MATH 305: MIDTERM 1: October 14th, 2011 (M. WARD)

Closed Book and Notes. 50 minutes. Total 50 points

PROBLEM 1: (12 Points) Find all solutions in the complex plane to the following:

$$\text{(i)}\quad z^4 = 8iz\,; \qquad \text{(ii)}\quad \sin z = \cosh 2\,; \qquad \text{(iii)}\quad e^{1/z} = e^{10}(1+i)\,.$$

(Hint: you will need the identity $\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$)

PROBLEM 2: (8 Points)

Let $f(z) = y^3 + 3x^2y - 3y + i\left(x^3 + 3xy^2 - 3x\right)$ where $z = x + iy$. Where is $f(z)$ differentiable in the complex plane? Where is $f(z)$ analytic? Explain your reasoning carefully.

PROBLEM 3: (18 Points) Establish the validity of each of the following statements. If it is true, then provide a proof. If it is false, carefully explain why.

i) $\text{Arg}(z^2) = 2\text{Arg}(z)$ for all $z \neq 0$.

ii) $\log(e^z) = z$ for all z.

iii) $|e^{z^2}| \leq e^{|z|^2}$ for all z.

iv) $\text{Re}\,(i/\bar{z}) = -\text{Im}(z)/|z|^2$ for all $z \neq 0$.

v) If $f(z) = u(x,y) + iv(x,y)$ is an entire function of $z = x + iy$, then $e^u \cos v$ is a harmonic function.

vi) $\text{Log}(z^5)$ is an analytic function everywhere in the complex z-plane except on the negative real axis.

PROBLEM 4: (12 Points) Find the image of the set S under the map $w = f(z)$ for each of the following:

i) $S = \left\{ z \;\middle|\; |z - i| \leq 2 \right\}$ and $f(z) = 2i(z+1)$

ii) $S = \left\{ z \;\middle|\; 1 \leq \text{Re}(z) \leq \frac{\pi}{2} + 1 \text{ with } \text{Im}(z) \geq 0 \right\}$ and $f(z) = e^{2i(z-1)}$.

MATH 305 MIDTERM

PROBLEM 1

(i) $z^4 = 8iz$

ONE ROOT IS $z=0$ SO THAT

$$z^3 = 8i = 8e^{\pi i/2}.$$

NOW PUT $z = \rho e^{i\phi}$ SO THAT

$$\rho^3 e^{3i\phi} = 8e^{\pi i/2}$$

HENCE TAKING MODULI $\longrightarrow \rho = 2$

AND $3\phi = \pi/2 + 2K\pi \quad K = 0,1,2.$

IN SUMMARY ROOTS ARE

$$z = 0 \quad \text{AND} \quad z_K = 2e^{i(\pi/6 + 2K\pi/3)}$$
$$K = 0,1,2.$$

(ii) $\sin z = \cosh 2$

WE WRITE

$$\sin(x+iy) = \sin x \cosh y + i\cos x \sinh y$$
$$= \cosh 2.$$

THUS $\cosh 2 = \sin x \cosh y$

$$0 = \cos x \sinh y$$

WE MUST HAVE $y \neq 0$ SO

$$x_n = (2n+1)\pi/2 \quad n = 0, \pm 1, \pm 2 ..$$

BUT we need $\sin(x_n) = 1 > 0.$

HENCE $x_n = (2n+1)\pi/2 \quad n = 0, \pm 2, \pm 4, ..$

AND $\cosh y = \cosh 2 \longrightarrow y = \pm 2.$

HENCE $z_n = \dfrac{(2n+1)\pi}{2} \pm 2i$

$$n = 0, \pm 2, \pm 4, ..$$

(iii) $e^{1/z} = e^{10}(1+i) = \sqrt{2}\, e^{10 + i\pi/4}$

let $w = 1/z$. THEN

$$e^w = \sqrt{2}\, e^{10}\, e^{i\pi/4}.$$

$$w = \log\left[\sqrt{2}\, e^{10}\, e^{i\pi/4}\right]$$

SO $w_K = \ln\left(\sqrt{2}\, e^{10}\right) + i\left(\dfrac{\pi}{4} + 2K\pi\right)$

$$K = 0, \pm 1, \pm 2, ..$$

THUS ROOTS ARE

$$z_K = \frac{1}{w_K} = \frac{1}{\ln\left(\sqrt{2}\, e^{10}\right) + i\left(\dfrac{\pi}{4} + 2K\pi\right)}$$

NOTICE THAT $|z_K| \longrightarrow 0$ AS $|K| \longrightarrow \infty.$

PROBLEM 2 $\quad f = y^3+3x^2y-3y+i(x^3+3xy^2-3x).$

$u = y^3+3x^2y-3y \qquad\qquad v = x^3+3xy^2-3x$

$u_x = 6xy \qquad\qquad v_y = 6xy$

$u_y = 3y^2+3x^2-3 \qquad\qquad v_x = 3x^2+3y^2-3$

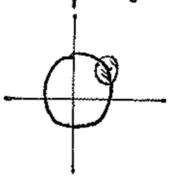
NOW $\quad u_x = v_y \;\rightarrow\; 6xy = 6xy \quad$ Always true

