1. Allen Cahn Equation

Energy: Phase transition model.
Let $\Omega \subseteq \mathbb{R}^N$ of a “binary mixture”: Two materials coexisting (or one material in two phases). We can take as an example of this: Water in solid phase (+1), and water in liquid phase (−1). The configuration of this mixture in $\Omega$ can be described as a function

$$u^*(x) = \begin{cases} 
+1 & \text{in } \Lambda \\
-1 & \text{in } \Omega \setminus \Lambda
\end{cases}$$

where $\Lambda$ is some open subset of $\Omega$. We will say that $u^*$ is the phase function.

Consider the functional

$$\frac{1}{4} \int_{\Omega} (1 - u^2)^2$$

minimizes if $u = 1$ or $u = -1$. Function $u^*$ minimize this energy functional. More generally this well happen for

$$\int_{\Omega} W(u) dx$$

where $W(u)$ minimizes at 1 and −1, i.e. $W(+1) = W(-1) = 0$, $W(x) > 0$ if $x \neq 1$ or $x \neq -1$, $W''(+1), W''(-1) > 0$.

1.1. The gradient theory of phase transitions. Possible configurations will try to make the boundary $\partial \Lambda$ as nice as possible: smooth and with small perimeter. In this model the step phase function $u^*$ is replaced by a smooth function $u_\varepsilon$, where $\varepsilon > 0$ is a small parameter, and

$$u_\varepsilon(x) \approx \begin{cases} 
+1 & \text{inside } \Lambda \\
-1 & \text{inside } \Omega \setminus \Lambda
\end{cases}$$

and $u_\varepsilon$ has a sharp transition between these values across a “wall” of width roughly $O(\varepsilon)$: the interface (thin wall).
In grad theory of phase transitions we want minimizers, or more generally, critical points \( u_\varepsilon \) of the functional
\[
J_\varepsilon(u) = \varepsilon \int_\Omega \left( \frac{\|\nabla u\|^2}{2} + \frac{1}{\varepsilon} \int_\Omega \frac{(1-u^2)^2}{4} \right)
\]
Let us observe that the region where \((1-u^2) > \gamma > 0\) has area of order \(O(\varepsilon)\) and the size of the gradient of \(u_\varepsilon\) in the same region is \(O(\varepsilon^2)\) in such a way \(J(u_\varepsilon) = O(1)\). We will find critical points \(u_\varepsilon\) to functionals of this type so that \(J(u_\varepsilon) = O(1)\).

Let us consider more generally the case in which the container isn’t homogeneous so that distinct costs are paid for parts of the interface
\[
J_\varepsilon(u) = \int_\Omega \left( \frac{\|\nabla u\|^2}{2} + \frac{1}{\varepsilon} \int_\Omega \frac{(1-u^2)^2}{4} \right) a(x) \, dx
\]
a(x) non-constant, \(0 < \gamma \leq a(x) \leq \beta\) and smooth.

1.2. **Critical points of \(J_\varepsilon\).** First variation of \(J_\varepsilon\) at \(u_\varepsilon\) is equal to zero.
\[
\frac{\partial}{\partial t} J_\varepsilon(u_\varepsilon + t\varphi) \bigg|_{t=0} = DJ_\varepsilon(u_\varepsilon)[\varphi] = 0, \quad \forall \varphi \in C^\infty_c(\Omega)
\]
We have
\[
J_\varepsilon(u_\varepsilon + t\varphi) = 0
\]
i.e. \(\forall \varphi \in C^\infty_c(\Omega)\)
\[
0 = DJ_\varepsilon(u_\varepsilon)[\varphi] = \varepsilon \int_\Omega (\nabla u_\varepsilon \nabla \varphi) a + \frac{1}{\varepsilon} \int_\Omega W'(u_\varepsilon) \varphi a.
\]
If \(u_\varepsilon \in C^2(\Omega)\)
\[
\int_\Omega \left( -\varepsilon \nabla \cdot (a \nabla u_\varepsilon) + \frac{a}{\varepsilon} W'(u_\varepsilon) \right) \varphi = 0, \quad \forall \varphi \in C^\infty_c(\Omega)
\]
This give us the weighted Allen Cahn equation in \(\Omega\)
\[
-\varepsilon \nabla \cdot (a \nabla u) + \frac{a}{\varepsilon} u(1-u^2) = 0 \text{ in } \Omega.
\]
We will assume in the next lectures \(\Omega = \mathbb{R}^N\), where \(N = 1\) or \(N = 2\).
If \(N = 1\) weight Allen Cahn equation is
\[
\varepsilon^2 u'' + \varepsilon^2 u' \frac{a'}{a} + (1-u^2)u = 0, \text{ in } (-\infty, \infty).
\]
Look for \(u_\varepsilon\) that connects the phases \(-1\) and \(+1\) from \(-\infty\) to \(\infty\).
Multiplying (1.1) against \(u'\) and integrating by parts we obtain
\[
\int_{-\infty}^{\infty} \frac{d}{dx} \left( \frac{\varepsilon u^2}{2} - \frac{(1-u^2)^2}{4} \right) + \int_{-\infty}^{\infty} \frac{a'}{a} u'^2 = 0
\]
Assume that $u(-\infty) = -1, u(\infty) = 1, u'(-\infty) = u'(\infty) = 0, a > 0$, then (1.2) implies that

\[
\frac{(1 - u^2)^2}{4} + \int_{-\infty}^{\infty} \frac{a'}{a} u^2 = 0
\]

from which we conclude that unless $a$ is constant, we need $a'$ to change sign. So: if $a$ is monotone and $a' \neq 0$ implies the non-existence of solutions as we look for. We need the existence (if $a' \neq 0$) of local maximum or local minimum of $a$. We will prove that under some general assumptions on $a(x)$, given a local max. or local min. $x_0$ of $a$ non-degenerate ($a''(x_0) \neq 0$), then a solution to (1.1) exists, with transition layer.

We consider first the problem with $a \equiv 1, \varepsilon = 1$:

(1.3) \quad $W'' + (1 - W^2)W = 0, \quad W(-\infty) = -1, W(\infty) = 1$.

The solution of this problem is

\[
W(t) = \tanh \left( \frac{t}{\sqrt{2}} \right)
\]

This solution is called “the heteroclinic solution”, and it’s the unique solution of the problem (1.3) up to translations.

**Observation 1.1.** This solution exists also for the problem

(1.4) \quad $w'' + f(w) = 0, \quad w(-\infty) = -1, w(\infty) = 1$

where $f(w) = -W'(w)$. Solutions satisfies $\frac{w^2}{2} - W(w) = E$, where $E$ is constant, and $w(-\infty) = -1$ and $w(\infty) = 1$ if and only if $E = 0$.

This implies

\[
\int_0^w \frac{ds}{\sqrt{2w(s)}} = t
\]

$t(w) \to \infty$ if $w \to 1$, and $t(w) \to -\infty$ if $w \to -1$, so the previous relation defines a solution $w$ such that $w(0) = 0$, and $w(-\infty) = -1, w(\infty) = 1$.

If we write the Hamiltonian system associated to the problem we have:

\[
p' = -f(q), \quad q' = p.
\]

Trajectories lives on level curves of $H(p, q) = \frac{p^2}{2} - W(q)$, where $W(q) = \frac{(1-q^2)^2}{4}$.

Let $x_0 \in \mathbb{R}$ (we will make assumptions on this point). Fix a number $h \in \mathbb{R}$ and set

\[
v(t) = u(x_0 + \varepsilon(t + h)), \quad v'(t) = \varepsilon u'(x_0 + \varepsilon(t + h))
\]
Using (1.1), we have

\[
\varepsilon^2 u''(x_0 + \varepsilon(t + h)) = -\varepsilon^2 \frac{a'}{a} u'(x_0 + \varepsilon(t + h)) - (1 - v^2(t))v(t)
\]

so we have the problem

(1.5)

\[
v''(t) + \varepsilon \frac{a'}{a} (x_0 + \varepsilon(t+h))v'(t) + (1 - v(t)^2)v(t)^2 = 0, \quad w(-\infty) = -1, \quad w(\infty) = 1.
\]

Let us observe that if \( \varepsilon = 0 \) the previous problem becomes formally in (1.3), so is natural to look for a solution \( v(t) = W(t) + \phi(t) \), with \( \phi \) a small error in \( \varepsilon \).

**Assumptions:**

1. There exists \( \beta, \gamma > 0 \) such that \( \gamma \leq a(x) \leq \beta, \forall x \in \mathbb{R} \)
2. \( \|a'\|_{L^\infty(\mathbb{R})}, \|a''\|_{L^\infty(\mathbb{R})} < +\infty \)
3. \( x_0 \) is such that \( a'(x_0) = 0, a''(x_0) \neq 0 \), i.e. \( x_0 \) is a non-degenerate critical point of \( a \).

**Theorem 1.1.** \( \forall \varepsilon > 0 \) sufficiently small, there exists a solution \( v = v_\varepsilon \) to (1.5) for some \( h = h_\varepsilon \), where \( |h_\varepsilon| \leq C\varepsilon \) and \( v_\varepsilon(t) = w(t) + \phi_\varepsilon(t) \) and \( \|\phi_\varepsilon\| \leq C\varepsilon \).

**Proof.** We write in (1.5) \( v(t) = w(t) + \phi(t) \). From now on we write \( f(v) = v(1 - v^2) \). We get

\[
w'' + \phi'' + \varepsilon \frac{a'}{a} (x_0 + \varepsilon(t+h))\phi' + \varepsilon \frac{a'}{a} (x_0 + \varepsilon(t+h))w' + f(w + \phi) - f(w) - f'(w)\phi + f(w) + f'(w)\phi = 0
\]

\[
\phi(-\infty) = \phi(\infty) = 0.
\]

It can be written in the following way

(1.6) \( \phi'' + f'(w(t))\phi + E + B(\phi) + N(\phi) = 0, \quad \phi(-\infty) = \phi(\infty) = 0 \)

where

\[
B(\phi) = \varepsilon \frac{a'}{a} (x_0 + \varepsilon(t+h))\phi',
\]

\[
N(\phi) = f(w + \phi) - f(w) - f'(w) = -3w\phi^2 - \phi^3,
\]

\[
E = \varepsilon \frac{a'}{a} (x_0 + \varepsilon(t+h))w'.
\]

We consider the problem

(1.7) \( \phi'' + f'(w(t))\phi + g(t) = 0, \quad \phi \in L^\infty(\mathbb{R}) \),

and we want to know when (1.7) is solvable. We will assume \( g \in L^\infty(\mathbb{R}) \). Multiplying (1.7) against \( w' \) we get

\[
\int_{-\infty}^{\infty} (w'' + f'(w)w')\phi + \int_{-\infty}^{\infty} gw' = 0
\]
the first integral is zero because (1.4). We conclude that a necessary condition is
\[ \int_{-\infty}^{\infty} gw' = 0. \]
This condition is actually sufficient for solvability. In fact, we write \( \phi = w'\Psi \), we have
\[ \phi'' + f'(w)\phi = g \iff w'\Psi + 2w''\Psi' = -g \]
Multiplying this last expression by \( w' \) (integration factor), we get
\[ (w^2\Psi')' = gw' \Rightarrow w'2\Psi'(t) = -\int_{-\infty}^{\infty} g(s)w'(s)ds \]
Let us choose
\[ \Psi(t) = -\int_{0}^{t} \frac{d\tau}{w^2(\tau)} \int_{-\infty}^{\Psi} g(s)w'(s)ds \]
Then the function
\[ \phi(t) = -w'(t)\int_{0}^{t} \frac{d\tau}{w^2(\tau)} \int_{-\infty}^{\Psi} g(s)w'(s)ds \]
Recall that
\[ w'(t) \approx 2\sqrt{2}e^{-\sqrt{2}|t|} \]
Claim: if \( \int_{-\infty}^{\infty} gw' = 0 \) then we have
\[ \|\phi\| \leq C\|g\|_\infty. \]
In fact, if \( t > 0 \)
\[ |\phi(t)| \leq |w'(t)| \int_{0}^{t} \frac{C}{e^{-2\sqrt{2}\tau}} \left| \int_{\tau}^{\infty} gw'ds \right| d\tau \leq C\|g\|_\infty e^{-\sqrt{2}t} \int_{0}^{t} e^{\sqrt{2}\tau} d\tau \leq C\|g\|_\infty. \]
For \( t < 0 \) a similar estimate yields, so we conclude
\[ |\phi(t)| \leq C\|g\|_\infty. \]
\[ \square \]
The solution of (1.7) is not unique because if \( \phi_1 \) is a solution implies that \( \phi_2 = \phi_1 + Cw'(t) \) is also a solution. The solution we found is actually the only one with \( \phi(0) = 0 \). For \( g \in L^\infty \) arbitrary we consider the problem
\[ (1.8) \quad \phi'' + f'(w)\phi + (g - cw') = 0, \text{ in } \mathbb{R}, \quad \phi \in L^\infty(\mathbb{R}) \]
where \( C = C(g) = \int_{-\infty}^{\infty} \frac{gw'}{w'^2} \).
Lemma 1.1. ∀g ∈ L∞(R) (1.8) has a solution which defines an operator ϕ = T[g] with

\[ \|T[g]\|_\infty \leq C\|g\|_\infty. \]

In fact if \( \hat{T}[\hat{g}] \) is the solution found in the previous step then \( \phi = \hat{T}[g - C(g)w] \) solves (1.8) and

(1.9) \( \|\phi\|_\infty \leq C\|g\|_\infty + |C(g)|C \leq C\|g\|_\infty \)

Proof. Back to the original problem: We solve first the projected problem

\[ \phi'' + f'(w)\phi + E + B(\phi) + N(\phi) = Cw', \quad \phi \in L^\infty(R) \]

where

\[ C = \frac{\int_R (E + B(\phi) + N(\phi))w'}{\int_R w'^2}. \]

We solve first (1.9) and then we find \( h = h_\varepsilon \) such that in (1.9) C=0 in such a way we find a solution to the original problem. We assume \( |h| \leq 1 \). It’s sufficient to solve

\[ \phi = T[E + B(\phi) + N(\phi)] := M[\phi]. \]

We have the following remark

\[ |E| \leq C\varepsilon^2, \quad \|B(\phi)\|_\infty \leq C\varepsilon\|\phi'\|_\infty, \quad \|N(\phi)\| \leq C(\|\phi^2\|_\infty + \|\phi^3\|_\infty) \]

where C is uniform on \( |h| \leq 1 \). We have

\[ \|M\|_\infty + \|\frac{d}{dt} M\|_\infty \leq C(\|E\|_\infty + \|B(\phi)\|_\infty + \|N(\phi)\|_\infty \leq C(\varepsilon^2 + \varepsilon\|\phi'\|_\infty + \|\phi^2\|_\infty + \|\phi^3\|_\infty) \]

then if \( \|\phi\|_\infty + \|\phi'\|_\infty \leq M\varepsilon^2 \) we have

\[ \|M\|_\infty + \|\frac{d}{dt} M\|_\infty \leq C^*\varepsilon^2. \]

We define the space \( X = \{ \phi \in C^1(R) : \|\phi\|_\infty + \|\phi'\|_\infty \leq C^*\varepsilon^2 \} \). Let us observe that \( M(X) \subset X \). In addition

\[ \|M(\phi_1)-M(\phi_2)\|_\infty + \|\frac{d}{dt}(M(\phi_1)-M(\phi_2))\|_\infty \leq C\varepsilon(|\phi_1-\phi_2|_\infty + |\phi'_1-\phi'_2|_\infty). \]

So if \( \varepsilon \) is small \( M \) is a contraction mapping which implies that there exists a unique \( \phi \in X \) such that \( \phi = M[\phi] \).

