Be sure that this examination has 2 pages.

The University of British Columbia

Final Examinations - December 2010

Mathematics 305

R. Froese, M. Ward

Closed book examination. No notes, texts, or calculators allowed. Time: $2\frac{1}{2}$ hours

Special Instructions: No notes, book, or calculator allowed

Marks

[40] **1.** Identify whether each of the following statements are true or false. You must give reasons for your answers.

(i) $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$.

(ii) $\text{Re}\,(i/\bar{z}) = -\text{Im}(z)/|z|^2$.

(iii) $\sin(n\theta) = \text{Im}\{(\cos\theta + i\sin\theta)^n\}$ where n is a positive integer.

(iv) $f(z) = |z|^2$ is analytic at $z = 0$ but not at any other point.

(v) $u = r^n \cos(n\theta)$ is a harmonic function, where n is a positive integer, $r^2 = x^2 + y^2$ and $\tan\theta = y/x$.

(vi) If $f(z) = u + iv$ is an entire function, then $u^2 - v^2$ is a harmonic function.

(vii) Let $M = \max(|e^{iz^2}|)$ over the disk $|z| \leq 2$. Then, $M = 1$.

(viii) $|\sin(z)|$ is bounded as $|z| \to \infty$.

(ix) the equation $\sqrt{z} + (1 - i) = 0$, where $\sqrt{z}$ is the principal branch of the square root function, has no solution.

(x) $|e^{z^2}| \leq e^{|z|^2}$ for all z.

(xi) $\log(e^z) = z$.

(xii) $\int_C z^{-1/2} \sin(\sqrt{z})dz = 0$ where C is the simple closed curve $|z| = 1$ oriented counterclockwise, and $\sqrt{z}$ is the principal branch of the square root function.

[15] **2.** Consider the function $f(z)$ defined by

$$f(z) = \frac{z}{z^2 - z - 2},$$

Continued on page 2

(i) Determine the Laurent series of $f(z)$ centered at $z_0 = 0$ that converges in the region $|z| > 2$.

(ii) By using the Laurent series in (i), and by integrating it term by term, evaluate $\int_C f(z)\, dz$ where C is the simple closed curve $|z| = 4$ oriented counterclockwise. Confirm your result by using the residue theorem applied to the function $f(z)$ on the region $|z| \leq 4$.

[15] **3.** Consider the following function $f(z)$ defined by

$$f(z) = \frac{1}{z\,(1 - \cos(\sqrt{z}))\,(z - \pi^2)}\,.$$

(i) Identify and then classify all of the singular points of $f(z)$ in the complex plane.

(ii) Calculate $\int_C f(z)dz$ where C is the circle $|z| = 10$ oriented in a counterclockwise sense.

[15] **4.** Let $a > 0$ with a real. By using residue theory, calculate values for the following integrals in as compact a form as you can:

$$\text{(i)}\quad I = \int_0^{2\pi} \frac{1}{a + \cos\theta} d\theta\,, \quad \text{with } a > 1; \qquad \text{(ii)}\quad I = \int_0^{\infty} \frac{x \sin x}{x^2 + a^2}\, dx\,.$$

[15] **5.** By using residuee theory, calculate the following integrals:

$$\text{(i)}\quad I = \int_0^{\infty} \frac{\sin x}{x(x^2 + 1)}\, dx\,; \qquad \text{(ii)}\quad I = \int_0^{\infty} \frac{\sqrt{x}}{(x^2 + 1)}\, dx\,.$$

[100] **Total Marks**

The End

1 (i) $\operatorname{ARG}(z_1 z_2) = \operatorname{ARG}(z_1) + \operatorname{ARG}(z_2)$

FALSE: IF $z_1 = e^{3\pi i/4}$ $z_2 = e^{3\pi i/4}$

THEN $\operatorname{ARG} z_1 + \operatorname{ARG} z_2 = 3\pi/4 + 3\pi/4 = 3\pi/2$

$\operatorname{ARG}(z_1 z_2) = \operatorname{ARG}(e^{3\pi i/2}) = -\pi/2$

(ii) $\operatorname{RE}(i/\bar{z}) = -\operatorname{IM}(z)/|z|^2$ true.

$$\frac{i}{\bar{z}}\frac{z}{z} = \frac{iz}{|z|^2} = \frac{ix - y}{|z|^2} \quad \text{so} \quad \operatorname{IM}\left(\frac{i}{\bar{z}}\right) = \frac{x}{|z|^2} = \operatorname{RE}\left(\frac{z}{|z|^2}\right)$$

$$\operatorname{RE}\left(\frac{i}{\bar{z}}\right) = -\operatorname{IM}(z)/|z|^2$$

(iii) $\sin(n\varphi) = \operatorname{IM}[(\cos\varphi + i\sin\varphi)^n]$ $n = 0, 1, 2, \ldots$

true let $z = re^{i\varphi}$

THEN $z^n = r^n e^{in\varphi} = r^n (e^{i\varphi})^n$.