$\qquad u_y = -v_x \;\rightarrow\; 6x^2+6y^2 = 6 \;\rightarrow\; x^2+y^2=1$

THUS CR EQUATIONS HOLD ON CIRCLE $x^2+y^2=1.$

- f is differentiable at each point on $|z|=1$

z-plane



- BUT f IS nowhere analytic since we cannot have ANY small disk centered at a point on $|z|=1$ FOR which f is differentiable everywhere inside disk

PROBLEM 3

(i) $\quad \mathrm{ARG}(z^2) = 2\,\mathrm{ARG}(z)$ is FALSE.

LET $z = e^{3\pi i/4}$. THEN $\mathrm{ARG}(z^2) = \mathrm{ARG}(e^{3\pi i/2}) = -\pi/2$

$2\,\mathrm{ARG}(z) = 2\,\mathrm{ARG}(e^{3\pi i/4}) = 2(3\pi/4) = 3\pi/2.$

(ii) $\quad \log(e^z) = z$ IS FALSE IN GENERAL.

NOTICE LHS IS MULTI-VALUED, WHILE RHS IS single-valued.

→

IN FACT IF $z = x + iy$ THEN

$$\log(e^z) = \log(e^{x+iy}) = \log(e^x e^{iy}) = \ln(e^x) + i(y + 2\pi k)$$

$$k = 0, \pm 1, \pm 2, \ldots$$

HENCE $\log(e^z) = z + 2\pi k$

(iii) $|e^{z^2}| \leq e^{|z^2|}$ IS TRUE.

LET $z = x + iy$, THEN $|e^{z^2}| = |e^{x^2 - y^2 + 2ixy}| = e^{x^2 - y^2} \leq e^{x^2 + y^2}$.

HENCE $|e^{z^2}| \leq e^{x^2+y^2} = e^{|z|^2}$.

(iv) $\mathrm{RE}\left(\frac{i}{\bar{z}}\right) = -\frac{\mathrm{IM}(z)}{|z|^2}$ FOR ALL $z \neq 0$ IS TRUE.

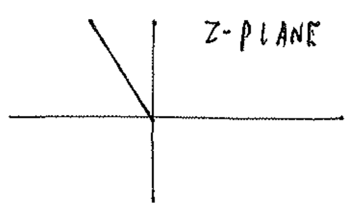
WE WRITE $\mathrm{RE}\left(\frac{i}{\bar{z}} \frac{z}{z}\right) = \mathrm{RE}\left(\frac{iz}{|z|^2}\right) = \frac{1}{|z|^2} \mathrm{RE}\left[i(x+iy)\right] = -\frac{y}{|z|^2}$.

THUS $\mathrm{RE}\left(\frac{i}{\bar{z}}\right) = -\frac{y}{|z|^2} = -\frac{\mathrm{IM}(z)}{|z|^2}$.

(vi) FALSE $\mathrm{LOG}(z^5)$ IS ANALYTIC EXCEPT ON PATHS FOR WHICH $\mathrm{RE}(z^5) < 0$ AND $\mathrm{IM}(z^5) = 0$.

IF WE SET $\mathrm{IM}(z^5) = 0 \rightarrow \sin(5\varphi) = 0 \rightarrow \varphi = \frac{n\pi}{5}$, $n = 0, \ldots, 9$.

CHOOSE THE PATH WITH $n = 3$. THEN $\varphi = 3\pi/5$ AS SHOWN.



ON THIS PATH,

$$\mathrm{RE}(z^5) = |z|^5 \cos\left(\frac{3\pi}{5} 5\right) = -|z|^5 < 0.$$

THIS IS A PATH, OTHER THAN $z < 0$, z REAL FOR WHICH $\mathrm{LOG}(z^5)$ IS NOT ANALYTIC.

(v) TRUE IF $f(z)$ IS ANALYTIC $\rightarrow g(z) = e^{f(z)} = e^{u+iv}$ IS ANALYTIC.

$\Longrightarrow \mathrm{RE}[g(z)] = e^u \cos v$ IS HARMONIC (since Real part of analytic function)

PROBLEM 4

(i) LET $F(z) = 2i(z+1)$ $\quad S = \{ z \mid |z-i| \le 2 \}$.