In summary: We found for each \( |h| \leq 1 \)

\[ \phi = \Phi(h), \text{ solution of 1.7} \]

. We recall that

\[ h \to \Phi(h) \]
is continuous (in $\|\cdot\|_{C^1}$). Notice that from where we deduce that $M$ is
continuous in $h$.

The problem is reduced to finding $h$ such that $C = 0$ in (1.7) for
$\phi \Phi(h) = .$. Let us observe that

$$C = 0 \Leftrightarrow \alpha \in C^1(h) \because \int_R (E_h + B[\Phi(h)]) + N[\Phi(h)]w' = 0.$$  

Let us observe that if we call $\psi(x) = \frac{\alpha}{\alpha}(x)$, then

$$\psi(x_0 + \varepsilon(t+h)) = \psi(x_0) + \psi'(x_0)\varepsilon(t+h) + \int_0^1 (1-s)\psi''(x_0 + s\varepsilon(t+h))\varepsilon^2(t+h)^2ds$$

We add the assumption $a'' \in L^\infty(\mathbb{R})$ in order to have $a'' \in L^\infty(\mathbb{R})$. We
deduce that

$$\int E_h w' = \varepsilon^2 \psi'(x_0) \int (t+h)w'(t)^2 + \varepsilon^3 \int_0^1 (1-s)\psi''(x_0 + s\varepsilon(t+h))ds(t+h)^2w'(t)dt$$

We recall that: $\int_\mathbb{R} tw'(t)^2$ and

$$|\int_\mathbb{R} (B[\Phi(h)] + N[\Phi(h)])w'| \leq C(\varepsilon \|\Phi(h)\|_{C^1} + \|\Phi(h)\|_{L^\infty}) \leq C\varepsilon^3.$$  

So, we conclude that

$$\alpha \in C^1(h) = \psi'(x_0)\varepsilon^2(h + O(\varepsilon))$$

and the term inside the parenthesis change sign. This implies that
$\exists h_\varepsilon: |h_\varepsilon| \leq M\varepsilon$ such that $\alpha \in C^1(h) = 0$, so $C = 0$.

Observe that

$$\mathcal{L}(\phi) = \phi'' - 2\phi + \varepsilon\psi + 3(1-w^2)\phi + \frac{1}{2}f''(w + s\phi)\phi + O(\varepsilon^3)e^{-\varepsilon^2|t|} = 0, \quad |t| > R$$

We consider $t > R$. Notice that $\frac{1}{2}f''(w + s\phi)\phi = O(\varepsilon^2)$. Then using

$$\dot{\phi} = \varepsilon e^{-\varepsilon^2} + \delta e^{\varepsilon^2}. \quad \text{Then using maximum principle and after taking}$$

$$\delta \to 0, \text{we obtain } \phi \leq \varepsilon e^{-\varepsilon^2|t|}.$$  

A property: We call

$$\mathcal{L}(\phi) = \phi'' + f'(w)\phi, \quad \phi \in H^2(\mathbb{R}).$$  

We consider the bilinear form associated

$$B(\phi, \phi) = -\int_\mathbb{R} \mathcal{L}(\phi)\phi = \int_\mathbb{R} \phi^2 - f'(w)^2\phi^2, \quad \phi \in H^1(\mathbb{R}).$$

Claim: $B(\phi, \phi) \geq 0, \forall \phi \in H^1(\mathbb{R})$ and $B(\phi, \phi) = 0 \Leftrightarrow \phi = cw'(t)$.

In fact: $J''(w)[\phi, \phi] = B(\phi, \phi)$. We give now the proof of the claim:
Take $\phi \in C^\infty_c(\mathbb{R})$. Write $\phi = w'\Psi \implies \Psi \in C^\infty_c(\mathbb{R})$. Observe that $L[w'\Psi] = \frac{1}{w'}(w^2\Psi')'$ and

$$B(\phi, \phi) = -\int w'(w^2\Psi')'w'\Psi = \int w^2\Psi'^2, \quad \forall \phi \in C^\infty_c(\mathbb{R})$$

Same is valid for all $\phi \in H^1(\mathbb{R})$, by density. So $B(\phi, \phi) = \int w' \Psi$ and $B(\phi, \phi) = 0 \Leftrightarrow \Psi = 0$ which implies $\phi = cw'$.

**Corollary 1.1.** Important for later porpuses There exists $r > 0$ such that if $\phi \in H^1(\mathbb{R})$ and $\int \phi w' = 0$ then

$$B(\phi, \phi) \geq \gamma \int \phi^2$$

**Proof.** If not there exists $\phi_n \int H^1(\mathbb{R})$ such that $0 \leq B(\phi_n, \phi_n) < \frac{1}{n} \int \phi_n^2$. We may assume without loss of generality $\int \phi_n^2 = 1$ which implies that up to subsequence

$$\phi_n \rightharpoonup \phi \in H^1(\mathbb{R})$$

and $\phi_n \to \phi$ uniformly and in $L^2$ sense on bounded intervals. This implies that

$$0 = \lim_{n \to \infty} \int \phi_n w' = \int \phi w'$$

On the other hand

$$\int |\phi_n'|^2 + 2 \int \phi_n^2 - 3 \int (1 - w^2)\phi_n^2 \to 0$$

and also $\int |\phi_n'|^2 + 2 \int \phi_n^2 - 3 \int (1 - w^2)\phi_n^2 \to \int |\phi'|^2 + 2 \int \phi^2 - 3 \int (1 - w^2)\phi^2$, so $B(\phi, \phi) = 0$, and $\int w' \phi = 0$ so $\phi = 0$. But also

$$2 \leq 3 \int (1 - w^2)\phi_n^2 + o(1)$$

which implies that $2 \leq 3 \int (1 - w^2)\phi^2$ and this means that $\phi \neq 0$, so we obtain a contradiction. \hfill \Box

**Observation 1.2.** If we choose $\delta = \frac{\gamma}{2\|f\|_\infty}$ then

$$\int \phi^2 - (1 + \delta) f'(w)\phi^2 \geq 0.$$ This implies in fact that

$$B(\phi, \phi) \geq \alpha \int \phi'^2.$$
2. Nonlinear Schrödinger equation (NLS)

\[ \varepsilon i \Psi_t = \varepsilon^2 \Delta \Psi - W(x) \Psi + |\Psi|^{p-1} \Psi. \]

A first fact is that \( \int_{\mathbb{R}^N} |\Psi|^2 = \text{constant}. \) We are interested into study solutions of the form \( \Psi(x, t) = e^{-iEt} u(x) \) (we will call this solutions standing wave solution). Replacing this into the equation we obtain

\[ \varepsilon E u = \varepsilon^2 \Delta u - W u - |u|^{p-1} u \]

whose transforms into

\[ \varepsilon^2 \Delta u - (W - \lambda) u + |u|^{p-1} u = 0, \quad u(x) \to 0, \quad \text{as} \ |x| \to \infty \]

choosing \( E = \frac{\lambda}{\varepsilon} \). We define \( V(x) = (W(x) - \lambda) \).

2.1. The case of dimension 1.

\[
\varepsilon^2 u'' - V(x) u + u^p = 0, \quad x \in \mathbb{R}, \quad 0 < u(x) \to 0, \quad \text{as} \ |x| \to \infty, p > 1.
\]

Assume: \( V \geq \gamma > 0, V, V', V'', V''' \in L^\infty, \) and \( V \in C^3(\mathbb{R}) \). Starting point

\[
w'' - w + w^p = 0, \quad w > 0, \quad w(\pm \infty) = 0, p > 1.
\]

There exists a homoclinic solution

\[ w(t) = \frac{C_p}{\cosh \left( \frac{p-1}{2} t \right)^{\frac{1}{p-1}}}, \quad C_p = \left( \frac{p+1}{2} \right)^{\frac{1}{p-1}} \]

Let us observe that \( w(t) \approx 2^{2/(p-1)} C_p e^{-|t|} \) as \( t \to \infty \) and also that \( W(t + c) \) satisfies same equation.

Staid at \( x_0 \) with \( V(x_0) = 1 \) we want \( u_\varepsilon(x) \approx w \left( \frac{x - x_0}{\varepsilon} \right) \) of the problem (2.1).

Observation 2.1. Given \( x_0 \) we can assume \( V(x_0) = 1 \). Indeed writing

\[ u(x) = \lambda^{\frac{2}{p-1}} v(\lambda x_0 + (1 - \lambda)x_0) \]

we obtain the equation

\[ \varepsilon^2 v''(y) - \hat{V}(y)v + v^p = 0 \]

where \( y = \lambda x_0 + (1 - \lambda)x_0 \), and \( \hat{V}(y) = V \left( \frac{y - (1-\lambda)x_0}{\lambda} \right) \). Then choosing \( \lambda = \sqrt{V(x_0)}, \) we obtain \( V(x_0) = 1 \).

Theorem 2.1. We assume \( V(x_0) = 1, V'(x_0) = 0, V''(x_0) \neq 0 \). Then there exists a solution to (2.1) with the form

\[ u_\varepsilon(x) \approx w \left( \frac{x - x_0}{\varepsilon} \right). \]
We define \( v(t) = u(x_0 + \varepsilon(t + h)) \), with \( |h| \leq 1 \). Then \( v \) solves the problem
\[
(2.3) \quad v'' - V(x_0 + \varepsilon(t + h)v + v^p = 0, \quad v(\pm\infty) = 0.
\]
We define \( v(t) = w(t) + \phi(t) \), so \( \phi \) solves
\[
(2.4) \quad \phi'' - \phi + pw^{p-1}\phi - (V(x_0 + \varepsilon(t + h)) - V(x_0))\phi + (w + \phi)^p - w^p - pw^{p-1}\phi
\]
\[
(2.5) \quad -(V(x_0 + \varepsilon(t + h)) - V(x_0))w(t) = 0
\]
So we want a solution of
\[
(2.6) \quad \phi'' - \phi + pw^{p-1}\phi + E + N(\phi) + B(\phi) = 0, \quad \phi(\pm) = 0.
\]
Observe that
\[
E = \frac{1}{2}V''(x_0 + \xi\varepsilon(t + h))\varepsilon^2(t + h)^2w(t),
\]
so \( |E| \leq C\varepsilon^2(t^2 + 1)e^{-|t|} \leq Ce^{-\sigma t} \) for \( 0 < \sigma < 1 \).

We won’t have a solution unless \( V' \) doesn’t change sign and \( V \neq 0 \).

For instance consider \( V'(x) \geq 0 \), and after multiplying the equation by \( u' \) and integrating by parts, we see that \( \int_{\mathbb{R}} u'w^2 = 0 \), which by ODE implies that \( u \equiv 0 \), because \( u \) and \( u' \) equals 0 on some point.

**2.2. Linear projected problem.**

\[
L(\phi) = \phi'' - \phi + pw^{p-1}\phi + g = 0, \quad \phi \in L^\infty(\mathbb{R})
\]

For solvability we have the necessary condition \( \int L(\phi)w' = 0 \). Assume \( g \) such that \( \int_{\mathbb{R}} gw' = 0 \). We define \( \phi = w'\Psi \), but we have the problem that \( w'(0) = 0 \). We conclude that \( (w^2\Psi)' + w'g = 0 \) for \( t \neq 0 \). We take for \( t < 0 \)
\[
\phi(t) = w'(t)\int_{1}^{-1} \frac{d\tau}{w'(\tau)^2} \int_{-\infty}^{\tau} g(s)w'(s)ds
\]
and for \( t > 0 \)
\[
\phi(t) = w'(t)\int_{1}^{t} \frac{d\tau}{w'(\tau)^2} \int_{\tau}^{\infty} g(s)w'(s)ds
\]
In order to have a solution of the problem we need \( \phi(0^-) = \phi(0^+) \).

\[
\phi(0^-) = \lim_{t \to 0^-} -\frac{\int_{-1}^{t} \frac{d\tau}{w'(\tau)^2} \int_{-\infty}^{\tau} g(s)w'(s)ds}{\frac{1}{w'(t)}} = \lim_{t \to 0^-} -\frac{1}{w'(t)} \int_{-\infty}^{t} gw' = \frac{1}{w''(0)} \int_{-\infty}^{0} gw'
\]
and
\[
\phi(0^+) = -\frac{1}{w''(0)} \int_{0}^{\infty} gw'
\]
and the condition is satisfies because of the assumption of orthogonality condition.

We get \( \| \phi \|_\infty \leq C \| g \|_\infty \). In fact we get also: \( \forall 0 < \sigma < 1, \exists C > 0 : \|

\| \phi e^{\sigma t} \|_{L^\infty} + \| \phi' e^{\sigma t} \|_{L^\infty} \leq C \| g e^{\sigma t} \| 

\)

Observation: We use \( g = g - cw' \). (Correct this part!!!!)

2.3. Method for solving. In this section we consider a smooth radial cut-off function \( \eta \in C^\infty(\mathbb{R}) \), such that \( \eta(s) = 1 \) for \( s < 1 \) and \( \eta(s) = 0 \) if \( s > 2 \). For \( \delta > 0 \) small fixed, we consider \( \eta_k,\varepsilon = \eta(\varepsilon |t| / k\delta), k \geq 1 \).