$\rightarrow$ let $r = 1$ AND EQUATE IMAGINARY PARTS.

$$\sin(n\varphi) = \operatorname{IM}[(\cos\varphi + i\sin\varphi)^n].$$

iv) $f(z) = |z|^2 = (x^2 + y^2)$ IS NOT ANALYTIC ANYWHERE SINCE CR EQUATIONS

$u_x = v_y \rightarrow x = 0$ ONLY HOLD AT THE ISOLATED
$u_y = -v_x \rightarrow y = 0$

FALSE

POINT $(0, 0)$ AND NOT IN NEIGHBORHOOD OF $(0, 0)$.

(v) $u = r^n \cos(n\varphi) = \operatorname{RE}[z^n]$ IS HARMONIC SINCE $f(z) = z^n$ IS ANALYTIC $\forall z$ AND $u = \operatorname{RE}[f(z)]$. true

(vi) true let $f = u + iv$ be entire. THEN

$f^2 = u^2 - v^2 + 2iuv$ IS entire FUNCTION.

$\rightarrow u^2 - v^2 = \operatorname{RE}[f^2]$ IS HARMONIC, SINCE f^2 IS entire.

(vii) FALSE

THE MAX OCCURS ON THE BOUNDARY BY MAX MODULUS PRINCIPAL.

THUS $M = \max_{|z|\le 2} |e^{iz^2}| = \max_{|z|=2} |e^{iz^2}| = e^4$ FALSE

let $iz^2 = 4 \rightarrow z^2 = -4i = 4e^{i\pi/2 + i\pi}$

$z^2 = 4e^{3\pi i/2}$

$\rightarrow z_{MAX} = 2e^{3\pi i/4}$

(viii) FALSE

$|\sin z|$ IS UNBOUNDED AS $|z| \rightarrow \infty$ WHEN $z = iy$,

WITH $y \rightarrow \infty$.

$$\sin(iy) = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{-e^{+y} + e^{-y}}{2i} = i\left(\frac{e^{y} - e^{-y}}{2}\right)$$

$\sin(iy) = i\sinh(y)$

SO $|\sin(iy)| = |\sinh y| \sim e^{y}/2$ AS $y \rightarrow +\infty$.

SO UNBOUNDED IF WE SET $x = 0$ AND LET $|y| \rightarrow \infty$.

(ix) $\sqrt{z} = r^{1/2} e^{i\varphi/2}$ $\quad -\pi < \varphi < \pi$ TRUE

$\mathrm{RE}(\sqrt{z}) = r^{1/2} \cos(\varphi/2) \ge 0$, IS GUARANTEED BY BRANCH choice.

SO $\mathrm{RE}(\sqrt{z}) + 1 = 0$ IS IMPOSSIBLE, SINCE $\mathrm{RE}(\sqrt{z}) \ge 0$.

(x) $|e^{w}| \le e^{|w|}$ IS TRUE. LET $w = z^2$ AND USE $|z^2| = |z|^2$.

PROOF: let $w = u + iv$. $\quad |e^{u+iv}| = e^{u} \le e^{|w|}$

NOTE $u \le |w|$

(xi) FALSE $\log(e^{z}) = \log(e^{x+iy})$

$$= \log(e^{x+iy}) = \ln(e^{x}) + y + 2k\pi i.$$

so $\log(e^{z}) = z + 2k\pi i.$

(xii) $\int_C z^{-1/2}\sin(\sqrt{z})\,dz = 0$ true.

NOTE: $\dfrac{\sin(\sqrt{z})}{\sqrt{z}} = \dfrac{\sqrt{z} - z^{3/2}/3! + z^{5/2}/5! \cdots}{\sqrt{z}}$

$$= 1 - z/3! + z^{2}/5! - z^{3}/7! + \cdots$$

Analytic $\forall z$.

