p \geq \frac{1}{\gamma_p} \int_{R^2} u_p^{p+1} \geq \frac{1}{C} \int_{R^2} u_p^{p+1} \geq \frac{C_1}{p}.$$

On the other hand, an interpolation argument shows

$$\|u_p\|_{L^p} \leq \|u_p\|_{L^2}^{(p-2)/(p+1-2)} \|u_p\|_{p+1}^{(p+1-p)/(p+1-2)}.$$

Therefore (2) and (3) imply

$$\int_{R^2} u_p^p \leq \frac{C_2}{p}$$

for C_2 independent of p when p is large. Therefore with the aid of Lemma 2.2(2) we have (4). ∎

5. On the Normalized u_p

Before we turn to the proof of Theorem 1.2, we first give a uniform exponential decay estimate in p of u_p.

Lemma 5.1. *There exists a constant C independent of p, such that for large p and $r \geq 1$*

$$u_p(r) \leq \frac{C}{p} e^{-r/2} r^{-1/2}.$$

Proof. Let $v_p = r^{1/2} u_p$. Then v_p satisfies

$$v_p'' = \left[q(r) - \frac{1}{4r^2}\right] v_p,$$

where $q(r) = 1 - u_p^{p-1}(r)$. Note for large p, by Lemma 2.4 and Corollary 4.2

$$q(1) - \frac{1}{4 1^2} \geq 1 - \left[\frac{2}{\sqrt{3\pi}} \|u_p\|_{L^2}\right]^{p-1} - \frac{1}{4} \geq \frac{3}{4} - \left(\frac{C}{p}\right)^{p-1} \rightarrow \frac{3}{4}$$

as $p \rightarrow \infty$. Hence for large p we have

$$q(r) - \frac{1}{4r^2} \geq \frac{1}{2}$$

for $r \geq 1$.

Let $w_p = v_p^2$; then w_p satisfies

$$\frac{1}{2} w_p'' = (v_p')^2 + \left[q(r) - \frac{1}{4r^2}\right] w_p.$$

Thus for $r \geq 1$, one has $w_p'' \geq w_p$ and $w_p \geq 0$.

Let $z_p = e^{-r}(w_p' + w_p)$. We have $z_p' = e^{-r}(w_p'' - w_p) \geq 0$; hence z_p is a nondecreasing function on $(1, \infty)$. If there exists $r_1 > 1$ such that $z_p(r_1) > 0$, then $z_p(r) \geq z_p(r_1) > 0$ for $r > r_1$. This implies that

$$w_p' + w_p \geq (z_p(r_1)) e^r$$

whence $w_p' + w_p$ is not integrable on (r_1, ∞). But v_p^2 and $v_p v_p'$ are integrable near ∞ for $u_p \in W^{1,2}(R^2)$, so that w_p' and w_p are also integrable, a contradiction.

Hence $z_p(r) \leq 0$ for $r \geq 1$. This implies that

$$(e^r w_p)' = e^{2r} z_p \leq 0$$

for $r \geq 1$. Hence

$$w_p \leq e^1 w_p(1) e^{-r}, \qquad u_p \leq e^{1/2} u_p(1) e^{-r/2} r^{-1/2}.$$

But $u_p(1)$ can be estimated by Lemma 2.4 and Corollary 4.2 as

$$u_p(1) \leq \frac{2}{\sqrt{3\pi}} \|u_p\|_{L^2(R^2)} \leq \frac{C}{p}.$$

Therefore

$$u_p(r) \leq \frac{C}{p} e^{-r/2} r^{-1/2}$$

for $r \geq 1$. ∎

Proof of Theorem 1.2. Let

$$A_\varepsilon = \left\{x \in R^2 \colon \varepsilon \leq |x| \leq \frac{1}{\varepsilon}\right\}.$$

Then by Lemma 2.4 and Corollary 4.2 we have that for large p

$$\left\| \frac{u_p}{\int u_p} \right\|_{L^\infty(A_\varepsilon)} \leq C_\varepsilon p \|u_p\|_{L^2} \leq C_\varepsilon.$$

Hence $\|u_p / \int u_p\|_{L^\infty(A_\varepsilon)}$ is bounded uniformly for large p.