2.3.1. The gluing procedure. Write \( \tilde{\phi} = \eta_{2,\varepsilon} \phi + \Psi \), then \( \phi \) solves (2.5) if and only if

\[
\eta_{2,\varepsilon} \left[ \phi'' + (pw^{p-1} - 1)\phi + B(\phi) + 2\phi' \eta_{2,\varepsilon}' \right]
\]

(2.7)

\[
+ \left[ \Psi'' + (pw^{p-1} - 1)\Psi + B(\Psi) \right] + E(\eta_{2,\varepsilon} \phi + \Psi) = 0.
\]

(\( \phi, \Psi \)) solves (2.8) if is a solution of the system

\[
\phi'' - (1 - pw^{p-1})\phi + \eta_{1,\varepsilon} E + \eta_{3,\varepsilon} B(\phi) + \eta_{1,\varepsilon} \eta_{1,\varepsilon} B(\phi) + \eta_{1,\varepsilon} N(\phi + \Psi) = 0
\]

(2.10)

\[
\Psi'' - (V(x_0 + \varepsilon(t + h)) - pw^{p-1}(1 - \eta_{1,\varepsilon})) \Psi
\]

\[
+(1 - \eta_{1,\varepsilon}) E + (1 - \eta_{1,\varepsilon}) N(\eta_{2,\varepsilon} \phi + \Psi) + 2\phi' \eta_{2,\varepsilon}' + \eta_{2,\varepsilon}' \phi = 0
\]

(2.11)

We solve first (2.11). We look first the problem

\[
\Psi'' - W(x) \Psi + g = 0
\]

where \( 0 < \alpha \leq W(x) \leq \beta \), \( W \) continuous and \( g \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). We claim that (2.3.1) has a unique solution \( \phi \in L^\infty(\mathbb{R}) \). Assume first that \( g \) has compact support and consider the well defined functional in \( H^1(\mathbb{R}) \)

\[
J(\Psi) = \frac{1}{2} \int_{\mathbb{R}} |\Psi'|^2 + \frac{1}{2} \int_{\mathbb{R}} w\Psi^2 - \int_{\mathbb{R}} g\Psi.
\]

Also, this functional is convex and coercive. This implies that \( J \) has a minimizer, unique solution of (2.3.1) in \( H^1(\mathbb{R}) \) and it is bounded. Now we consider the problem

\[
\Psi''_{R} - W_{R} \Psi_{R} + gn(\frac{|t|}{R}) = 0
\]
Let us see that $\Psi_R$ has a uniform bound. Take $\varphi(t) = \left\| g \right\|_\infty + \rho \cosh \left( \sqrt{\alpha^2/2} |t| \right)$ for $\rho > 0$ very small. Since $\Psi_R \in L_\infty(\mathbb{R})$ we have

$$\Psi_R \leq \varphi(t), \quad \text{for } |t| > t_{\rho,R}.$$ 

Let us observe that in $[-t_{\rho,R}, t_{\rho,R}]$

$$\varphi'' - W\varphi + g\eta \left( \frac{|t|}{R} \right) < 0.$$ 

From (2.3.1), we see that $\gamma = (\Psi_R - \varphi)$ satisfies

$$\gamma'' - W\gamma > 0.$$ 

Claim: $\gamma \leq 0$ on $\mathbb{R}$. It’s for $|t| > t_{\rho,R}$ if $\gamma(\bar{t}) > 0$ there is a global maximum positive $\gamma \in [-t_{\rho,R}, t_{\rho,R}]$. This implies that $\gamma''(t) \leq 0$ which is a contradiction with (2.12). This implies that $\Psi_R(t) \leq \left\| g \right\|_\infty + \rho \cosh \left( \frac{\sqrt{\alpha^2/2} t}{2} \right)$. Taking the limit $\rho$ going to 0 we get $\Psi_R \leq \left\| g \right\|_\infty$, and similarly we can conclude that

$$\|\Psi_R\|_{L_\infty} \leq \frac{\|g\|_\infty}{\alpha}, \quad \forall R$$

Passing to a subsequence we get a solution $\Psi = \lim_{R \to \infty} \Psi_R$, and the convergence is uniform over compacts sets, to (2.3.1) with

$$\|\Psi\|_{\infty} \leq \frac{\|g\|_\infty}{\alpha}.$$ 

Also, the same argument shows that the solution is unique (in $L_\infty$ sense). Besides: We observe that if $\|e^{\sigma|t|}g\|_\infty < \infty$, $0 < \sigma < \sqrt{\alpha}$ then

$$\|e^{\sigma|t|}\Psi\|_{\infty} \leq C\|e^{\sigma|t|}g\|$$

The proof of this fact is similar to the previous one. Just take as the function $\varphi$ as follows

$$\varphi = M \frac{\|e^{\sigma|t|}g\|_\infty}{\alpha} e^{-\sigma|t|} + \rho \cosh \left( \frac{\sqrt{\alpha}}{2} |t| \right).$$

Observe now that $\Psi$ satisfies (2.11) if and only if

$$\Psi = \left( -\frac{d^2}{dt^2} + W \right)^{-1} [F[\Psi, \phi]]$$

where $W(z) = V(x_0 + \varepsilon (t + h)) - pu^{\beta-1}(1 - \eta_{1,\varepsilon})$ and $F[\phi] = (1 - \eta_{1,\varepsilon})E + (1 - \eta_{1,\varepsilon})N(y_{2,\varepsilon} \phi + \Psi) + 2\phi' y_{2,\varepsilon} + \eta''_{2,\varepsilon} \phi$. The previous result tell us that the inverse of the operator $\left( -\frac{d^2}{dt^2} + W \right)$ is well define. Assume that $\|\phi\|_{C^1} := \|\phi\|_\infty + \|\phi'\|_\infty \leq 1$, for some $\sigma < 1$ and $\|\Psi\|_{\infty} \leq \rho$, where $\rho$
is a very small positive number. Observe that \(\|(1 - \eta_1, \varepsilon)E\|_\infty \leq e^{-c\delta/\varepsilon}\). Furthermore, we have

$$|F(\Psi, \phi)| \leq e^{-c\delta/\varepsilon} + c\varepsilon \|\phi\|_{C^1} + \|\phi\|_{\infty}^2 + \|\Psi\|_{\infty}^2$$

This implies that

$$\|M[\Psi]\| \leq C_*[\mu + \|\Psi\|_{\infty}^2]$$

where \(\mu = e^{-c\delta/\varepsilon} + c\varepsilon \|\phi\|_{C^1} + \|\phi\|_{\infty}^2\). If we assume \(\mu < \frac{1}{4C_*}\), and choosing \(\rho = 2C_*\mu\), we have

$$\|M[\Psi]\| < \rho.$$  

If we define \(X = \{\Psi|\|\Psi\|_{\infty} < \rho\}\), then \(M\) is a contraction mapping in \(X\). We conclude that

$$\|M[\Psi_1] - M[\Psi_2]\| \leq C_*C\|\Psi_1 - \Psi_2\|,$$  

where \(C_*C < 1\).

Conclusion: There exists a unique solution of (2.11) for given \(\phi\) (small in \(C^1\)-norm) such that

$$\|\Psi(\phi)\|_{\infty} \leq \left[ e^{-c\delta/\varepsilon} + \varepsilon \|\phi\|_{C^1} + \|\phi\|_{\infty}^2 \right]$$

Besides: If \(\|\phi\| \leq \rho\), independent of \(\varepsilon\), we have

$$\|\Psi(\phi_1) - \Psi(\phi_2)\|_{\infty} \leq o(1)\|\phi_1 - \phi_2\|.$$  

Next step: Solver for (2.9), with \(\|\phi\|\) very small, the problem (2.13)

\[ \phi'' - (1 - p\omega^p)\phi + \eta_{1, \varepsilon}E + \eta_{3, \varepsilon}B(\phi) + \eta_{1, \varepsilon}p\omega^{-1}\Psi + \eta_{1, \eta}N(\phi + \Psi) = cw' \]

where \(c = \frac{1}{\int_{x_0}^x \int_{x_1}^x (\eta_{3, \varepsilon}B(\phi) + \eta_{1, \varepsilon}p\omega^{-1}\Psi + \eta_{1, \eta}N(\phi + \Psi))w'}\). To solve (2.13) we write it as

\[ \phi = T[\eta_{3, \varepsilon}B\phi] + T[N(\phi + \Psi(\phi)) + p\omega^{-1}\Psi(\phi)] + T[E] =: Q[\phi] \]

Choosing \(\delta\) sufficiently small independent of \(\varepsilon\) we conclude that \(Q(x) \subseteq X\), and \(Q\) is a contraction in \(X\) for \(\|\cdot\|_{C^1}\). This implies that (2.13) has a unique solution \(\phi\) with \(\|\phi\|_{C^1} < M\varepsilon^2\). Also the dependence \(\phi = \Phi(h)\) is continuous. Now we only need to adjust \(h\) in such a way that \(c = 0\). After some calculations we obtain

\[ 0 = K\varepsilon^2 V''(x_0)h + O(\varepsilon^3) + O(\delta\varepsilon^2). \]

So we can find \(h = h_\varepsilon\) and \(|h_\varepsilon| \leq C\varepsilon\), such that \(c = 0\).
3. Schrödinger equation in dimension $N$

(3.1) \[
\begin{cases}
\varepsilon^2 \Delta u - V(y)u + w^p = 0 & \text{in } \mathbb{R}^N \\
0 < u \text{ in } \mathbb{R}^n & u(x) \to 0, \text{ as } |x| \to \infty
\end{cases}
\]

We consider $1 < p < \infty$ if $N \leq 2$, and $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$. The basic problem that we consider is

\[
\begin{cases}
\Delta w - w + w^p = 0 & \text{in } \mathbb{R}^N \\
0 < u \text{ in } \mathbb{R}^n & w(x) \to 0, \text{ as } |x| \to \infty
\end{cases}
\]

We look for a solution $w = w(|x|)$, a radially symmetric solution. $w(r)$ satisfies the ordinary differential equation

(3.2) \[
\begin{cases}
w'' + \frac{N-1}{r}w' - w + w^p = 0 & r \in (0, \infty) \\
w'(0) = 0, 0 < w \text{ in } (0, \infty) & w(|x|) \to 0, \text{ as } |x| \to \infty
\end{cases}
\]

**Proposition 3.1.** There exist a solution to (3.2).

**Proof.** Let us consider the space

$$H^1_r = \{ u = u(|x|) : u \in H^1(\mathbb{R}^N) \}$$

with the norm $\| u \|_{H^1} = \int_0^{\infty} (|u'|^2 + |u|^2) r^{N-1} dr$. Let

$$S = \inf_{u \neq 0, u \in H^1_r} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + u^2}{\left( \int_{\mathbb{R}^N} |u|^{p+1}/2(p+1) \right)^2}$$

We recall that $H^1(\mathbb{R}^N) \to L^{p+1}(\mathbb{R}^N)$ continuously, which means that $S > 0$ (the larger constant such that $c\|u\|_{L^{p+1}} \leq \|u\|_{H^1}$). Strategy: Take $u_n \geq 0$ a minimizing sequence for $S$. We may assume $\|u_n\|_{L^{p+1}} = 1$. This means that $\|u_n\|^2_{H^1} \to S$. This means that the sequence is bounded in $H^1$. We may assume $u_n \rightharpoonup u \in H^1$. We have by lower weak s.c.i.

$$\int |\nabla u|^2 + u^2 \leq \liminf_n \int |\nabla u_n|^2 + u_n^2 = S.$$ 

We could get existence of a minimizer for $S$ if we prove that $\|u\|_{L^{p+1}} = 1$. This is indeed the case thanks to:

**Strauss Lemma:** There exist a constant $C$ such that $\forall u \in H^1_r(\mathbb{R}^N)$:

$$|u(|x|)| \leq \frac{C}{|x|^{N-1}} \|u\|_{H^1}$$

The proof of this fact is the following: Let $u \in C^\infty_c(\mathbb{R}^N)$, $u = u(|x|)$.

$$w^2(r) = -2 \int_r^\infty u(s)u'(s)ds \leq 2 \int_r^\infty |u(s)||u'(s)| \frac{s^{N-1}}{r^{N-1}}ds$$
\[ (3.3) \leq \frac{2}{r^{N-1}} \left( \int_0^\infty |u|^2 s^{n-1} ds \right)^{1/2} \left( \int_0^\infty |u'|^2 s^{N-1} ds \right)^{1/2} \leq \frac{C}{r^{N-1}} \|u\|_{H^1}^2 \]

By density we conclude the proof.

Let us observe that
\[ \|u_n\|_{L^{p+1}(R^N)}^{p+1} = \|u_n\|_{L^{p+1}(B_R)}^{p+1} + \|u_n\|_{L^{p+1}(B_R^c)}^{p+1} \]
and
\[ \|u_n\|_{L^{p+1}(B_R^c)}^{p+1} \leq \|u_n\|_{L^{p+1}(\{x|>R\})} \int_{R^N} u_n^2 \leq \varepsilon \]
if \( R \geq R_0(\varepsilon) \) (here we use the lemma of Strauss). On the other hand:
\[ u_n \to u \]
strong in \( L^{p+1}(B_R) \) since \( H^1(B_R) \to L^{p+1}(B_R) \) compactly. This implies that
\[ 1 \leq \lim_{n \to \infty} \|u_n\|_{L^{p+1}(B_R)} + \varepsilon = \|u\|_{L^{p+1}(B_R)} + \varepsilon \leq \|u\|_{L^{p+1}(R^N)} + \varepsilon \]
This implies that \( \|u\|_{L^{p+1}} \geq 1 \) and we conclude \( \|u\|_{L^{p+1}} = 1 \).

\( u \) is a minimizer for \( S, u \geq 0, u \neq 0 \). We define \( \Phi(v) = \|v\|_{H^1}/(\int |v|^{p+1})^{2/p+1} + \varepsilon \). So \( u \) is a minimizer for \( \Phi \). This means that \( u \) is a weak solution of the problem
\[ -\Delta u + u = \alpha u^p \]
where \( \alpha = \|u\|_{H^1} \). We define \( u = \alpha^{1/p} \tilde{u} \), then \( \tilde{u} \) is a solution of
\[ -\Delta \tilde{u} + \tilde{u} = \tilde{u}^p \]
And, with the aid of maximum strong principle we can conclude that \( \tilde{u} \) is in fact strictly positive everywhere. This concludes the proof \( \square \)

**Observation 3.1.** There no exist a solution of class \( C^2 \) for \( p \geq \frac{N+2}{N-2} \).

The proof of this fact is an application of Pohozaev identity.

We claim that \( w(r) \approx C_T r^{-\frac{N-1}{2}} e^{-r} \). This can be proved with the change of variables \( k = r^{-\frac{N+1}{2}} h \). The equation that satisfies \( h \) is like \( h'' - h(1 + \frac{c}{r^2}) = 0 \), and the solution of this equation is like \( e^{-r} \).

**Theorem 3.1.** Kwong, 1989 The radial solution of (3.2) is unique.

### 3.1. Linear problem.

Consequence of the proof of Kwong: We define
\[ L(\phi) = \Delta \phi + pw(x)^{p-1} \phi - \phi \]
Let us consider the problem
\[ L(\phi) = 0, \quad \phi \in L^\infty(R^N) \]
A known fact is that if \( \phi \) is a solution of this problem, then \( \phi \) is a linear combination of the functions \( \frac{\partial w}{\partial x_j}(x), j = 1, \ldots, N \). This is known as non degeneracy of \( w \).
We assume as always $0 < \alpha \leq V \leq \beta$. We want to solve the problem

\begin{equation}
\epsilon^2 \Delta \tilde{u} - V(y) \tilde{u} + \tilde{u}^p = 0 \quad \text{in } \mathbb{R}^N
\end{equation}

\begin{equation*}
0 < \tilde{u} \text{ in } \mathbb{R}^n \quad \tilde{u}(x) \to 0, \text{ as } |x| \to \infty
\end{equation*}

We fix a point $\xi \in \mathbb{R}^N$. Observe that $U_\epsilon(y) = V(\xi)^{\frac{1}{p-1}} \left( \sqrt{V(\xi)} \frac{w-\xi}{\epsilon} \right)$, is a solution of the problem equation

\begin{equation*}
\epsilon^2 \Delta u - V(\xi)u + u^p = 0.
\end{equation*}

We will look for a solution of (3.4) $u_\epsilon(x) \approx U_\epsilon(y)$. We define $w_\lambda = \lambda^{\frac{1}{p-1}} w(\sqrt{\lambda} x)$.

Let us observe that if $\tilde{u}$ satisfies (3.4), then $u(x) = \tilde{u}(\epsilon z)$ satisfies the problem

\begin{equation}
\begin{cases}
\Delta u - V(\epsilon z) u + u^p = 0 & \text{in } \mathbb{R}^N \\
0 < u \text{ in } \mathbb{R}^n & u(x) \to 0, \text{ as } |x| \to \infty
\end{cases}
\end{equation}

Let $\xi' = \frac{\xi}{\epsilon}$. We want a solution of (3.5) with the form $u(z) = w_\lambda(z - \xi') + \phi(z)$, with $\lambda = V(\xi)$ and $\phi$ small compared with $w_\lambda(z - \xi')$.