BY C-G $\int_C z^{-1/2}\sin(\sqrt{z})\,dz = 0.$

PROBLEM 2

$$f(z) = \frac{z}{(z-2)(z+1)} = \frac{A}{z-2} + \frac{B}{z+1} \longrightarrow z = A(z+1) + B(z-2)$$

(i) let $z=2 \longrightarrow A = 2/3$

$z=-1 \longrightarrow B = 1/3$.

$$f(z) = \frac{2}{3(z-2)} + \frac{1}{3(z+1)} = \frac{2}{3z(1-2/z)} + \frac{1}{3z(1+1/z)}$$

$$f(z) = \frac{2}{3z}\sum_{j=0}^{\infty}(2/z)^{j} + \frac{1}{3z}\sum_{j=0}^{\infty}(-1)^{j}(1/z)^{j} \quad \text{converges in } |z|>2$$

$$f(z) = \frac{2}{3z}\left(1 + 2/z + 4/z^{2} + \cdots\right) + \frac{1}{3z}\left(1 - 1/z + 1/z^{2} - 1/z^{3} \cdots\right)$$

(ii) Now integrate term by term since $|z| = 4$ is in zone of convergence. We recall $\int_C \frac{1}{z^{p}}\,dz = 0$ for $p = 2, 3, 4, \ldots$

AND $\int_C \frac{1}{z}\,dz = 2\pi i.$

THUS $\quad I = \displaystyle\int_{|z|=4} f(z)\,dz = \int_{|z|=4} \left(\tfrac{2}{3z} + \tfrac{1}{3z} \right) dz = 2\pi i.$

NOW USE RESIDUE THEOREM.

$$\int_{|z|=4} f(z)\,dz = 2\pi i \left(\mathrm{Res}[f;-1] + \mathrm{Res}(f;2) \right)$$
$$= 2\pi i \left(\frac{z}{2z-1}\Big|_{1} + \frac{z}{2z-1}\Big|_{2} \right) = 2\pi i \left(\frac{+1}{3} + \frac{2}{3} \right) = 2\pi i.$$

PROBLEM 3

(i) • $z = \pi^2$ IS A SIMPLE POLE

~~so~~ $\cancel{z = (2k-1)^2}$

• $\cos(\sqrt{z}) = 1$ WITH $z \neq 0$ IS A SIMPLE POLE

$$\sqrt{z} = 2k\pi \qquad z = 4k^2\pi^2, \quad k = 1, 2, 3, \ldots \text{ SIMPLE POLE.}$$

• NEAR $z = 0$ WE GET

$$\cos(\sqrt{z}) \sim 1 - z/2 + z^2/4! + \cdots$$

$$1 - \cos(\sqrt{z}) \sim z/2 - z^2/24 = z/2 \left(1 - z/12 \right)$$

SO NEAR $z=0$ $\qquad f(z) \approx \dfrac{1}{z\left[z/2 - z^2/24 + \cdots\right]\left[z - \pi^2\right]} \sim c/z^2$

$z = 0$ is a pole of ORDER 2.

(ii) NOW INSIDE $|z| = 10$ WE HAVE A SIMPLE POLE AT $z = \pi^2$ AND A DOUBLE pole AT $z = 0$.

SO $\quad \displaystyle\int_{|z|=10} f(z)\,dz = 2\pi i\, \mathrm{Res}[f;0] + 2\pi i\, \mathrm{Res}[f;\pi^2].$

AND NOW AT $Z = \pi^2$

$$f(z) \approx \frac{\left[1/\pi^2 \left(1 - \cos(\pi)\right) \right]}{z - \pi^2} \approx \frac{1}{2\pi^2 (z - \pi^2)}$$

$$\text{RES}\left[f ; \pi^2 \right] = 1/2\pi^2 .$$

NOW NEAR $Z = 0$.

$$f(z) \approx \frac{1}{z\left[z/2 - z^2/24 \right](z - \pi^2)} = \frac{-1}{\frac{z^2}{2}\left(1 - \frac{z}{12}\right)\left(\pi^2 - z\right)}$$

$$f(z) \approx \frac{-1}{\frac{z^2}{2}\pi^2\left(1 - z/12\right)\left(1 - z/\pi^2\right)} = \frac{-2}{\pi^2 z^2 \left(1 - z/12\right)\left(1 - z/\pi^2\right)}$$

$$f(z) \approx -\frac{2}{\pi^2 z^2}\left[1 + z/12 \right]\left[1 + z/\pi^2 \right] + \cdots$$

$$f(z) \approx -\frac{2}{\pi^2 z^2}\left(1 + z\left(1/12 + 1/\pi^2 \right) \right)$$

$$\text{RES}\left[f ; 0 \right] = -\frac{2}{\pi^2}\left(\frac{1}{12} + \frac{1}{\pi^2} \right).$$

SO
$$\int_C f(z)\, dz = 2\pi i \left[\frac{1}{2\pi^2} - \frac{1}{6\pi^2} - \frac{2}{\pi^4} \right]$$

$$C: |z| = 10 \qquad = 2\pi i \left[\frac{1}{3\pi^2} - \frac{2}{\pi^4} \right].$$