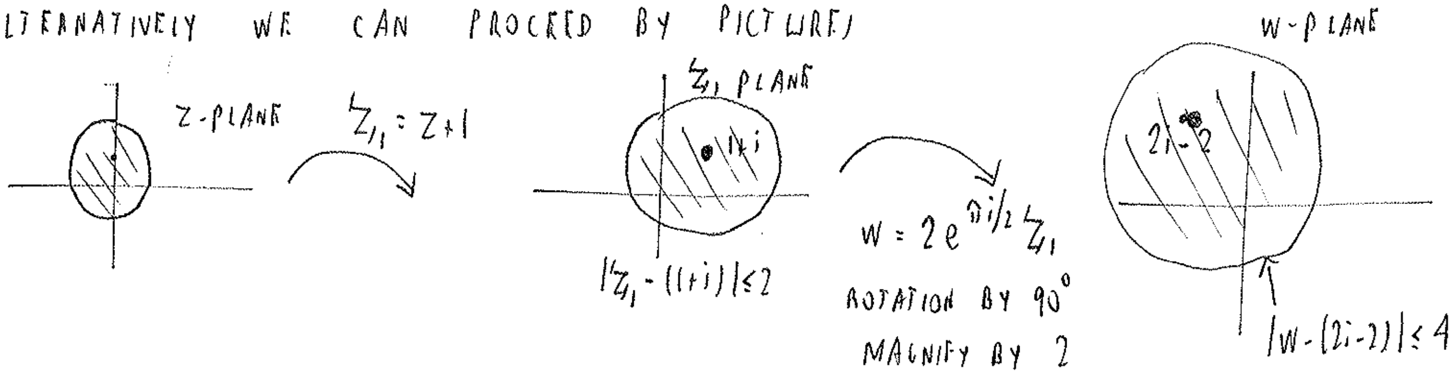
DEFINE $w = 2i(z+1)$ SO $z = -1 + w/2i$. $\rightarrow$ $|z-i| \le 2$ YIELDS $\left| -1 + \frac{w}{2i} - i \right| \le 2$.

HENCE $S' = \{ w \mid | w/2i - 1 - i | \le 2 \}$.

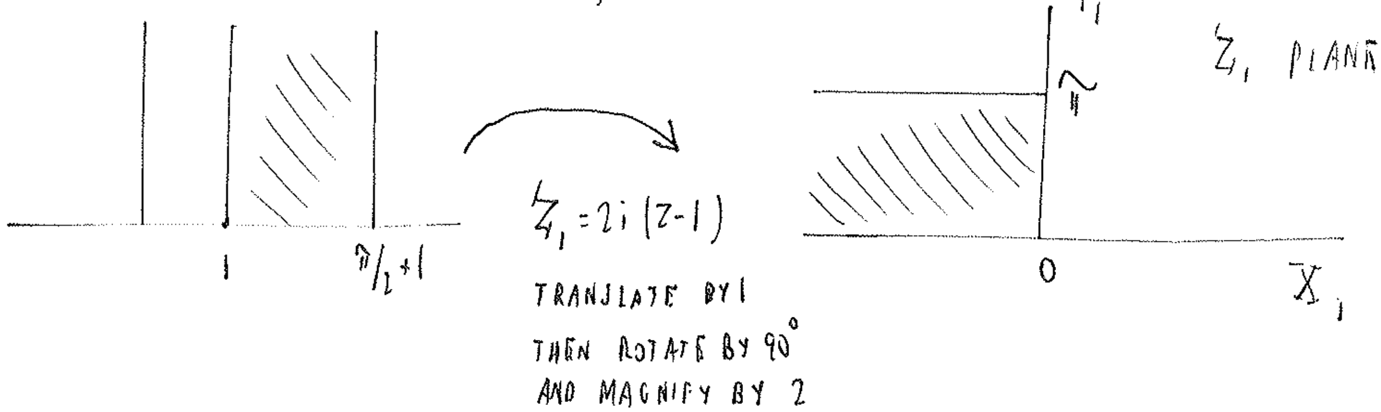
NOW $\left| \dfrac{w + 2i(-1-i)}{2i} \right| = \dfrac{1}{2} | w - (2i-2) | \le 2 \rightarrow | w - (2i-2) | \le 4$.

HENCE $S' = \{ w \mid | w - (2i-2) | \le 4 \}$.

ALTERNATIVELY WE CAN PROCEED BY PICTURES



(ii) $S = \{ z \mid 1 \le \mathrm{Re}(z) \le \pi/2 + 1 \}$ WITH $\mathrm{Im}\, Z \ge 0$



NOW LET $w = e^{Z_1}$ SO $u + iv = e^{X_1} \cos y + i e^{X_1} \sin y$

THIS GIVES $U = e^{X_1} \cos Y_1 \quad 0 \le Y_1 \le \pi$

$V = e^{X_1} \sin Y_1 \quad -\infty < X_1 \le 0$ $\quad \Longrightarrow \quad U^2 + V^2 = (e^{X_1})^2, \; V \ge 0$.

NOW e^{X_1} RANGES FROM $(0, 1)$

AS $-\infty < X_1 \le 0$ $\quad \Longrightarrow$

SO $S' = \{ w \mid |w| \le 1$ WITH $\mathrm{Im}\, w \ge 0 \}$.