### 3.2. Equation in terms of $\phi$. $\phi(x) = \tilde{\phi}(\xi' - x)$.

Then $\phi$ satisfies the equation $\Delta_w[w_\lambda(x) + \phi(x)] - V(\xi + \epsilon z)[w_\lambda(x) + \phi(x)] + [w_\lambda(x) + \phi(x)]^p = 0$.

We can write this equations as

\begin{equation*}
\Delta \phi - V(\xi) \phi + pw_\lambda^{p-1}(x) \phi - E + B(\phi) + N(\phi) = 0
\end{equation*}

where $E = (V(\xi + \epsilon x) - V(\xi))w_\lambda(x)$, $B(\phi) = (V(\xi) - V(\xi + \epsilon x))\phi$ and $N(\phi) = (w_\lambda + \phi)^p - w_\lambda^p - pw_\lambda^{p-1}\phi$. We consider the linear problem for $\lambda = V(\xi)$,

\begin{equation}
\begin{cases}
L(\phi) = \Delta \phi - V(\xi + \epsilon x) \phi + pw_\lambda(x) \phi = g - \sum_{i=1}^{N} c_i \partial_{x_i} \\
\int_{\mathbb{R}^N} \phi \partial_{x_i} = 0, \quad i = 1, \ldots, N
\end{cases}
\end{equation}

The $c_i$'s are defined as follows

\begin{equation*}
\int L(\phi)(w_\lambda)_{x_i} = \int L(\phi)(w_\lambda)_{x_i} + \int (V(\xi) - V(\xi + \epsilon x))\phi(w_\lambda)_{x_i}
\end{equation*}

$w = w(|x|)$. $(w_\lambda)_{x_i}(x) = w_\lambda \frac{x_i}{|x|^2}$. This implies that

\begin{equation*}
\int (w_\lambda)_{x_i}(w_\lambda)_{x_j} = \int w_\lambda(|x|)^2 x_i x_j \frac{1}{|x|^2}
\end{equation*}

This integral is 0 if $i \neq j$ and equals to $\int_{\mathbb{R}^N} w_\lambda(|x|)^2 x_i x_j \frac{1}{|x|^2} dx = 1/N \int |\nabla w_\lambda|^2 = \gamma$. Then $c_i = \int g(w_\lambda)_{x_i} + \int_{\mathbb{R}^N} [V(\xi + \epsilon x) - V(\xi)]\phi(w_\lambda)_{x_i} \frac{1}{|w_\lambda(x)|}$.

Problem: Given $g \in L^\infty(\mathbb{R}^N)$ we want to find $\phi \in L^\infty(\mathbb{R}^N)$ solution to the problem (3.6). Assumptions: We assume $V \in C^1(\mathbb{R}^N)$, $\|V\|_{C^1} < \infty$. We assume in addition that $|\xi| \leq M_0$ and $0 < \alpha \leq V$. 


Proposition 3.2. There exists $\varepsilon_0$, $C_0 > 0$ such that $\forall 0 < \varepsilon \leq \varepsilon_0$, $\forall |\xi| \leq M_0$, $\forall g \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, there exist a unique solution $\phi \in L^\infty(\mathbb{R}^N)$ to (3.6), $\phi = T[g]$ satisfies

$$\|\phi\|_{C^1} \leq C_0\|g\|_{\infty}$$

Proof. Step 1: A priori estimates on bounded domains: There exist $R_0$, $\varepsilon_0$, $C_0$ such that $\forall \varepsilon < \varepsilon_0$, $R > R_0$, $|\xi| \leq M_0$ such that $\forall \phi, g \in L^\infty$ solving $L(\phi) = g - \sum_i c_i (w_\lambda)_{x_i}$ in $B_R$, $\int_{B_R} \phi(w_\lambda)_{x_i} = 0$ and $\phi = 0$ on $\partial B_R$, we have

$$\|\phi\|_{C^1(B_R)} \leq C_0\|g\|_{\infty}$$

We prove first $\|\phi\|_{\infty} \leq C_0\|g\|_{\infty}$. Assume the opposite, then there exist sequences $\phi_n$, $g_n$, $\varepsilon \to 0$, $R_n \to \infty$, $|\xi_n| \leq M_0$ such that

$$L(\phi_n) = g_n - c_i^0 \frac{\partial w_\lambda}{\partial x_i}$$

The first fact is that $c_i^0 \to 0$ as $n \to \infty$. This fact follows just after multiplying the equation against $(w_\lambda)_{x_i}$ and integrating by parts.

Observation: If $\Delta \phi = g$ in $B_2$ then there exist $C$ such that

$$\|\nabla \phi\|_{L^\infty(B_1)} \leq C[\|g\|_{L^\infty(B_2)} + \|\phi\|_{L^\infty(B_2)}]$$

Where $B_1$ and $B_2$ are concentric balls. This implies that $\|\nabla \phi_n\|_{L^\infty(B)} \leq C$ a given bounded set $B$, $\forall n \geq n_0$. This implies that passing to a subsequence $\phi_n \to \phi$ uniformly on compact sets, and $\phi \in L^\infty(\mathbb{R}^N)$. Observe that $\|\phi_n\|_{\infty} = 1$, and this implies that $\|\phi\|_{\infty} \leq 1$. We can assume that $\xi_n \to \xi_0$.

$\phi$ satisfies the equation $\Delta \phi - V(\xi_0)\phi + pw_\lambda^{p-1}(x)\phi = 0$, where $\lambda_0 = V(\xi_0)$, and this implies that $\phi \in \text{Span}\left\{\frac{\partial w_\lambda}{\partial x_1}, \ldots, \frac{\partial w_\lambda}{\partial x_N}\right\}$, but also $\int_{\mathbb{R}^N} \phi(w_\lambda)_{x_i} = 0$, $i = 1, \ldots, N$. Then $\phi = 0$ and this implies that $\|\phi_n\|_{L^\infty(B_M(0))} \to 0$, $\forall M < \infty$. Maximum principle implies that $\|\phi_n\|_{L^\infty(B_{R_n}\setminus B_{M_0})} \to 0$, just because $|\phi_n| = o(1)$ on $\partial B_{R_n} \setminus B_{M_0}$ and $\|g_n\|_{\infty} \to 0$. Therefore $\|\phi_n\|_{\infty} \to 0$, a contradiction. This implies that $\|\phi\|_{L^\infty(B_R)} \leq C_0\|g\|_{L^\infty(B_R)}$ uniformly on large $R$. The $C^1$ estimate follows from elliptic local boundary estimates for $\Delta$.

Step 2: Existence: $g \in L^\infty$. We want to solve (3.6). We claim that solving (3.6) is equivalent to finding $\phi \in X = \{\psi \in H^1_0 : \int \psi(w_\lambda)_{x_i} = 0, i = 1, \ldots, N\}$ such that

$$\int \nabla \phi \nabla \psi + \int V(\xi + \varepsilon x)\phi \psi - pw_\lambda^{p-1}\phi \psi + \int g \psi = 0, \quad \forall \psi \in X.$$
Take general $\Psi \in H^1_0$, $\Psi = \psi + \sum_i \alpha_i(w_\lambda)_{x_i}$, with $\alpha_i = \int \frac{\Psi(w_\lambda)_{x_i}}{\int(w_\lambda)_{x_i}}$. We have

$$-\int \Delta(\sum_i \alpha_i(w_\lambda)_{x_i})\nabla \phi + \int V(\xi)(\sum_i \alpha_i(w_\lambda)_{x_i})\phi - pw^{p-1}(\sum_i \alpha_i(w_\lambda)_{x_i})\phi = 0$$

Which implies that

$$\int \nabla \phi \nabla \Psi + \int V(\xi)\phi \Psi - pw^{p-1}\phi \Psi - \int (V(\xi) - V(\xi + \varepsilon x))\psi - \sum_i \alpha_i(w_\lambda)_{x_i} + \int g(\Psi - \sum_i \alpha_i(w_\lambda)_{x_i})$$

Then

$$\int [(V(\xi + \varepsilon x) - V(\xi))\phi + g](\Psi - \sum_i \alpha_i(w_\lambda)_{x_i})$$

and $\Pi_X(\Psi) = \sum_i \alpha_i(w_\lambda)_{x_i}$, then the previos integral is equal to

$$\int \Pi_X([(V(\xi + \varepsilon x) - V(\xi))\phi + g]\phi)$$

This implies that

$$-\Delta \phi + V(\xi)\phi - pw^{p-1}\phi + \Pi_X([(V(\xi + \varepsilon x) - V(\xi))\phi + g]\phi) = 0.$$
We use the notation $W_j = W_{\lambda_j}(x - \xi'_j)$, $\lambda_j = V(\xi_j)$ and $W = \sum_{j=1}^{n} W_j$. Look for a solution $v = W + \phi$, so $\phi$ solves the problem

$$\Delta \phi - V(\varepsilon x)\phi + pW^{p-1}\phi + E + N(\phi) = 0$$

where

$$E = \Delta W - VW + W^p, \quad N(\phi) = (W + \phi)^p - W^p - pW^{p-1}\phi.$$ 

Observe that $\Delta W = \sum_{j} \Delta W_j = \sum_{j} \lambda_j W_j - W_j^p$. So we can write

$$E = \sum_{j} (\lambda_j - V(\varepsilon x))W_j + (\sum_{j} W_j)^p - \sum_{j} W_j^p.$$

3.3. Linearized (projected) problem. We use the following notation $Z_j^i = \frac{\partial W_j}{\partial x_i}$. The linearized projected problem is the following

$$\Delta \phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g = \sum_{i,j} c_{ij}^i Z_j^i,$$

with the orthogonality condition $\int \phi Z_j^i = 0, \forall i,j$. The $Z_j^i$’s are “nearly orthogonal” if the centers $\xi'_j$ are far away one to each other. The $c_{ij}^i$’s are, by definition, the solution of the linear system

$$\int_{\mathbb{R}^N} (\Delta \phi - V(\varepsilon x)\phi + pW^{p-1}\phi + g)Z_{j_0}^{i_0} = \sum_{i,j} c_{ij}^i \int_{\mathbb{R}^N} Z_j^i Z_{j_0}^{i_0},$$

for $i_0 = 1, \ldots, N$, $j_0 = 1, \ldots, k$. The $c_{ij}^i$’s are indeed uniquely determined provided that $|\xi'_i - \xi'_j| > R_0 \gg 1$, because the matrix with coefficients $\alpha_{i,j,i_0,j_0} = \int Z_j^i Z_{j_0}^{i_0}$ is “nearly diagonal”, this means

$$\alpha_{i,j,i_0,j_0} = \begin{cases} \frac{1}{N} \int |\nabla W_{j_0}^i|^2 & \text{if } (i,j) = (i_0,j_0), \\ o(1) & \text{if not} \end{cases}$$

Moreover:

$$|c_{j_0}^{i_0}| \leq C \sum_{i,j} \int |\phi||\lambda_j - V + p|W^{p-1} - W_j^{p-1}||Z_j^i| + \int |g||Z_j^i| \leq C(||\phi||_{\infty} + ||g||_{\infty})$$

with $C$ uniform in large $R_0$. Even more, if we take $x = \xi' + y$

$$|\lambda_j - V(\varepsilon x)||Z_j^i| \leq |\left(\frac{\partial W_{\lambda_j}}{\partial y_k}\right)W_j^{p-1}| \leq C e^{-\frac{\sqrt{\lambda_j}}{\lambda_j}}|y|,$$

because $|\frac{\partial W_j}{\partial y_k}| \leq C e^{-|y|\sqrt{\lambda_j}}|y|^{-(N-1)/2}$. Observe also that

$$|(W^{p-1} - W_j^{p-1})Z_j^i| = \left|\left(1 - \sum_{l \neq j} W_l/W_j\right)^p - 1\right||W_j^{p-1}Z_j^i|.$$
We first prove the a priori estimate implies that

$$\frac{W_i(x)}{W_j(x)} \approx \frac{e^{-\sqrt{\lambda_i}|x-x'|}}{e^{-\sqrt{\lambda_j}|x-x'|}}.$$  

If $$\delta_0 \ll 1$$ but fixed, we conclude that

$$e^{-\sqrt{\lambda_j} \min_{j_1 \neq j_2} |\xi_j - x_j'|} \approx e^{-\rho \min_{j_1 \neq j_2} |\xi_j - x_j'|}.$$  

This implies that

$$|W^{p-1} - W_j^{p-1}| \leq e^{-\rho \min_{j_1 \neq j_2} |\xi_j - x_j'|} e^{-\frac{\rho}{2}|x-x'|}.$$  

If $$|x-x'| > \delta_0 \min_{j_1 \neq j_2} |\xi_j - x_j'|,$$ then

$$|W^{p-1} - W_j^{p-1}| \leq C|\phi_j^0| \leq C(e + e^{-\rho \min_{j_1 \neq j_2} |\xi_j - x_j'|})\|\phi\|_{\infty} + \|g\|_{\infty}. \tag{3.1}$$

**Lemma 3.1.** Given $$k \geq 1,$$ there exist $$R_0, C_0, \varepsilon_0$$ such that for all points $$\xi_j^i$$ with $$|\xi_j^i - \xi_j^{i+1}| > R_0,$$ $$j = 1, \ldots, k$$ and all $$\varepsilon < \varepsilon_0$$ there exist a unique solution $$\phi$$ to the linearized projected problem with

$$\|\phi\|_{\infty} \leq C_0\|g\|_{\infty}.$$  

**Proof.** We first prove the a priori estimate $$\|\phi\|_{\infty} \leq C_0\|g\|_{\infty}.$$ If not there exist $$\varepsilon_n \to 0,$$ $$\|\phi_n\|_{\infty} = 1,$$ $$\|g_n\| \to 0,$$ $$\xi_j^m$$ with $$\min_{j_1 \neq j_2} |\xi_j^m - \xi_j^{m+1}| \to \infty.$$ We denote $$W_n = \sum_j W_n,$$ and we have

$$\Delta \phi_n - V(\varepsilon_n x)\phi_n + pW^{p-1}\phi_n + g_n = \sum_{i,j} (c_j^i)(z_j^i) n$$

First observation: $$(c_j^i)n \to 0$$ (follows from estimate for $$c_j^m.$$ Second: $$\forall R > 0 \|\phi_n\|_{L^\infty(B(\xi_j^m, R))} \to 0,$$ $$j = 1, \ldots, k.$$ If not, there exist $$j_0$$

$$\|\phi_n\|_{L^\infty(B(\xi_j^m, R_0))} \geq \gamma > 0.$$  

We denote $$\phi_{j}(y) := \phi_n(\xi_j^m + y).$$ We have

$$\|\phi_n\|_{L^\infty(B(0, R_0))} \geq \gamma > 0.$$  

Since $$|\Delta \phi_n| \leq C,$$ $$\|\phi_n\|_{\infty} \leq 1.$$ This implies that $$\|\nabla \phi_n\| \leq C.$$ Passing to a subsequence we may assume $$\phi_n \to \phi$$ uniformly on compact sets. Observe that also

$$V(\varepsilon_n(\xi_j^m + y)) = V(\varepsilon_n(\xi_j^m + y)) + O(\varepsilon_n|x|) \to \lambda$$

over compact sets and $$W_n(\xi_j^m + y) \to W_{\lambda_{j_0}}(y)$$ uniformly on compact sets. This implies that $$\phi$$ is a solution of the problem

$$\Delta \phi - \lambda \phi + pW^{p-1}\phi \sim 0, \quad \int \phi \frac{\partial W_{\lambda_{j_0}}}{\partial y_i} dy = 0, i = 1, \ldots, N.$$  