PROBLEM 4

(i) $$I = \int_0^{2\pi} \frac{1}{a+\cos\varphi}\, d\varphi \qquad a > 1.$$

$$\cos\varphi = \frac{z + 1/z}{2} \qquad d\varphi = \frac{dz}{iz}$$

$$I = \int_C \frac{1}{\left(a + \frac{(z+z^{-1})}{2}\right)} \frac{dz}{iz} = -i \int_C \left(\frac{1}{az + \frac{z^2+1}{2}}\right) dz$$

$$I = -2i \int_C \frac{dz}{z^2 + 2az + 1} \qquad C: |z| = 1 \text{ counter-clockwise}$$

ROOTS ARE simple poles at $z_\pm = \frac{-2a \pm \sqrt{4a^2-4}}{2} = -a \pm \sqrt{a^2-1}$.

ONLY ROOT inside is at $z_+ = -a + \sqrt{a^2-1}$ since $|z_-| > 1$

THUS $$I = -2i\left[2\pi i \, \mathrm{RES}(f; z_+)\right] = 4\pi \frac{1}{2z_+ + 2a}$$

$$I = 2\pi \frac{1}{z_+ + a} = \frac{2\pi}{\sqrt{a^2-1}}. \quad \text{VALID FOR } a > 1.$$

(ii) $$I = \frac{1}{2} \mathrm{IM}(J) \qquad J = \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2 + a^2}\, dx.$$

NOW BY JORDAN'S LEMMA, we integrate over semi-circle IN upper 1/2 plane

$$J + \lim_{R\to\infty} \int_{C_R} \frac{z e^{iz}}{z^2 + a^2}\, dz = 2\pi i \, \mathrm{RES}\left(\frac{z e^{iz}}{z^2+a^2}; ia\right)$$

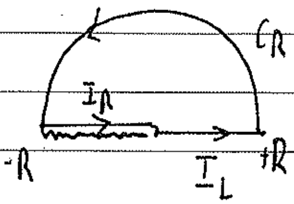
$$= 2\pi i \left(\frac{\cancel{ia}\, e^{-a}}{2\cancel{ia}}\right) = \pi i e^{-a}.$$

$$I = \frac{1}{2} \mathrm{IM}\left(\pi i e^{-a}\right) = \pi e^{-a}/2.$$

PROBLEM 5

(ii). WE LET $\sqrt{z}$ BE PRINCIPAL BRANCH OF $\sqrt{\ }$.

we integrate as follows: over top of branch cut.



$$\lim_{R\to\infty}\left(I_L + I_R + C_R\right) = 2\pi i \operatorname{RES}\left(\frac{\sqrt{z}}{z^2+1}; i\right) = 2\pi i \frac{e^{\pi i/4}}{2i} = \pi e^{\pi i/4}$$

NOW $\lim_{R\to\infty}\left|\int_{C_R}\frac{\sqrt{z}}{z^2+1}\,dz\right| \leq \left(\frac{R^{1/2}}{R^2}\right)\pi R \to 0$ AS $R\to\infty$.

NOW ON I_R: $z = re^{i\pi}$. $dz = e^{i\pi}\,dr$.

SO $I_R = \displaystyle\int_{\infty}^{0}\frac{-\left(r^{1/2}e^{i\pi/2}\right)}{r^2+1}\,dr = i\int_0^{\infty}\frac{r^{1/2}}{r^2+1}\,dr.$

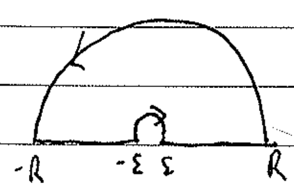
THUS let $I = \displaystyle\int_0^{\infty}\sqrt{x}/_{x^2+1}\,dx.$

THEN $(1+i)\,I = \pi(1+i)/\sqrt{2}$.

$\longrightarrow I = \pi/\sqrt{2}$.

(i) $I = \dfrac{1}{2}\operatorname{IM}\left(\displaystyle\int_{-\infty}^{\infty}\frac{e^{ix}}{x(x^2+1)}\,dx\right)$. LET $J = \text{P.V.}\displaystyle\int_{-\infty}^{\infty}\frac{e^{ix}}{x(x^2+1)}\,dx.$

WE NEED INDENTED CONTOUR AS SHOWN:



$$J + \lim_{\varepsilon\to 0}\int_{C_\varepsilon}\frac{e^{iz}}{z(z^2+1)}\,dz = 2\pi i \operatorname{RES}\left(\frac{e^{iz}}{z(z^2+1)}; i\right)$$

$$J - i\pi = \cancel{2}\pi \cancel{i}\left[\frac{e^{-1}/i}{\cancel{2}\cancel{i}}\right] = \frac{\pi e^{-1}}{i} = -i\pi e^{-1}.$$

so $J = i\pi\left(1-e^{-1}\right).$

$$I = \frac{1}{2}\,\mathrm{IM}(J)$$

so $I = \frac{\pi}{2}\left(1-e^{-1}\right).$

Be sure that this examination has 2 pages.