On the other hand, by Lemma 2.4 and Corollary 4.2

$$\left\| \frac{u_p^p}{\int u_p} \right\|_{L^\infty(A_\varepsilon)} \leq Cp \left(\frac{C_2}{p}\right)^p \varepsilon^{-p}$$

which is bounded uniformly for large p. So the elliptic L^p-estimate shows that $u_p / \int u_p$ is bounded in $C^\alpha(A_\varepsilon)$ for all $\alpha \in (0, 1)$. Using the Schauder estimate, we have that $u_p / \int u_p$ is bounded uniformly in $C^{2,\alpha}(A_\varepsilon)$ for all $\alpha \in$

(0, 1). Therefore a subsequence of $\{u_p/\int u_p\}$, still denoted by $\{u_p/\int u_p\}$, will approach a function, say G', in $C^{2,\alpha}_{loc}(R^2\backslash\{0\})$. So we almost have (1) except that we need to show $G' = G$ where G is the fundamental solution given in Remark 1.3.

We now prove

$$\frac{u_p}{\int u_p} \to G'$$

in the sense of distribution and

$$-\Delta G' + G' = \delta.$$

Let $\phi(x) \in C_0^\infty(R^2)$, $\varepsilon > 0$. A standard kernel argument shows that (note $\int u_p^p = \int u_p$ by Lemma 2.2), with the aid of Lemma 2.4

$$\left|\int_{R^2} \phi(x)\frac{u_p^p}{\int u_p}\,dx - \phi(0)\right| \le \int_{R^2} |\phi(x) - \phi(0)|\,\frac{u_p^p}{\int u_p}$$

$$\le \int_{B_\delta} |\phi(x) - \phi(0)|\,\frac{u_p^p}{\int u_p} + 2\max_{x\in R^2}(\phi(x))\int_{R^2\backslash B_\delta}\frac{u_p^p}{\int u_p}$$

$$\le \max_{B_\delta}(\phi(x) - \phi(0)) + 2\max_{R^2}[\phi(x)]\,Cp\int_\delta^\infty \frac{1}{r^p}\,r\left(\frac{C}{p}\right)^p$$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

if we choose δ small enough first, then choose p large enough.

Hence,

$$\lim_{p\to\infty}\int \phi(x)\frac{u_p^p}{\int u_p} = \phi(0).$$

But

$$\int \phi(x)\frac{u_p^p}{\int u_p} = \int [(-\Delta + 1)\phi(x)]\frac{u_p}{\int u_p}. \tag{5.1}$$

To pass the limit in Eq. (5.1), we need a dominating function.

Let us first consider $u_p/\int u_p$ in B_1, the unit disc. Lemma 2.4 and Corollary 4.2 imply that

$$\frac{u_p}{\int u_p} \le \frac{2C}{|x|}. \tag{5.2}$$

So we take the latter to be the dominating function in B_1.

Then we use Lemma 5.1 to get the following decay estimate on $R^2 \backslash B_1$

$$\frac{u_p(x)}{\int u_p} \le Ce^{-|x|/2}|x|^{-|x|/2}, \tag{5.3}$$

where the latter gives a dominating function on $R^2 \backslash B_1$.

Combining (5.2) with (5.3), we can pass the limit in (5.1); hence

$$\begin{aligned}\int [(-\Delta + 1)\phi(x)]G'(x) &= \lim_{p\to\infty} \int [(-\Delta + 1)\phi(x)] \frac{u_p(x)}{\int u_p} \\ &= \lim_{p\to\infty} \int \phi(x) \frac{u_p(x)}{\int u_p} = \phi(0).\end{aligned}$$

Therefore G' satisfies

$$-\Delta G' + G' = \delta;$$

hence

$$-\Delta(G - G') + (G - G') = 0,$$

where G is the fundamental solution given in Remark 1.3. Because $G - G'$ decays, actually decays exponentially by Remark 1.3 and (5.3), at infinity, the maximum principle implies $G = G'$; hence we get (1) and (2) simultaneously. ∎

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