Non degeneracy of $$w_{\lambda_{j_0}}$$ implies that $$\phi = \sum_i \alpha_i \frac{\partial \phi_{\lambda_{j_0}}}{\partial y_i}.$$ The orthogonality condition implies that $$\alpha_i = 0, \forall i = 1, \ldots, N.$$ This implies that
\(\hat{\phi} = 0\) but \(\| \hat{\phi} \|_{L^{\infty}(B(0, R))} \geq \gamma > 0\), a contradiction. Now we prove: \(\| \phi_n \|_{L^{\infty}(\mathbb{R}^N \setminus \Omega_n, R))} \to 0\), provided that \(R \gg 1\) and fixed so that \(\phi_n \to 0\) in the sense of \(\| \phi_n \|_{\infty}\) (again a contradiction). We will denote \(\Omega_n = \mathbb{R}^N \setminus \Omega_n, R))\). For \(R \gg 1\) the equation for \(\phi_n\) has the form

\[\Delta \phi_n - Q_n \phi_n + g_n = 0\]

where \(Q_n = V(\varepsilon x) - pW^{p-1}_n \geq \frac{\alpha}{2} > 0\) for some \(R\) sufficiently large (but fixed). Let’s take for \(\sigma^2 < \alpha/2\)

\[\hat{\phi} = \delta \sum_{j} e^{\sigma|x - \xi_j^n|} + \mu_n,\]

We denote \(\varphi(y) = e^{\sigma|y|}, r = |y|\). Observe that \(\Delta \varphi - \alpha/2 \varphi = e^{\sigma|y|}(\sigma^2 + N-1|y|^2 - \alpha/2) < 0\) if \(|y| > R \gg 1\). Then

\[-\Delta \hat{\phi} + Q_n \hat{\phi} - g_n > -\Delta \phi + \frac{\alpha}{2} \hat{\phi} - \|g_n\|_{\infty} > \frac{\alpha}{2} \mu_n - \|g_n\|_{\infty} > 0\]

if we choose \(\mu_n \geq \|g_n\|_{\infty} \frac{2}{\alpha}\). In addition we take \(\mu_n = \sum_j \|\phi_n\|_{L^{\infty}(B(\xi^n_j, R))} + \|g_n\|_{\infty} \frac{2}{\alpha}\). Maximum principle implies that \(\phi_n(x) \leq \hat{\phi}\) for all \(x \in \Omega_n\). Taking \(\delta \to 0\) this implies that \(\phi_n(x) \leq \hat{\phi}\) for all \(x \in \Omega_n\). Also true that \(|\phi_n(x)| \leq \mu_n\) for all \(x \in \Omega^n_c\), and this implies that \(\|\phi_n\|_{L^{\infty}(\mathbb{R}^N)} \to 0\).

**Observation 3.2.** If in addition we have \(\theta_n = \|g_n\|_{\infty} \left(\sum_j e^{-\rho|x - \xi_j^n|}\right)^{-1} \|\infty \to 0\) with \(\rho < \alpha/2\). Then we can use as a barrier

\[\hat{\phi} = \delta \sum_{j} e^{\sigma|x - \xi_j^n|} + \mu_n \sum_{j} e^{-\rho|x - \xi_j^n|}\]

with \(\mu_n = e^{\rho R} \sum_j \|\phi_n\|_{L^{\infty}(B(\xi^n_j, R))} + \theta_n\), then \(\hat{\phi}\) is a super solution of the equation and we have \(|\phi_n| \leq \hat{\phi}\), and letting \(\delta \to 0\) we get \(|\phi_n(x)| \leq \mu_n \sum_{j} e^{-\rho|x - \xi_j^n|}\). As a conclusion we also get the a priori estimate

\[\|\phi \left(\sum_{j=1}^k e^{-\rho|x - \xi_j^n|}\right)^{-1} \|_{\infty} \leq C\|g\| \left(\sum_{j=1}^k e^{-\rho|x - \xi_j^n|}\right)^{-1} \|_{\infty}\]

provided that \(0 \leq \rho < \alpha/2, |\xi_j^n - \xi_{j'}^n| > R_0 \gg 1, \varepsilon < \varepsilon_0\).

We now give the proof of existence

**Proof.** Take \(g\) compactly supported. The weak formulation for

\[(3.7) \quad \Delta \phi - V(\varepsilon x) \phi + pW^{p-1} \phi + g = \sum_{i,j} \epsilon_{ij} Z_j^i, \int \phi Z_j^i, \forall i, j\]
is find $\phi \in X = \{ \phi \in H^1(\mathbb{R}^N) : \int \phi Z^i_j = 0, \forall i, j \}$ such that

(3.8) \[
\int_{\mathbb{R}^N} \nabla \phi \nabla \psi + V \phi \psi - p W^{p-1} \phi \psi - g \psi = 0, \quad \forall \psi \in X.
\]

Assume $\phi$ solves (3.7). For $g \in L^2$, write $g = \tilde{g} + \Pi[g]$ where $\int \tilde{g} Z^i_j = 0$, for all $i, j$. $\Pi$ is the orthogonal projection of $g$ onto the space spanned by the $Z^i_j$'s. Take $\psi \in H^1(\mathbb{R}^N)$ arbitrary and use $\psi - \Pi[\psi]$ as a test function in (3.8). Then if $\varphi \in C_0^\infty(\mathbb{R}^N)$, then

\[
\int_{\mathbb{R}^N} \nabla \varphi \nabla (\Pi[\psi]) = - \int_{\mathbb{R}^N} \Delta \varphi \Pi[\psi] = - \int_{\mathbb{R}^N} \Pi[\Delta \varphi] \psi.
\]

But $\Pi[\Delta \varphi] = \sum_{i,j} \alpha_{i,j} Z^i_j$, where

\[
\sum \alpha_{i,j} \int Z_{i0} Z_{j0} = \int \Delta \varphi Z_{i0} = \int \varphi \Delta Z_{i0}.
\]

Then $\|\Pi[\Delta \varphi]\|_{L^2} \leq C \|\varphi\|_{H^1}$. By density is true also for $\varphi \in H^1$ where $\Delta \varphi \in H^{-1}$. Therefore

\[
\int \nabla \varphi \nabla \psi + \int (V \phi - p W^{p-1} \phi - g) \psi = \int \Pi(V \phi - p W^{p-1} \phi + g) \psi
\]

then $\phi$ solves in weak sense

\[-\Delta \phi + V \phi - p W^{p-1} \phi - g = \Pi[-\Delta \phi + V \phi - p W^{p-1} \phi - g]
\]

and $\Pi[-\Delta \phi + V \phi - p W^{p-1} \phi - g] = \sum_{i,j} c_{i,j} Z_{i,j}$. Therefore by definition $\phi$ solves (3.8) implies that $\phi$ solves (3.8). Classical regularity gives that this weak solution is solution of (3.7) in strong sense, in particular $\phi \in L^\infty$ so that

\[
\|\phi\|_{\infty} \leq C \|g\|_{\infty}
\]

. Now we give the proof of existence for (3.7). We take $g$ compactly supported. The equation (3.8) can be written in the following way (using Riesz theorem):

\[
\langle \phi, \psi \rangle_{H^1} + \langle B[\phi], \psi \rangle_{H^1} = \langle \tilde{g}, \psi \rangle_{H^1}
\]

or $\phi + B[\phi] = \tilde{g}$, $\phi \in X$. We claim that $B$ is a compact operator. Indeed if $\phi_n \rightharpoonup 0$ in $X$, then $\phi_n \to 0$ in $L^2$ over compacts.

\[
|\langle B[\phi_n], \psi \rangle| \leq \int p W^{p-1} \phi_n \psi \leq (\int p W^{p-1} \phi^2_n)^{1/2}(\int p W^{p-1} \psi^2)^{1/2}
\]

then

\[
|\langle B[\phi_n], \psi \rangle| \leq c(\int p W^{p-1} \phi^2_n)^{1/2} \|\psi\|_{H^1}
\]
Take $\psi = B[\phi_n]$, which implies
\[ \|B[\phi_n]\|_{H^1} \leq c(\int pW^{p-1}\phi_n^2)^{1/2} \to 0. \]
This implies that $B$ is a compact operator. Now we prove existence with the aid of fredholm alternative. Problem is solvable if for $\hat{g} = 0$ implies that $\phi = 0$. But $\phi + B[\phi] = 0$ implies solve (3.7)(strongly) with $g = 0$. This implies $\phi \in L^\infty$, and the a priori estimate implies $\phi = 0$. Considering $g \Xi_{B_\mathcal{R}(0)}$ we conclude that
\[ \|\phi_R\|_\infty \leq \|g\|_\infty \]
Taking $R \to \infty$ then along a subsequence $\phi_R \to \phi$ uniform over compacts.

We take $g \in L^\infty$. We have $\phi = T_\xi'[g]$, where $\xi' = (\xi'_1, \ldots, \xi'_k)$. We want to analyze derivatives $\partial_{\xi'_i} T_\xi'[g]$. We know that $\|T_\xi'[g]\| \leq C_0 \|g\|_\infty$. First we will make a formal differentiation. We denote $\Phi = \bar{\sigma}_{\xi_{0j0}}$.

We have $\Delta \phi - V\phi + pW^{p-1}\phi + g = \sum_{i,j} c^j_i Z^i_j$ and $\int \phi Z^i_j = 0$, for all $i,j$. Formal differentiation yields
\[ \Delta \Phi - V\Phi + pW^{p-1}\Phi + + \partial_{\xi_{0j0}}(W^{p-1})\phi - \sum_{i,j} c^j_i \partial_{\xi_{0j0}} Z^j_i = \sum_{i,j} c^j_i Z^i_j \]
where formally $c^j_i = \partial_{\xi_{0j0}} c^j_i$. The orthogonality conditions traduces into
\[ \int_{\mathbb{R}^N} \Phi Z^j_i \, dz^j = \left\{ \begin{array}{ll} 0 & \text{if } j \neq j_0 \\ -\int \phi \partial_{\xi_{0j0}} Z^j_i & \text{if } j = j_0 \end{array} \right. \]
Let us define $\tilde{\Phi} = \Phi - \sum_{i,j} \alpha_{i,j}^0 Z^j_i$. We want $\int \tilde{\Phi} Z^j_i = 0$, for all $i,j$. We need
\[ \sum_{i,j} \alpha_{i,j} \int \Phi Z^j_i \, dz^j = \left\{ \begin{array}{ll} 0 & \text{if } \tilde{j} \neq j_0 \\ -\int \phi \partial_{\xi_{0j0}} Z^j_i & \text{if } \tilde{j} = j_0 \end{array} \right. \]
The system has a unique solution and $|\alpha_{i,j}| \leq C\|\phi\|_\infty$ (since the system is almost diagonal). So we have the condition $\int \Phi Z^j_i = 0$, for all $i,j$. We add to the equation the term $\sum_{i,j} \alpha_{i,j} (\Delta - V + pW^{p-1})Z^j_i$, so $\tilde{\Phi}$ satisfies the equation $\Delta \tilde{\Phi} - V\tilde{\Phi} + pW^{p-1}\tilde{\Phi} + \partial_{\xi_{0j0}}(W^{p-1})\phi - \sum_{i,j} c^j_i \partial_{\xi_{0j0}} Z^j_i = \sum_{i,j} c^j_i Z^j_i - \sum_{i,j} \alpha_{i,j} (\Delta - V + pW^{p-1})Z^j_i$

This implies $\|\tilde{\Phi}\| \leq C(\|h\| + \|g\|) \leq C\|g\|_\infty$. This implies $\|\Phi\| \leq C\|g\|_\infty$. We do this in a discrete way, and passing to the limit all these calculations are still valid. Conclusion: The map $\xi \to \partial_\xi \phi$ is well
We have \( \|\partial_\xi \phi\|_\infty \leq C\|g\|_\infty \), and this implies
\[
\|\partial_\xi T_\xi [\phi]\| \leq C\|g\|
\]

3.4. **Nonlinear projected problem.** Consider now the nonlinear projected problem
\[
\Delta \phi - V \phi + p\mu^{p-1} \phi + E + N(\phi) = \sum_{i,j} c_i^j Z_i^j, \quad \int \phi Z_i^j = 0, \forall i, j
\]
We solve this by fixed point. We have \( \phi = T(E + N(\phi)) =: M(\phi) \). We define \( \Lambda = \{\phi \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) : \|\phi\|_\infty \leq M\|E\|_\infty\} \). Remember that \( E = \sum_i (\lambda_i - V(\epsilon x))W_j + (\sum_j W_j)^p - \sum_j W_j^p \). Observe that
\[
|E| \leq \epsilon \sum_i e^{-\sigma|x-x_i|} + ce^{-\delta_0 \min_{1 \neq 2} |\xi'_1 - \xi'_2|} \sum_j e^{-\sigma|x-x'_j|}
\]
so, for existence we have \( \|E\| \leq C(\epsilon e^{-\delta_0 \min_{1 \neq 2} |\xi'_1 - \xi'_2|} + \rho \) (see that \( \rho \) is small). Contraction mapping implies unique existence of \( \phi = \Phi(\xi) \) and \( \|\Phi(\xi)\| \leq M\rho \).

3.5. **Differentiability in \( \xi' \) of \( \Phi(\xi') \).** We have
\[
\Phi - T'_\xi (E'_\xi + N'_\xi(\phi)) = A(\Phi, \xi') = 0
\]
If \( (DA)(\Phi(\xi'), \xi') \) is invertible in \( L^\infty \), then \( \Phi(\xi') \) turns out to be of class \( C^1 \). This is a consequence of the fixed point characterization, i.e., \( DA(\Phi(\xi'), \xi') = I + o(1) \) (the order \( o(1) \) is a direct consequence of fixed point characterization). Then is invertible. Theorem and the \( C^1 \) derivative of \( A(\Phi, \xi') \) in \( (\phi, \xi') \). This implies \( \Phi(\xi') \) is \( C^1 \). \( \|D'_\xi \Phi(\xi')\| \leq C\rho \) (just using the derivate given by the implicit function theorem).