The University of British Columbia

Final Examinations - December 2011

Mathematics 305

M. Ward

Closed book examination. No notes, texts, or calculators allowed. Time: $2\frac{1}{2}$ hours

Marks

[30] **1.** Identify whether each of the following statements are true or false. You must give reasons for your answers to receive credit. (Hint: very little calculation is needed to solve these).

(i) $\text{Log}(z^2) = 2\,\text{Log}(z)$.

(ii) $|e^{-z^3}| \le 1$ when $|\text{Arg}(z)| \le \pi/2$.

(iii) $f(z) = |z|^2$ is differentiable at $z = 0$ but is not analytic at $z = 0$.

(iv) If $f(z) = u + iv$ is an entire function, then uv is a harmonic function.

(v) Let $f(z) = z(z - i)$. Then $\max_{|z| \text{ in } D} |f(z)| = 2$, where D is the region $|z| \le 1$.

(vi) Suppose that $f(z)$ is an entire function that satisfies $|f(z)| > 1$ for all z. Then, $f(z)$ must be the constant function.

[10] **2.** Let $f(z) = (z^2+1)^{1/3}$. We seek to construct a branch of $f(z)$ that is analytic in $|z| < 1$, with branch cuts on portions of the imaginary axis, and that satisfies $f(0) = 1$.

(i) Show how to construct this branch by specifying the range of angles $\arg(z - i)$ and $\arg(z + i)$ appropriately.

(ii) Next, define this branch of $f(z)$ in terms of the principal value of some logarithm function.

(iii) For this branch of $f(z)$ calculate $f(2 + 2i)$.

Continued on page 2

[16] **3.** Calculate each of the following integrals over the simple closed curve C:

(i) $\int_C z^5/(z^6+2z)\,dz$ where C is the counter-clockwise circle $|z|=2$.

(ii) $\int_C z^3 e^{1/z}\,dz$ where C is the counter-clockwise circle $|z|=1$.

(iii) $\int_C z^{-4}\sin(3z)\,dz$ where C is the counter-clockwise circle $|z|=1$.

(iv) $\int_C (z-2i)^{-2}\,\mathrm{Log}(z)\,dz$, where $\mathrm{Log}(z)$ denotes the principal value of the logarithm function, and C is the counter-clockwise circle $|z-2i|=1$.

[14] **4.** Consider the function $f(z)$ defined by

$$f(z)=\frac{\sin(iz/4)}{z^2(1-e^z)}\,.$$

(i) Identify and then classify all of the singular points of $f(z)$ in the complex plane.

(ii) Calculate the first two terms in the Laurent expansion of $f(z)$ in powers of z which converges in $0<|z|<r_1$. What is the radius r_1 of convergence of this series?

(iii) Calculate $I=\int_C f(z)\,dz$ where C is the counter-clockwise circle $|z|=1$.

[20] **5.** Calculate the following integrals in as explicit a form as you can:

$$\text{(i)}\quad I=\int_0^\infty \frac{x^{1/3}}{x^2-4x+8}\,dx \qquad\qquad \text{(ii)}\quad I=\int_0^{2\pi}\frac{\cos(n\theta)}{1+k\cos(\theta)}\,d\theta\,.$$

In (ii), n is a non-negative integer and k is real with $k^2<1$. (Hint: In (ii) it may be helpful to first write $\cos(n\theta)=\mathrm{Re}(e^{in\theta})$.)

[10] **6.** Suppose that $p(z)=a_0+a_1z+\cdots+a_Nz^N$ is a polynomial of degree $N\geq 2$ with $a_N\neq 0$ and $a_0\neq 0$. Suppose that $z_1,\ldots,z_N$ are distinct roots of $p(z)=0$. By using residue theory applied to the integral

$$\int_C \frac{p'(z)}{z^2 p(z)}\,dz\,,$$

where the contour C is to be chosen appropriately, derive an explicit formula for the sum

$$S=\sum_{j=1}^N \frac{1}{z_j^2}\,,$$

in terms of some of the coefficients $a_0,\ldots,a_N$ of the polynomial. Does your formula still work if the roots of the polynomial $p(z)$ are not distinct?

[100] **Total Marks**

The End

MATH 305 EXAM

PROBLEM 1

(i) $\log(z^2) = 2\log(z)$ is FALSE

let $z = e^{3\pi i/4}$ $\qquad \log(z^2) = -i\pi/2$

$\qquad\qquad 2\log(z) = 3\pi i/2$

(ii) $|e^{-z^3}| \leq 1$ WHEN $|\arg z| \leq \pi/2$ FALSE.

let $z = re^{i\varphi}$ so $|e^{-z^3}| = |e^{-r^3\cos(3\varphi) - ir^3\sin(3\varphi)}| = e^{-r^3\cos(3\varphi)} \leq 1$

WHEN $\cos(3\varphi) \geq 0 \longrightarrow -\pi/2 \leq 3\varphi \leq \pi/2 \longrightarrow -\pi/6 \leq \varphi \leq \pi/6$.

(iii) TRUE:

$f = x^2 + y^2 + i0$ so $u = x^2 + y^2$, $v = 0$

$u_x = v_y$, $u_y = -v_x$ AT $(0,0)$ ONLY.

. f is differentiable at $z = 0$

. NOT analytic at $z = 0$.

(iv) $f(z) = z(z-i)$

BY MAX MODULUS PRINCIPLE

$$\max_{z \text{ in } D} |f(z)| = \max_{|z|=1} |z(z-i)| = \max_{|z|=1} |z-i| = 2,$$ OCCURS WHEN $z = -i$

(v) TRUE. THERE is NO POINT z_0 WHERE $f(z_0) = 0$. (SINCE $|f(z_0)| > 1$).

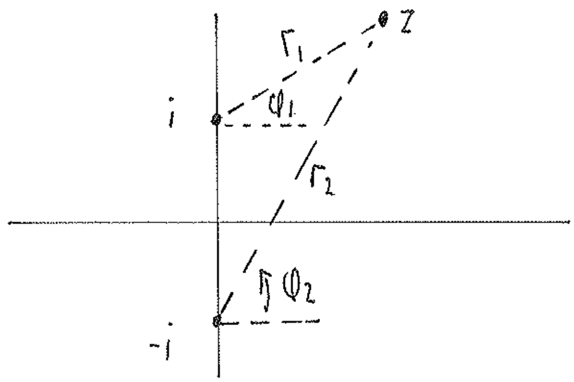
THUS $g(z) = 1/f(z)$ is AN ANALYTIC FUNCTION THAT SATISFIES

$|g(z)| \leq 1 \ \forall z \longrightarrow$ BY Liouville's theorem, $g(z)$ AND

hence $f(z)$ is the constant FUNCTION.

PROBLEM 2

(i) $f(z) = (z+i)^{1/3}(z-i)^{1/3} = (r_1 r_2)^{1/3} e^{i(\varphi_1+\varphi_2)/3}$.



CHOOSE $-\pi/2 \le \varphi_2 \le 3\pi/2$

$-3\pi/2 \le \varphi_1 \le \pi/2$

THEN AT $z=0$; $\varphi_1 = -\pi/2$, $\varphi_2 = \pi/2$

$\longrightarrow f(0) = 1$.

(ii) $(z^2+1)^{1/2} = e^{\frac{1}{2}\log(z^2+1)}$

TAKE $f(z) = e^{\frac{1}{2}\log(z^2+1)}$

NOW THIS IS ANALYTIC EXCEPT where $\mathrm{RE}(z^2)+1<0$ $\uparrow$ $|y|\ge 1$.

$\mathrm{IM}(z^2)=0 \rightarrow z = iy$

(iii) NOW FOR $z = 2+2i$.

$$f(2+2i) = e^{\frac{1}{2}\log(1+8i)} = e^{\frac{1}{2}\ln[65] + \frac{i}{2}\tan^{-1}(8)}$$

$$f(2+2i) = \sqrt{65}\, e^{\frac{i}{2}\tan^{-1}(8)}$$

PROBLEM 3

(i) $I = \int_C \frac{z^5}{z^6+2z}\, dz \qquad C: |z| = 2.$

singularities at $z = 0$ AND $z^6 = -2 \rightarrow |z| = 2^{1/6} < 2.$

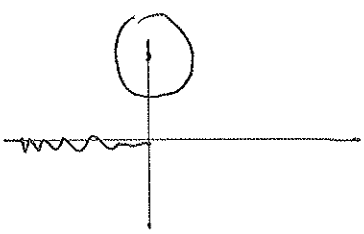
ALL singularities inside C.

so $I \sim \int_C \frac{z^5}{z^6}\, dz = 2\pi i.$

(ii) $I = \int_C z^3 \left(1 + 1/z + 1/2!\,z^2 + 1/3!\,z^3 + 1/4!\,z^4\right) dz = \frac{2\pi i}{4!} = \frac{\pi i}{12}$