3.6. **Variational reduction.** We want to find \( \xi' \) such that the \( c_j^i = 0 \), for all \( i, j \), to get a solution to the original problem. We use a procedure that we call Variational Reduction in which the problem of finding \( \xi' \) with \( c_j^i = 0 \), for all \( i, j \), is equivalent to finding a critical point of a functional of \( \xi' \). Recall:
\[
J(v) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + V(\epsilon x)v^2 - \frac{1}{p+1} \int_{\mathbb{R}^{N+1}} v^{p+1}_x
\]
is defined in \( H^1(\mathbb{R}^N) \), since \( 1 < p < \frac{N+2}{N-2} \). \( v \) is a solution of \( \Delta v - Vv + v^p = 0, v \rightarrow 0 \) if and only if \( v \in H^1(\mathbb{R}^N) \) and \( J'(v) = 0 \). Observe that
\[
\langle J'(v), \varphi \rangle = \int \nabla v \nabla \varphi + Vv \varphi - v^p_x \varphi.
\]
The following fact happens: \( v = W_{\xi'} + \phi(\xi') \) is a solution of the original problem (for \( \rho \ll 1 \)) if and only if
\[
\partial_{\xi'} J(W_{\xi'} + \phi(\xi'))|_{\xi' = \xi'} = 0.
\]
Indeed, observe that $v(\xi') := W_\varepsilon + \phi(\xi')$ solves the problem $\Delta v(\xi') - V(\varepsilon x)v(\xi') + v(\xi')^p = \sum_{i,j} c_{ij}^j Z_{ij}^j$ and also that

$$
\partial_{\xi_{0}^{\varepsilon j}} J(v(\xi')) = \langle J'(v(\xi')) , \partial_{\xi_{0}^{\varepsilon j}} v(\xi') \rangle = - \sum_{i,j} c_{ij}^j \int Z_{ij}^j \partial_{\xi_{0}^{\varepsilon j}} v = - \sum_{i,j} c_{ij}^j \int \frac{Z_{ij}^j}{\partial_{\xi_{0}^{\varepsilon j}}} W_\varepsilon + \partial_{\xi_{0}^{\varepsilon j}} \phi
$$

Remember that $W_\varepsilon = \sum_{j=1}^k w_{\lambda_j}(x - \xi_j)$,

$$
\partial_{\xi_{0}^{\varepsilon j}} W_\varepsilon' = \partial_{\xi_{0}^{\varepsilon j}} w_{\lambda_j(\xi')} (x - \xi)' = (\partial_\lambda w_\lambda(x - \xi'_j))|_{\lambda = \lambda_j} - \partial_{x_0} w_{\lambda_j} (x - \xi'_j) = O(e^{-d|x-\xi'_j|})\phi(\varepsilon) - Z_{ij}^j.
$$

This because $\partial_\lambda w_\lambda = O(e^{-d|x-\xi'_j|})$. On the other hand $|\partial_{\xi_{0}^{\varepsilon j}} \phi| \leq C\rho e^{-d|x-\xi'_j|}$. Finally, observe that

$$
-\int Z_{ij}^j (\partial_{\xi_{0}^{\varepsilon j}} W_\varepsilon' + \partial_{\xi_{0}^{\varepsilon j}} \phi) = \int Z_{ij}^j Z_{ij}^j + O(\rho)
$$

The matrix of these numbers is invertible provided $\rho \ll 1$.

A consequence (D, Felmer 1996): Assume $j = 1$ and that there exist an open, bounded set $\Lambda \subset \mathbb{R}^N$ such that

$$
\inf_{\Lambda} \varepsilon V > \inf_{\Lambda} V,
$$

then there exist a solution to the original problem, $v_\varepsilon$ with $v_\varepsilon(x) = W_\varepsilon V(\xi)(x - \varepsilon \xi)/\varepsilon + o(1)$ and $V(\xi) \rightarrow \min_\Lambda V, \xi = \xi_\varepsilon$.

Another consequence (D, Felmer 1998): $\Lambda_1, \ldots, \Lambda_k$ disjoint bounded with $\inf_{\Lambda_j} V < \inf_{\Lambda_j} V$, for all $j$. For the problem $\varepsilon^2 \Delta u - V(x)u + u^p = 0$, $0 < u \rightarrow 0$ at $\infty$, there exist a solution $u_\varepsilon$ with $u_\varepsilon(x) \approx \sum_{j=1}^k W_{\varepsilon}'(\xi')(x - \xi_j'/\varepsilon)$, $\xi_j' \in \Lambda_j$ and $V(\xi_j') \rightarrow \inf_{\Lambda_j} V$ (in the case of non-degeneracy minimal or more generally non-degenerate critical points the result is due to Oh (1990)).

**Proof.** First result: $j = 1$. $v(\xi') = W_\varepsilon + \phi(\xi')$. Then

$$
J(W_\varepsilon(\xi')) = J(W_\varepsilon + \phi(\xi')) + \langle J'(W_\varepsilon + \phi), -\phi \rangle + \frac{1}{2} J''(W_\varepsilon + (1 - t)\phi)[\phi]^2
$$

(Taylor expansion of the function $a(t) = J(W_\varepsilon + (1 - t)\phi)$). Observe that $\langle J'(W_\varepsilon + \phi), -\phi \rangle = \sum_{i,j} c_{ij}^j \int Z_{ij}^j \phi = 0$. Also observe that

$$
J''(W_\varepsilon + (1 - t)\phi)[\phi]^2 = \int |\nabla \phi|^2 + V(\varepsilon x)\phi^2 - p(W_\varepsilon' + (1 - t)\phi)\phi^2 = O(\varepsilon^2)
$$

uniformly on $\xi'$ because $\nabla \phi, \phi = O(\varepsilon e^{-d|x-\xi'|})$. We call $\Phi(\xi) := J(v(\xi')) = J(W_\varepsilon) + O(\varepsilon^2)$, and

$$
J(W_\varepsilon) = \frac{1}{2} \int |\nabla W_\varepsilon|^2 + V(\xi)W_\varepsilon^2 - \frac{1}{p + 1} \int W_\varepsilon^{p + 1}\varepsilon \int (V(\varepsilon x) - V(\xi')) W_\varepsilon^2
$$
Taking $\lambda = V(\xi)$, we have that
\[
\int |\nabla w_\lambda(x)|^2 = \lambda^{-N/2} \int |\nabla (\lambda^{1/2}x)|^2 \lambda^{1+2/(p-1)} \lambda^{N/2} dx = \lambda^{-N/2+p+1/p-1} |\nabla w(y)|^2 dy
\]
and
\[
\lambda \int w_\lambda^2(x) = \lambda^{-N/2+p+1/p-1} \int w(y)^{p+1} dy
\]
This implies that
\[
\frac{1}{2} \int |\nabla W_\xi'|^2 + V(\xi)W_\xi'^2 - \frac{1}{p+1} \int W_\xi'^{p+1} = V(\xi)^{p+1/p-1-N/2} c_{p,N}.
\]
also
\[
\int (V(\varepsilon x) - V(\xi')) x - \xi')^2 = O(\varepsilon)
\]
uniformly on $\xi$. In summary $\Phi(\xi) = J(v(\xi')) = V(\xi)^{p+1/p-1-N/2} c_{p,N} + O(\varepsilon)$ and $\frac{p+1}{p-1} - \frac{N}{2} > 0$. Then $\forall \varepsilon \ll 1$ we have
\[
\inf_{\xi \in \Lambda} \Phi(\xi) < \inf_{\xi \in \partial \Lambda} \Phi(\xi)
\]
therefore $\Phi$ has a local minimum $\xi_\varepsilon \in \Lambda$ and $V(\xi_\varepsilon) \to \min_\Lambda V$. Same thing works at a maximum.

For several spikes separated: $|\xi_{j_1} - \xi_{j_2}| > \delta$, for all $j_1 \neq j_2$. $\rho = e^{-\delta_0 \min_{j_1 \neq j_2} |\xi_{j_1}' - \xi_{j_2}'| + \varepsilon} \leq e^{-\delta_0 \delta/\varepsilon} + \varepsilon < 2\varepsilon$, so we have
\[
|\nabla_x \phi(\xi')| + |\phi(\xi')| \leq C\varepsilon \sum_j e^{-\delta_0 |x-\xi_j'|}
\]
Now we get
\[
J(v(\xi')) = \sum_j V(\varepsilon_j)^{p+1/p-1-N/2} c_{p,N} + O(\varepsilon)
\]
$\xi' = 1/\varepsilon(\xi_1, \ldots, \xi_k)$ implies for several minimal on the $\Lambda_j$ we have the result desired. \(\square\)

Result at one non-degenerate critical point: if $\xi_0$ is a non-degenerate critical point of $V$ ($V'(\xi_0) = 0$ and $V''(\xi_0)$ invertible), then there exist a solution $u_\varepsilon(x)$ such that
\[
u_\varepsilon(x) \approx W_{V(\xi_\varepsilon)}(x - \xi_\varepsilon)/\varepsilon, \quad \xi_\varepsilon \to \xi_0.
\]
For small $\delta$ we have that $J(v)$ has degree different from 0 in a ball centered at $x_0$ and of radius $\delta$. 

4. Back to Allen Cahn in \( \mathbb{R}^2 \)

We consider the functional

\[
J(u) = \int_{\mathbb{R}^2} \left( \varepsilon^2 \frac{\lvert \nabla u \rvert^2}{2} + \frac{(1 - u^2)^2}{4} \right) a(x) dx.
\]

Critical points of \( J \) are solutions of

\[
\varepsilon^2 \text{div}(a(x) \nabla u) + a(x)(1 - u^2)u = 0,
\]

where we suppose \( 0 < \alpha \leq a(x) \leq \beta \). This equation is equal to

\[
(4.1) \quad \varepsilon^2 \Delta u + \varepsilon^2 \frac{\nabla a}{a}(x) \nabla u + (1 - u^2)u = 0.
\]

Using the change of variables \( v(x) = u(\varepsilon x) \), we find the equation

\[
(4.2) \quad \Delta v + \varepsilon \frac{\nabla a}{a}(x) \nabla v + (1 - v^2)v = 0.
\]

We will study the problem: Given a curve \( \Gamma \) in \( \mathbb{R}^2 \) we want to find a solution \( u_\varepsilon(x) \) to (4.1) such that \( u_\varepsilon(x) \approx w(\frac{x}{\varepsilon}) \), for points \( x = y + z\nu(y) \), \( y \in \Gamma, \lvert z \rvert < \delta \), where \( \nu(y) \) is a vector perpendicular to the curve and \( w(t) = \tanh(\frac{t}{\sqrt{2}}) \), which solves the problem

\[
w'' + (1 - w^2)w = 0, \quad w(\pm \infty) = \pm 1.
\]

First issue: Laplacian near \( \Gamma \), which we will consider as smooth as we need.

Assume: \( \Gamma \) is parametrized by arc-length

\[
\gamma : [0, l] \to \mathbb{R}^2, \ s \to \gamma(s), \ l = \lvert \gamma \rvert = 1, l = \lvert \Gamma \rvert.
\]

Convention: \( \nu(s) \) inner unit normal at \( \gamma(s) \). We have that \( \lvert \nu(s) \rvert^2 = 1 \), which implies that \( 2\nu \dot{\nu} = 0 \), so we take \( \dot{\nu}(s) = -k(s)\dot{\gamma}(s) \), where \( k(s) \) is the curvature.

Coordinates: \( x(s, t) = \gamma(s) + z\nu(s), \ s \in (0, l) \) and \( \lvert z \rvert < \delta \). If we take a compact supported function \( \psi(x) \) near \( \Gamma \), and we call \( \dot{\psi}(s, z) = \psi(\gamma(s) + z\nu(s)) \), then \( \frac{\partial \dot{\psi}}{\partial s} = \nabla \psi \cdot \dot{\gamma} + z\nu \), \( \frac{\partial \dot{\psi}}{\partial z} = \nabla \psi \cdot \nu \). Observe that \( \nabla \psi = (\nabla \psi \cdot \dot{\gamma})(\nabla \cdot \nu) \nu \). This means that

\[
\nabla \psi = \frac{1}{1 - k\dot{z}^2} \frac{\partial \dot{\psi}}{\partial s} \dot{\gamma} + \frac{\partial \dot{\psi}}{\partial z} \nu, \quad \text{and} \quad \lvert \nabla \psi \rvert^2 = \frac{1}{(1 - k\dot{z}^2)^2} \lvert \dot{\psi}_s \rvert^2 + \lvert \dot{\psi}_z \rvert^2.
\]

Then

\[
\int_{\mathbb{R}^2} \lvert \nabla \psi(x) \rvert^2 dx = \iint \left( \frac{1}{(1 - k\dot{z}^2)^2} \lvert \dot{\psi}_s \rvert^2 + \lvert \dot{\psi}_z \rvert^2 \right) (1 - k\dot{z}) ds dz
\]

\( \psi \to \psi + t\varphi \) and differentiating at \( t = 0 \) we get

\[
\int \nabla \psi \nabla \varphi dx = \iint \frac{1}{(1 - k\dot{z})} \dot{\psi}_s \varphi_s + \dot{\psi}_z \varphi_z (1 - k\dot{z}) ds dz
\]
So
\[-\int \Delta \psi \phi dx = - \iint \frac{1}{(1 - k z)} \left( \left( \frac{1}{1 - k z} \tilde{\psi}_s \right)_s + \left( \tilde{\psi}_z (1 - l z) \right)_z \right) \tilde{\phi} (1 - k z) dsdz \]
then
\[\Delta \tilde{\psi} = \frac{1}{(1 - k z)} \frac{\partial}{\partial s} \left( \frac{1}{1 - k z} \tilde{\psi}_s \right) + \tilde{\psi}_{zz} - \frac{k}{1 - k z} \tilde{\psi}_z \]
We just say
\[\Delta \tilde{\psi} = \frac{1}{1 - k z} \left( 1 - k z \tilde{\psi}_s \right)_s + \tilde{\psi}_{zz} - \frac{k}{1 - k z} \tilde{\psi}_z \]
Near \( \Gamma (x = \gamma(s) + z \nu(s)) \), we have the new equation for \( u \rightarrow \tilde{u}(s, z) \)
\[S[u] = \varepsilon^2 \frac{1}{1 - k z} \left( \frac{1}{1 - k z} u_s \right)_s + \varepsilon^2 u_{zz} + (1 - a^2) u - \varepsilon^2 k a_s u_s + \frac{\varepsilon^2}{1 - k z} a_z u_z = 0 \]
we want a solution \( u(s, z) \approx w(\frac{z}{\varepsilon}) \).
\[S[w(\frac{z}{\varepsilon})] = \varepsilon \left[ \frac{a_z}{a} - \frac{k(s)}{1 - k(s) \frac{z}{\varepsilon}} \right] w'(\frac{z}{\varepsilon}) \]
The condition we ask (geodesic condition) is \( \frac{a_z}{a}(s, 0) = k(s) \). In \( v \)
language we want
\[\Delta v + \varepsilon \nabla a \left( \varepsilon x \right) \cdot \nabla v + f(v) = 0 \]
transition on \( \Gamma_{\varepsilon} = \frac{1}{\varepsilon} \Gamma \). we use coordinates relative to \( \Gamma_{\varepsilon} \) rather than \( \Gamma \)
\[X_{\varepsilon}(s, z) = \frac{1}{\varepsilon} \gamma(\varepsilon s) + z \nu(\varepsilon s), \quad |z| < \delta / \varepsilon \]
Laplacian for coordinates relative to \( \Gamma_{\varepsilon} \) are
\[\Delta \psi = \frac{1}{(1 - \varepsilon k(\varepsilon s) z)} \left( \frac{1}{1 - \varepsilon k(\varepsilon s) z} v_s \right)_s + \psi_{zz} - \varepsilon \frac{k(\varepsilon s)}{1 - \varepsilon k(\varepsilon s) z} \frac{a_s}{a} v_s + \frac{1}{1 - \varepsilon k(\varepsilon s) z} v_z + \varepsilon \frac{a_z}{a} v_z \]
where we use the computation \( \frac{\partial \gamma(\varepsilon s)}{\partial s} = -k(\varepsilon) \gamma_s(\varepsilon) \), where \( k_\varepsilon = \varepsilon k(\varepsilon s) \)
Hereafter we use \( \tilde{s} \) instead of \( s \) and \( \tilde{z} \) instead of \( z \). Observation: The
operator is closed to the Laplacian on \( (\tilde{s}, \tilde{z}) \) variables, at least on the
curve \( \Gamma \), if we assume the validity of the relation
\[a_{\tilde{z}}(\tilde{s}, 0) = k(\tilde{s}) a(\tilde{s}, 0), \quad \forall \tilde{s} \in (0, l). \]
We can write this relation also like \( \partial_{\tilde{s}} a = k a \) on \( \Gamma \) (Geodesic condition).
This relation means that \( \Gamma \) is a critical point of curve length weighted
by \( a \). Let \( L_a[\Gamma] = \int_\Gamma a dl \). Consider a normal perturbation of \( \Gamma \), say
\( \Gamma_h := \{ \gamma(\tilde{s}) + h(\tilde{s}) \nu(\tilde{s}) | \tilde{s} \in (0, l) \} \), \| h \|_{C^2(\Gamma)} \ll 1 \). We want: first
variation along this type of perturbation be equal to zero. This is
\[DL_a[\Gamma_h]|_{h=0} = 0 \]
This means
\[ \frac{\partial}{\partial \lambda} L[\Gamma_{\lambda h}]|_{h=0} = 0 \]
or just \( DL(\Gamma), h = 0 \) for all \( h \). Observe that
\[ L(\Gamma_{\lambda h}) = \int_0^l a(\gamma(\tilde{s}) + h(\tilde{s})\nu(\tilde{s})) \cdot |\dot{\gamma}(\tilde{s})_{\lambda h}| d\tilde{s} \]
and also \( \dot{\gamma}_{\lambda h}(\tilde{s}) = \dot{\gamma}(\tilde{s}) + \dot{\lambda} h \nu + \lambda h \dot{\nu}, \) and \( \dot{\nu} = -k \dot{\gamma} \). With the Taylor expansion
\[ (1 - 2k \lambda h + \lambda^2 k^2 h^2 + \lambda^2 h^2)^{1/2} = 1 + \frac{1}{2} (1 - 2k \lambda h + \lambda^2 k^2 h^2 + \lambda^2 h^2) - \frac{1}{8} 4k^2 \lambda^2 h^2 + O(\lambda^2 h^3) \]
and
\[ a(\gamma(\tilde{s}) + \lambda h(\tilde{s})\nu(\tilde{s})) = a(\tilde{s}, \lambda h(\tilde{s})) = a(\tilde{s}, 0) + \lambda a_z(\tilde{s}, 0) h(\tilde{s}) + \frac{1}{2} \lambda^2 a_{zz}(\tilde{s}, 0) h(\tilde{s})^2 + O(\lambda^3 h^3). \]
we conclude
\[ L_h[\Gamma_{\lambda h}] = L_a(\Gamma) = \lambda \int_0^l (-ka + a_z)(\tilde{s}, 0) h(\tilde{s}) d\tilde{s} + \lambda^2 \int_0^l (\frac{\dot{h}}{2} + a_z k^2 h^2 + \frac{1}{2} a_{zz} h^2) + O(\lambda^3 h^3) \]
This tells us:
\[ \frac{\partial}{\partial \lambda} L_h[\Gamma_{\lambda h}]|_{\lambda=0} = 0 \Leftrightarrow k(\tilde{s}) a(\tilde{s}, 0) = a_z(\tilde{s}, 0), \]
the geodesic condition. Also we conclude that
\[ \frac{\partial^2}{\partial \lambda^2} L(\Gamma_{\lambda h})|_{\lambda=0} = \int_0^l (a \dot{h}^2 - 2k^2 a + a_{zz} h^2) d\tilde{s} = -\int_0^l (a(\tilde{s}, 0) \dot{\tilde{s}} h) + (2a(\tilde{s}, 0) k \ddot{a} - a_{zz}(\tilde{s}, 0) h) h \]
This can be expressed as \( D^2 L(\Gamma) = J_a \), which means \( D^2 L(\Gamma)[h]^2 = -\int_0^l J_a[h] h \). \( J_a[h] \) is called the Jacobi operator of the geodesic \( \Gamma \). Assumption: \( J_a \) is invertible.
We assume that if \( h(\tilde{s}), \tilde{s} \in (0, l) \) is such that \( h(0) = h(l) \), \( \dot{h}(0) = \dot{h}(l) \) and \( J_a[h] = 0 \) then \( h \equiv 0 \). \( Ker(J_a) = \{0\} \), in the space of \( l \)-periodic \( C^2 \) functions. This implies (exercise) that the problem
\[ J_a[h] = g, g \in C(0, l), g(0) = g(l), h(0) = h(l), \dot{h}(0) = \dot{h}(l) \]
has a unique solution \( \phi \). Moreover \( ||\phi||_{C^{2,\alpha}(0, l)} \leq C ||g||_{C^{\alpha}(0, l)} \).
Remember that the equation in coordinates \((s, z)\) is
\[ E(v) = \frac{1}{(1 - \varepsilon k(\varepsilon s) z)} \left( \frac{1}{(1 - \varepsilon k(\varepsilon s) z)} v_s \right)_s + v_{zz} - \frac{\varepsilon k(\varepsilon s)}{(1 - \varepsilon k(\varepsilon s) z)} v_z + \frac{\varepsilon a_z}{a} (1 - \varepsilon k(\varepsilon s) z)^2 v_s + \varepsilon a_z v_z + f(v) = 0 \]
Change of variables: Fix a function $h \in C^{2,\alpha}(0,1)$ with $\|h\| \leq 1$ and do the change of variables $z = h(\varepsilon s) = t$ and take as first approximation $v_0 \equiv w(t)$. Let us see that $v_0(s, z) = w(z - h(\varepsilon s))$ so

$$E(v_0) = \frac{1}{1 - \varepsilon k z} \left( \frac{1}{1 - \varepsilon k z} w'(-\dot{h}(\varepsilon s, \varepsilon z)) + w'' + f(w) \right) + \varepsilon \left( \frac{a \varepsilon}{s} \right) w' - \varepsilon h + \frac{\varepsilon}{(1 - \varepsilon k z)^2} a \varepsilon w'$$

Error in terms of coordinates $(s, t)$ $z = t + h(\varepsilon s)$:

$$E(v_0)(s, t) = \varepsilon w'(t) \left[ \frac{a \varepsilon}{s} (\varepsilon s, \varepsilon (t + h)) - \frac{k(\varepsilon s)}{1 - k(\varepsilon s)(t + h)\varepsilon} \right] - \frac{\varepsilon^2 w'}{(1 - \varepsilon k(t + h))^2} h''$$

In fact

$$|E(v_0)(t, s)| \leq C \varepsilon^2 e^{-\sigma |t|}$$

$\sigma < 1$, and

$$\|e^{|t|} E(v_0)\|_{C^{0,\alpha}(|t|^{\frac{1}{\varepsilon}})} \leq C \varepsilon^2$$

Formal computation: We would like $\int_{-\delta/\varepsilon}^\delta E(v_0)(s, y)w'(t)dt \approx 0$. Observe that

$$-\varepsilon^2 h''(\varepsilon s) \int_{|t| < \delta/\varepsilon} \frac{w'^2}{(1 - k\varepsilon(t + h)\varepsilon)} = -\varepsilon^2 h'' \int \omega^2 dt + O(\varepsilon^3)$$

Also

$$\dot{h}^2 \varepsilon^2 \int \frac{1}{1 - \varepsilon k(t + h)} w'' w' dt = 0 + O(\varepsilon^3).$$

$$\varepsilon^2 \dot{h} \int \frac{a \varepsilon}{s} (\varepsilon s, \varepsilon (t + h)) w'^2 / (1 + k\varepsilon(t + h))^2 = \varepsilon^2 h \frac{a \varepsilon}{s} (\varepsilon s, 0) \int \omega^2 + O(\varepsilon^3)$$

and finally

$$\varepsilon \int_{|t| < \delta/\varepsilon} \frac{w'^2 (a \varepsilon)(\varepsilon s, \varepsilon (t + h))}{1 - k(\varepsilon s)(t + h)\varepsilon} = \varepsilon^2 \int \frac{w'(t)^2 (\varepsilon^2)((a \varepsilon)(\varepsilon s, 0) - k^2) h(\varepsilon s) + O(\varepsilon)}{(a \varepsilon)}$$

Then

$$\int \frac{E w' dt}{\varepsilon^2 \omega^2} = h'' + \dot{h}^2 \frac{a \varepsilon}{s} - \left( \frac{a \varepsilon}{s} \right) (\varepsilon s, 0) - k^2) h + O(\varepsilon)$$

We call $\dot{s} = \varepsilon s$, and we conclude that the right hand side of the above equality is equal to

$$\frac{1}{a(\dot{s}, 0)}((a(\dot{s}, 0)h'') + (2k^2 a(\dot{s}, 0) - a_{\ddot{s}}(\dot{s}, 0)))h + O(\varepsilon)$$
and this is equal to
\[ \frac{1}{a(\tilde{s}, 0)} (J_0[h] + O(\varepsilon)) \]

We need the equation for \( v(s, z) = \tilde{v}(s, z - h(\varepsilon s)) \). We have
\[ \frac{\partial v}{\partial s} = \frac{\partial \tilde{v}}{\partial s} - \frac{\partial \tilde{v}}{\partial t} \frac{h}{\partial s} \]

We write \( z = t + h \), so we have
\[ S(\tilde{v}) = \frac{1}{1 - \varepsilon k z} \left( \frac{\partial}{\partial s} \frac{-\varepsilon k \frac{\partial}{\partial t}}{1 - \varepsilon k (t + h)} \left( \frac{\partial}{\partial s} - \varepsilon k \frac{\partial}{\partial t} \right) \right) \tilde{v} + \tilde{v}_u \]
\[ \varepsilon \left[-\frac{k}{1 - \varepsilon k z} \frac{\partial}{\partial s} + \frac{\partial}{\partial t} \right] \tilde{v}_t + \varepsilon \frac{a_z}{a} \frac{1}{1 - k \varepsilon z} [\tilde{v}_s - \varepsilon h \tilde{v}_t] + f(\tilde{v}) = 0 \]

The first term of this equation is equal to
\[ \frac{1}{1 - \varepsilon k z} \left( \varepsilon \left(-\frac{k}{1 - \varepsilon k t (t + h)} \right) + \frac{1}{1 - \varepsilon k (t + h)} \left(-\varepsilon^2 h'' v_t - 2\varepsilon h v_t s + \frac{1}{1 + \varepsilon k (t + h)} \right) \right) \tilde{v}_s + \tilde{v}_u \]

Let us observe that for \( |t| < \delta / \varepsilon, \delta \ll 1 \)
\[ S[\tilde{v}](s, t) = \tilde{v}_{ss} + \tilde{v}_u + O(\varepsilon) \partial_s \tilde{v} + O(\varepsilon) \partial_s \tilde{v} + O(\varepsilon) \partial_s \tilde{v} + f(v) = 0 \]

We will call the operator that appears in the equation \( B[\tilde{v}] \). We look for a solution of the form \( \tilde{v}(s, t) = w(t) + \phi(s, t) \). The equation for \( \phi \) is
\[ \phi_{ss} + \phi_u + f'(w(t)) \phi + E + B(\phi) + N(\phi) = 0, \quad |t| < \delta / \varepsilon \]

where \( E = S(w(t)) = O(\varepsilon^2 e^{-\sigma t}), \ N(\phi) = f(w + \phi) - f(w) - f'(w) \phi, s \in (0, 1 / \varepsilon) \). We use the notation \( L(\phi) = \phi_{ss} + \phi_u + f'(w(t)) \phi \). We also need the boundary condition \( \phi(0, t) = \phi(1 / \varepsilon, t) \) and \( \phi_s(0, t) = \phi_s(1 / \varepsilon, t) \).

It is natural to study the linear operator in \( \mathbb{R}^2 \) and the linear projected problem
\[ \phi_{ss} + \phi_u + f'(w(t)) \phi + g(t, s) = c(s) w'(t) \]

where \( c(s) = \frac{\int_s^t g(t, s) w'(t) dt}{\int_s^t w'(t)^2 dt} \) and under the orthogonally condition
\[ \int_{-\infty}^{\infty} \phi(s, t) w'(t) dt = 0, \quad \forall s \in \mathbb{R} \]

Basic ingredient: (Even more general) Consider the problem in \( \mathbb{R}^m \times \mathbb{R} \), with variables \( (y, t) \):
\[ \Delta_y \phi + \phi_u + f'(w(t)) \phi = 0, \quad \phi \in L^\infty(\mathbb{R}^m \times \mathbb{R}) \]
If \( \phi \) is a solution of the above problem, then \( \phi(y, t) = \alpha w'(t) \) some \( \alpha \in \mathbb{R} \). Ingredient: \( \exists \gamma > 0 : \int_{\mathbb{R}} p'(t)^2 - f'(w(t))p(t)^2 \geq \gamma \int_{\mathbb{R}} p^2(t) dt \)
for all \( p \in H^1 \) with \( \int_{\mathbb{R}} pu' = 0 \). \( \psi(y) = \int_{\mathbb{R}} \phi^2(y,t) dt \). This is well defined (as we will see). Indeed: It turns out that \( |\phi(y,t)| \leq Ce^{-\sigma t}, \) \( \sigma < \sqrt{2} \), thanks to the fact that \( \phi \in L^\infty \). We use \( x = (y,t) \) and we obtain

\[
\Delta_x \phi - (2 - 3(1 - w(t)^2)) \phi = 0
\]

Observe that \( 1 - w(t)^2 \) is small if \( |t| \gg 1 \). Fix \( 0 < \sigma < \sqrt{2}, \) for \( |t| > R_0 \) we have \( 2 - 3(1 - w^2(t)) > \sigma^2 \). Let

\[
\bar{\phi}_\rho(y,t) = \rho \sum_{i=1}^n \cosh(\sigma y_i) + \rho \cosh(\sigma t) + \|\phi\|_\infty e^{\sigma R_0} e^{-\sigma |t|}.
\]

We have that

\[
\phi(y,t) \leq \bar{\phi}_\rho(y,t), \quad \text{for } |t| = R_0
\]

also true that for \( |t| + |y| > R_\rho \gg 1 \), \( \phi(y,t) \leq \bar{\phi}_\rho \).

\[-\Delta_x \phi + (2 - 3(1 - w(t)^2)) \bar{\phi} = (2 - \sigma^2 - 3(1 - w(t)^2) \bar{\phi}_\rho) > 0 \]

for \( |t| > R_0 \). So is a supersolution of the operator

\[-\Delta_x \phi + (2 - 3(1 - w(t)^2)) \phi \]

in \( D_\rho \), which implies that \( \phi \leq \bar{\phi}_\rho \) for \( |t| > R_0 \). This implies that \( |\phi(x)| \leq C\bar{\phi}_\rho \) for all \( x \), and we conclude the assertion taking \( \rho \to 0 \). If \( \phi \) solves \(-\Delta \phi + (1 - 3w^2) \phi = 0, \) then \( \|\phi\|_{C^{2,\alpha}(B_1(x_0))} \leq C\|\phi\|_{L^\infty(B_2(x_0))} \).