(iii) $I = \int_C \frac{1}{z^4} \sin 3z = \int_C \frac{1}{z^4}\left(3z - \frac{(3z)^3}{3!}\right) = -\frac{27}{6}(2\pi i) = -9\pi i$

(iv) $I = \int_C (z-2i)^{-2} \log(z)\, dz = 2\pi i \left[\frac{d}{dz} \log(z)\right]\Big|_{2i} = \frac{2\pi i}{2i} = \pi.$



PROBLEM 4

(i) $z = 0$ is a double pole, $z = 2m\pi i \quad m = \cancel{0}, \pm 1, \pm 2, \ldots$

NOW IF m = ODD THEN simple pole

m = even THEN Removable singularity.

(ii) RADIUS OF CONVERGENCE is 2π.

A) $z \to 0$,

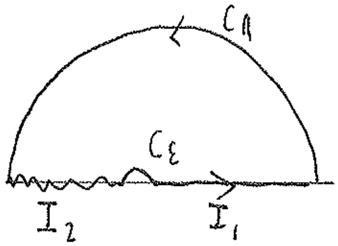
$$f(z) \sim \frac{(iz/4) - (iz/4)^3/3!}{z^2\left(1 - \left(1 + z + \frac{z^2}{2} + \ldots\right)\right)} = \frac{(iz/4)}{z^2}\frac{\left[1 + z^2/96\right]}{\left[-z - \frac{z^2}{2} + \ldots\right]}$$

$$f(z) \sim \frac{i}{-4z^2}\left[1 + z^2/96\right]\left[1 - \frac{z}{2}\ldots\right] \sim \frac{i}{-4z^2}\left(1 - \frac{z}{2}\right) \sim \frac{i}{-4z^2} + \frac{i}{8z} + \ldots$$

THUS $a_{-1} = i/4 \longrightarrow \int_C f(z)\,dz = 2\pi i(i/4) = -\pi/4$.

PROBLEM 5

(i)



$$\lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \left(\int_{C_\varepsilon} + \int_{C_R} + \int_{I_1} + \int_{I_2}\right) = 2\pi i \operatorname{RES}\Big| \quad \Big)$$

NOW $\int_{C_\varepsilon} \to 0$, $\int_{C_R} \to 0$.

$$z^2 - 4z + 8 = 0 \longrightarrow z_\pm = \frac{4 \pm \sqrt{16 - 32}}{2} = 2 \pm 2i.$$

NOTICE THAT $z_+ = 2 + 2i$ is inside C. $\quad z_+ = \sqrt{8}e^{i\pi/4}$.

THUS $\quad 2\pi i \operatorname{RES}\left(\quad ; z_+\right) = 2\pi i\left(\frac{z_+^{1/3}}{2z_+ - 4}\right) = \frac{2\pi i \sqrt{8}^{1/3} e^{i\pi/12}}{4i}$

NOW ON I_2: $z = re^{i\pi}$. $dz = -dr \longrightarrow \int_{I_2} = -\int_\infty^0 \frac{r^{1/3}e^{i\pi/3}}{r^2 + 4r + 8}\,dr$.

THUS $\quad e^{i\pi/3} J + I = \frac{\pi}{2}\sqrt{8}^{1/3} e^{i\pi/12} \qquad J = \int_0^\infty \frac{r^{1/3}}{r^2 + 4r + 8}$.

SOLVE FOR I. $\qquad I = \int_0^\infty \frac{r^{1/3}}{r^2 - 4r + 8}\,dr$.

$$J + e^{-i\pi/3} I = \frac{\pi}{2} \sqrt{8}^{1/3} e^{-i\pi/4}$$

THEN $$\mathrm{IM}\left(e^{-i\pi/3} I \right) = \frac{\pi}{2} \sqrt{8}^{1/3} \, \mathrm{IM}\left(e^{-i\pi/4} \right)$$

$$I \sin\left(\pi/3 \right) = \frac{\pi}{2} \sqrt{8}^{1/3} \sin\left(\pi/4 \right)$$

$$I \frac{\sqrt{3}}{2} = \frac{\pi}{2} \sqrt{8}^{1/3} \frac{\sqrt{2}}{2} = \frac{\pi}{4} \sqrt{2} \left[8^{1/6} \right]$$

SO $$I \sqrt{3} = \frac{\pi}{2} \, 2^{1/2} (2^3)^{1/6} = \frac{\pi}{2} \, 2 = \pi.$$

SO $I = \pi/\sqrt{3}$.