This implies that also

\[|\phi_y| + |\phi_{yy}| \leq C e^{-\sigma t}.\]

Let \( \phi(y,t) = \phi(y,t) - \frac{1}{w^2} \int_{\mathbb{R}} \phi(y,r) w'(r) dr w'. \) We call \( \beta(y) = \frac{1}{w^2} \int_{\mathbb{R}} \phi(y,r) w'(r) dr \)

\[
\Delta \tilde{\phi} + f'(w) \tilde{\phi} = \Delta \phi + f'(w) \phi + (\Delta_y \beta) w' + \beta(\Delta w' + f'(w)) w' = 0
\]

because \( \Delta_y \beta = 0 \) by integration by parts. Let \( \psi(y) = \int_{\mathbb{R}} \tilde{\phi}^2 dt. \)

\[
\Delta_y \psi = \int_{\mathbb{R}} \nabla_y (2\tilde{\phi} \nabla_y \tilde{\phi}) dt = 2 \int |\nabla_y \tilde{\phi}|^2 dt + 2 \int \phi \Delta_y \tilde{\phi} = 2 \int |\nabla_y \tilde{\phi}|^2 - 2 \int \tilde{\phi} \phi_t + f'(w) \tilde{\phi} dt
\]

Using \( 2 \int |\nabla_y \tilde{\phi}|^2 dt + 2 \int (\tilde{\phi}_t^2 - f'(w) \tilde{\phi})^2 \) This implies that \( \Delta \psi \geq 2\gamma \psi \) which implies \( -\Delta \psi + 2\gamma \psi \leq 0, \) \( 0 \leq \psi \leq c. \)

We obtain that \( \psi \equiv 0 \) and this implies \( \tilde{\phi} = 0 \). This implies that \( \phi(t) = (\int \phi w') w' = \beta(y) w' \) and \( \Delta \beta = 0, \) \( \beta \in L^\infty. \) Liouville implies that \( \beta = \text{constant} \) so \( \phi = \text{constant} w'. \)

Lemma: \( L^\infty \) a priori estimates for the linear projected problem:

\[\exists C: \|\phi\|_\infty \leq C\|g\|_\infty.\]

Proof: If not exists \( \|g_n\|_\infty \to 0 \) and \( \|\phi_n\|_\infty = 1. \)

\[L[\phi_n] = -g_n + c_n(t)w'(t) = h_n(t)\]
and \( h_n \to 0 \) in \( L^\infty \). \( \| \phi_n \| = 1 \) which implies that \( \exists (y_n, t_n): |\phi(y_n, t_n)| \geq \gamma > 0 \). Assume that \( |t_n| \leq C \) and define \( \tilde{\phi}(y, t) = \phi_n(y_n + y, t) \). Then

\[
\Delta \tilde{\phi}_n + f'(w(t))\tilde{\phi}_n = \tilde{h}_n
\]

but \( f'(w(t))\tilde{\phi}_n \) is uniformly bounded and the right hand side goes to 0. This implies that \( \| \tilde{\phi} \|_{C^1(\mathbb{R}^{m+1})} \leq C \) This implies that \( \tilde{\phi}_n \to \tilde{\phi} \) passing to subsequence, and the convergence is uniformly on compacts, where \( \Delta \tilde{\phi} + f'(w)\tilde{\phi} = 0, \tilde{\phi} \in L^\infty \). We conclude after a classic argument that \( \tilde{\phi} = 0 \). We have also that \( \| e^{\sigma |t|} \phi \|_\infty \leq C \| e^{\sigma |t|} g \|_\infty, 0 < \sigma < \sqrt{2} \). Elliptic regularity implies that \( \| e^{\sigma |t|} \phi \|_{C^2, \sigma} \leq \| e^{\sigma |t|} g \|_{C^{0, \sigma}} \).

Existence: Assume \( g \) has compact support and take the weak formulation: Find \( \phi \in H \) such that

\[
\int_{\mathbb{R}^{m+1}} \nabla \phi \nabla \psi - f'(w)\phi \psi = \int g y, \text{for all } \psi \in H,
\]

where \( H = \{ f \in H^1(\mathbb{R}^{m+1}) \mid \int \psi w' dt = 0, \forall y \in \mathbb{R}^m \} \).

Let us see that \( a(\psi, \psi) = \int |\nabla \psi|^2 - f'(w) \psi^2 \geq \gamma \int \psi^2 + \psi^2 \). So \( a(\psi, \psi) \geq C \| \psi \|_{H^1(\mathbb{R}^{m+1})} \) This implies the unique existence solution. Observe that

\[
\int (\Delta \phi + f'(w)\phi + g)\psi = 0
\]

for all \( \psi \in H \). Let \( \psi \in H^1 \) and \( \psi = \tilde{\psi} - \int \frac{\psi w' dt}{\int w \sigma} w' = \Pi(\tilde{\psi}) \). We have that

\[
\int dy \int g \Pi(\tilde{\psi}) dt = \int \Pi(g) \psi
\]

which implies that \( \Pi(\Delta \phi + f'(w)\phi + g) = 0 \) if and only if \( \Delta \phi + f'(w)\phi + g = \frac{\int (\Delta \phi + f'(w)\phi + g) w} {\int w^2} \) Regularity implies that \( \phi \in L^\infty \) and \( \| \phi \|_\infty \leq C \| g \|_\infty \). Approximating \( g \in L^\infty \) by \( g_R \in C^\infty (\mathbb{R}^N) \) locally over compacts. This implies existence result.

We can bound \( \phi \) in other norms. For example if \( 0 < \sigma < \sqrt{2} \), then

\[
\| e^{\sigma |t|} \phi \|_\infty \leq C \| e^{\sigma |t|} g \|_\infty.
\]

Indeed, \( f'(w) < -\sigma^2 - \eta \) if \( |t| > R \), with \( \eta = (2 - \sigma^2)/2 \). We set

\[
\tilde{\phi} = M e^{-\sigma |t|} + \rho \sum_{i=1}^n \cosh(\sigma y_i) + \rho \cosh(\sigma t).
\]

Therefore

\[
-\Delta \tilde{\phi} + (-f'(w))\tilde{\phi} \geq -\delta \tilde{\phi} + (\sigma^2 + \eta) \tilde{\phi} = \eta \tilde{\phi} > \tilde{g} = -g + c(y)w'(t)
\]

if \( M \geq \frac{A}{\eta} \| e^{\sigma |t|} g \|_\infty \). In addition we have \( \tilde{\phi} \geq \phi \) on \( |t| = R \) if \( M \geq \| \phi \|_\infty e^{\sigma R} \). By an standard argument based on maximum principle, we conclude that \( \phi \leq \tilde{\phi} \). This means, letting \( \rho \to 0, \phi \leq Me^{-\sigma |t|} \), where \( M \geq C \max \{ \| \phi \|_\infty, \| ge^{\sigma |t|} \|_\infty \} \). Since \( \| \phi \|_\infty \leq C \| g \|_\infty \leq C \| ge^{\sigma |t|} \|_\infty \), we can take \( M = C \| ge^{\sigma |t|} \|_\infty \). Finally, we conclude \( \| \phi e^{\sigma |t|} \|_\infty \leq \| ge^{\sigma |t|} \|_\infty \).
Our setting:

In order to prove this result we define

\[ g = \frac{\rho f}{\sigma}. \]

The proof of this fact is very similar to the previous one (use that

\[ \Delta (\delta y) = \rho^{-2} \Delta (\rho^{-1} (\delta y)) + \delta^2 (\rho^{-1}) (\delta y) = f' (w) + g - cw' \]

We get \( L [\tilde{\phi}] = O (\delta^2) \tilde{\phi} + \rho (g - cw'). \) We get

\[ \| \Delta \tilde{\phi} \| \| \tilde{\phi} \| \| \Delta \tilde{\phi} \| + \| \tilde{\phi} \| \leq C (\delta^2 \| \tilde{\phi} \| \| \rho g \| \| \rho g \| \). \]

If \( \delta \) is small we conclude that

\[ \| \tilde{\phi} \| \| \Delta \tilde{\phi} \| \| \tilde{\phi} \| \leq C (\| \rho g \| \| \rho g \| \). \]

and we obtain

\[ \| \rho \tilde{\phi} \| \leq \| \rho g \|. \]

Our setting:

\[ \epsilon^2 [\delta u + \nabla a \cdot \nabla u] + f (u) = 0 \]
We want a solution to (4.3) \( u(x) \approx W(z/\varepsilon) \). Writing \( x = y + z\gamma(y) \), |\( z \| < \delta \), we have

\[
\Delta v + \nabla a(\varepsilon x)/a \cdot \nabla v + f(v) = 0,
\]

in \( \Gamma_\varepsilon = \frac{1}{\varepsilon} \Gamma \): \( x = y + z\nu(\varepsilon y) \), which means \( x = \frac{1}{\varepsilon} \gamma(\varepsilon s) + \varepsilon \nu(\varepsilon s) \). Remember that \( |\gamma(\hat{s})| = 1 \) which implies \( \hat{\nu}(\hat{s}) = -k(\hat{s})\hat{\gamma}(\hat{s}) \). We also set \( z = h(\varepsilon s + t \cdot x = \frac{1}{\varepsilon} \gamma(\varepsilon s) + (t + h(\varepsilon s))\nu(\varepsilon s) \). We assume \( \|h\|_{\alpha,(0,t)} \leq 1 \), for \( 0 < \alpha < 1 \). We wrote \( \Delta_x \) in terms of this coordinates \( (t,s) \) and the equations \( S(v) = 0 \) is rewritten taking as first approximation \( w(t) \). We evaluated \( S(w(t)) \) and got that \( S(w(t)) = 0 \).

From the expression of \( \Delta_x \) we get \((x = \frac{1}{\varepsilon} \gamma(\varepsilon s) + (t + h(\varepsilon s))\nu(\varepsilon s))\)

\[
\Delta_x v = \partial_{ss} + \partial_t + \varepsilon[b_1^s(t,s)\partial_{ss} + b_2^s\partial_t + b_4^s\partial_t + b_5^s\partial_t + b_6^s\partial_t]
\]

\(|\varepsilon b_1| \leq C\delta \) in the region \(|t| < \delta/\varepsilon \). The coefficients are periodic (same values at \( s = 0 \) and \( s = l/\varepsilon \)). Our equation reads

\[
\partial_{ss} v + \partial_t v + B(\varepsilon v) + f(v) = 0, \quad \text{for } s \in (0,l/\varepsilon), \ |t| < \delta/\varepsilon.
\]

This expression does not make sense globally. We consider \( \delta \ll 1 \). We define

\[
H(x) = \begin{cases} 
-1 & \text{in } \Omega_-^x \\
+1 & \text{in } \Omega_+^x
\end{cases}
\]

where \( \Omega_-^x \) is a bounded component of \( \mathbb{R}^2 \setminus \Gamma \), and \( \Omega_+^x \) the other. For the equation

\[
\Delta v + \varepsilon \frac{\nabla a}{a} \cdot \nabla v + f(v) = 0
\]

we take as first (global) approximation

\[
v_0(x) = w(t)\eta_3 + (1 - \eta_4)H(x)
\]

where

\[
\eta_t(x) = \begin{cases} 
\eta \left( \frac{|t|}{2\delta} \right) & \text{if } |t| < 2\delta l/\varepsilon \\
0 & \text{otherwise}
\end{cases}
\]

Look for a solution of the form \( v = v_0 + \tilde{\phi} \), so

\[
\Delta_x \tilde{\phi} + \varepsilon \frac{\nabla a}{a} \cdot \nabla \tilde{\phi} + f'(v_0)\tilde{\phi} + E + N(\tilde{\phi}) = 0
\]

where \( E = S(v_0) \) and \( N(\tilde{\phi}) = f(v_0 + \tilde{\phi}) - f(v_0) - f'(v_0)\tilde{\phi} \). We write \( \tilde{\phi} = \eta_3\phi + \psi \). We require that \( \phi \) and \( \psi \) solve the system

\[
\eta_3 \left[ \Delta_x \phi + f'(w(t))\phi + \eta_4(2 + f'(w(t)))\psi + \eta_4 E + \eta_1 N(\phi + \psi) + \varepsilon \frac{\nabla a}{a} \cdot \nabla \phi \right] = 0.
\]
We need that the $\phi$ above satisfies the equation just for $|t| < 6\delta/\varepsilon$. We assume that $\phi(s, t)$ is defined for all $s$ and $t$ (and it is $l/\varepsilon$- periodic in $s$). We require that $\phi$ satisfies globally

$$
\phi_{tt} + \phi_{ss} + \eta_6 B_\varepsilon[\phi] + f'(w(t))\phi + \eta_1 E + \eta_1 N(\phi + \psi) + \eta_1(2 + f'(w))\psi = 0
$$

and $\phi \in L^\infty(\mathbb{R}^n + 1)$ and periodic in $s$. Notice that $\phi_{tt} + \phi_{ss} + \eta_6 B_\varepsilon[\phi] = \Delta_x \phi$ inside the support of $\eta_3$. Rather than solving this problem directly we solve the projected problem

(4.4)

$$
\phi_{tt} + \phi_{ss} + \eta_6 B_\varepsilon[\phi] + f'(w(t))\phi + \eta_1 E + \eta_1 N(\phi + \psi) + \eta_1(2 + f'(w))\psi = c(s)w'(t)
$$

and $\int_{\mathbb{R}} \phi w'(t) dt = 0$. We solve (4)-(4.4) first, then we find $h$ such that $c(s) \equiv 0$. We consider $\phi$ with $\|\phi\|_\infty + \|\nabla \phi\|_\infty \leq \varepsilon$. The operator $-\Delta \psi + 2\psi$ is invertible $L^\infty(\mathbb{R}^3) \to C^1(\mathbb{R}^2)$. We conclude that if $g \in L^\infty$ the exist a unique solution $\psi = T[g] \in C^1(\mathbb{R}^2)$ with $\|\phi\|_{C^1} \leq C\|g\|_\infty$ of equation $-\Delta \psi + 2\psi = g$ in $\mathbb{R}^2$. Observe that (4) is equivalent to

$$
\psi = T[(2+f'(v_0))(1-\eta_1)\psi + \varepsilon \frac{\nabla a}{a} \nabla \psi + (1-\eta_1)E + (1-\eta_1)N(\eta_3 \phi + \psi) + \nabla \eta_3 \nabla \phi + \nabla \eta_3 \nabla \phi + \varepsilon \frac{\nabla a}{a} \nabla \phi]
$$

Using contraction mapping in $C^1$ on $\|\psi\|_{C^1} \leq C\varepsilon$, we conclude that there exist a unique solution of the this problem $\psi = \psi(\phi, h)$ such that

$$
\|\psi\| \leq C[\varepsilon^2 + \varepsilon \|\phi\|_{C^1}].
$$

Even more, $\|\psi(\phi_1, h) - \psi(\phi_2, h)\|_{C^1} \leq C\varepsilon \|\phi_1 - \phi_2\|_{C^1}$.