(ii) $$I = \int_0^{2\pi} \frac{\cos(n\varphi)}{1 + \kappa \cos\varphi} d\varphi = \mathrm{RE}\left(\int_0^{2\pi} \frac{e^{in\varphi}}{1 + \kappa \cos\varphi} d\varphi \right).$$

Let $z = e^{i\varphi}$ so $$I = \mathrm{RE}\left(\int_C \frac{z^n}{1 + \frac{\kappa}{2}(z + 1/z)} \frac{1}{iz} dz \right)$$

$$I = \mathrm{RE}\left[-i \int_C \frac{z^n}{\frac{\kappa}{2} z^2 + z + \frac{\kappa}{2}} dz \right] = \frac{2}{\kappa} \mathrm{RE}\left[-i \int_C \frac{z^n}{z^2 + \frac{2z}{\kappa} + 1} dz \right].$$

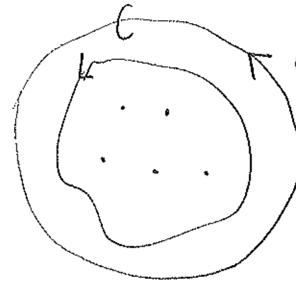
NOW pole at $$z_\pm = \frac{-\frac{2}{\kappa} \pm \sqrt{\frac{4}{\kappa^2} - 4}}{2} = -\frac{1}{\kappa} \pm \sqrt{\frac{1}{\kappa^2} - 1}$$ z_+ inside C.

$$I = \frac{2}{\kappa} \mathrm{RE}\left[-i \, 2\pi i \, \mathrm{RES}\left(\frac{z^n}{z^2 + \frac{2z}{\kappa} + 1} ; z_+ \right) \right] = \frac{4\pi}{\kappa} \mathrm{RE}\left(\frac{z_+^n}{2z_+ + \frac{2}{\kappa}} \right)$$

NOW $2z_+ + 2/\kappa = 2\sqrt{1/\kappa^2 - 1}$ $$I = \frac{2\pi}{\kappa} \frac{\left[-\frac{1}{\kappa} + \sqrt{\frac{1}{\kappa^2} - 1} \right]^n}{\sqrt{1/\kappa^2 - 1}}$$

So $$I = \frac{2\pi}{\kappa^n} \frac{\left[-1+\sqrt{\kappa^2-1}\right]^n}{\sqrt{1-\kappa^2}}.$$

PROBLEM 6



LET C enclose $z=0$ AND all zeroes of $p(z)$.

THEN SINCE $\left|\frac{p'(z)}{p(z)}\right| \le d/R$ FOR $|z| = R >> 1$

WE DEFORM $\int_C$ TO $\int_{C_R}$ BY CIT AND LET $R \to \infty$

$$\int_C \frac{p'}{z^2 p}\,dz = \lim_{R\to\infty} \int_{C_R} \frac{p'}{z^2 p}\,dz = 0 \quad \text{SINCE } \left|\int_{C_R} \frac{p'}{z^2 p}\,dz\right| \le \frac{d}{R}\frac{1}{R^2} 2\pi R \to 0$$

AS $R \to \infty$.

THEN BY residue theorem

$$0 = \int_C \frac{p'}{z^2 p}\,dz = 2\pi i \sum_{j=1}^{N} \underbrace{\operatorname{RES}\left(\frac{p'}{z^2 p}; z_j\right)}_{1/z_j^2} + 2\pi i \operatorname{RES}\left(\frac{p'}{z^2 p}; 0\right).$$

THUS (*) $$\sum_{j=1}^{N} 1/z_j^2 = -\operatorname{RES}\left[\frac{p'}{z^2 p}; 0\right] \qquad z=0 \text{ is a double pole.}$$

Use L-series,

$$\frac{p'}{z^2 p} = \frac{a_1 + 2a_2 z + \cdots}{z^2(a_0 + a_1 z + \cdots)} = \frac{[a_1 + 2a_2 z][1 - a_1/a_0\, z]}{a_0 z^2}$$

$$= \frac{a_1}{a_0 z^2} + \frac{1}{z}\left[\frac{-a_1^2}{a_0^2} + \frac{2a_2}{a_0}\right].$$

THUS $$\boxed{\sum_{j=1}^{N} 1/z_j^2 = a_1^2/a_0^2 - 2a_2/a_0}$$

check $p = (2z-1)(4z+1)(z-1) = 0$.

$z_1 = 1/2$, $z_2 = -1/4$, $z_3 = 1 \to \sum 1/z_j^2 = 21$

NOW $p = (8z^2 - 2z - 1)(z-1) = 8z^3 - 2z^2 - z - 8z^2 + 2z + 1 = 8z^3 - 10z^2 + z + 1$.

$a_0 = 1$, $a_1 = 1$, $a_2 = -10 \to$ WORKS