SERIES

DEFINITION An infinite series has form $c_0 + c_1 + c_2 + \dots = \sum_{j=0}^{\infty} c_j$ where c_j are complex numbers. The n^{th} partial sum is $S_n = \sum_{j=0}^{n} c_j$. If $\lim_{n \to \infty} S_n = S$, then the series converges to S and we write $S = \sum_{j=0}^{\infty} c_j$. A series that does not converge is said to diverge.

EXAMPLE The geometric series $\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}$ if $|z| < 1$.

PROOF $(1-z)(1+z+\dots+z^n) = (1+z+\dots+z^n) - (z+z^2+\dots+z^{n+1}) = 1 - z^{n+1}$.

Thus $1 + z + \dots + z^n = \frac{1-z^{n+1}}{1-z}$.

Then since $|z|<1$ we let $n \to \infty$ and use $|z|^n \to 0$ as $n \to \infty$ to obtain

$$1 + z + z^2 + \dots = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{in } |z| < 1.$$

LEMMA (RATIO TEST) Consider $\sum_{j=0}^{\infty} c_j$. Assume that $\lim_{j \to \infty} \frac{|c_{j+1}|}{|c_j|}$ exists and that $\lim_{j \to \infty} \frac{|c_{j+1}|}{|c_j|} = L$.

- The series $\sum_{j=0}^{\infty} c_j$ converges if $L < 1$ and diverges if $L > 1$
- No information on convergence\divergence if $L = 1$.

(The proof of this is omitted)

A POWER SERIES ABOUT z_0 IS A SERIES OF THE FORM

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{WHERE} \quad z_0, a_1, a_2, \ldots \text{ ARE COMPLEX CONSTANTS.}$$

FOR ANY POWER SERIES WE HAVE EITHER

(i) IT CONVERGES ONLY FOR $z = z_0$

OR (ii) IT CONVERGES FOR ALL z

OR (iii) IT CONVERGES FOR SOME $z \neq z_0$, BUT NOT FOR ALL z.

WE WOULD LIKE TO DETERMINE A NUMBER R, CALLED THE RADIUS OF CONVERGENCE OF THE POWER SERIES, FOR WHICH

$$\text{IF } |z-z_0| < R \longrightarrow \sum_{n=0}^{\infty} a_n (z-z_0)^n \text{ CONVERGES}$$

$$\text{IF } |z-z_0| > R \longrightarrow \sum_{n=0}^{\infty} a_n (z-z_0)^n \text{ DIVERGES.}$$

THIS CAN BE DONE BY USING THE RATIO TEST.

EXAMPLE FIND RADIUS OF CONVERGENCE FOR $\sum_{n=0}^{\infty} \frac{i^n n^2}{2^{2n}} z^n$ ABOUT $z=0$.

SOLUTION $c_n = \frac{i^n n^2}{2^{2n}} z^n$. BY RATIO TEST, $\left|\frac{c_{n+1}}{c_n}\right| = \frac{(n+1)^2 |z|^{n+1}}{2^{2(n+1)}} \frac{2^{2n}}{n^2 |z|^n}$

LET $n \to \infty$, $\left|\frac{c_{n+1}}{c_n}\right| \to \frac{|z|}{4}$. THUS WE NEED $\frac{|z|}{4} < 1$ FOR CONVERGENCE.

$\to$ THE SERIES CONVERGE INSIDE $|z| < 4$. RADIUS OF CONVERGENCE $= 4$.

THEOREM CONSIDER $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$. SUPPOSE THAT THE RADIUS OF CONVERGENCE OF THIS SERIES IS R; i.e. CONVERGES IN $|z-z_0| < R$.

THEN, IN $|z-z_0| < R$, $f(z)$ IS AN ANALYTIC FUNCTION, AND WE CAN DIFFERENTIATE OR INTEGRATE THE SERIES TERM-BY-TERM AS OFTEN AS WE WISH.

$$\text{i.e. } f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}, \quad f''(z) = \sum_{n=2}^{\infty} n(n-1) a_n (z-z_0)^{n-2}$$

IN $|z-z_0| < R$. PROOF: (NEXT PAGE)

THEOREM CONSIDER $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$

SUPPOSE THE SERIES HAS A POSITIVE OR INFINITE RADIUS OF CONVERGENCE R. WE WILL SHOW THAT $f(z)$ IS ANALYTIC IN $|z-z_0| < R$ AND ITS DERIVATIVE IS

$$(*) \quad f'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}.$$

PROOF THE SERIES IN $(*)$ CONVERGES, AND HAS RADIUS OF CONVERGENCE AT LEAST R.

LET $|z-z_0| = r < R$ AND LET $r < s < R$. THEN FOR $n \geq N$,

$$n r^{n-1} \leq s^n \qquad n \geq N \qquad \text{SINCE} \quad \lim_{n\to\infty} n \left(r/s \right)^n = 0.$$

THUS

$$n |a_n| |z-z_0|^{n-1} \leq |a_n| s^n \qquad \text{FOR} \quad n \geq N.$$

NOW SINCE $s < R$ THE SERIES $\sum_{n=0}^{\infty} |a_n| s^n$ CONVERGES. HENCE

$$\sum_{n=1}^{\infty} n |a_n| |z-z_0|^{n-1} \text{ CONVERGES} \longrightarrow \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1} \text{ CONVERGES.}$$

NOW LET

$$g(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}. \qquad \text{NEED TO SHOW} \quad g(z) = f'(z).$$

FOR SIMPLICITY LET $z_0 = 0$. THEN LET

$$|z| < R \quad \text{AND} \quad \delta = \frac{1}{2} \left[R - |z| \right] \qquad |h| < \delta.$$

CALCULATE

$$f(z+h) - f(z) = \sum_{n=1}^{\infty} a_n \left[(z+h)^n - z^n \right]$$

NOW WE GET,

$$\frac{f(z+h) - f(z)}{h} = \sum_{n=1}^{\infty} a_n \left[\frac{(z+h)^n - z^n}{h} \right]$$

NOW SUBTRACT $g(z)$;

$$\frac{f(z+h) - f(z)}{h} - g(z) = \sum_{n=1}^{\infty} \left(a_n \left[\frac{(z+h)^n - z^n}{h} \right] - n a_n z^{n-1} \right)$$

THUS

$$\frac{f(z+h) - f(z)}{h} - g(z) = \sum_{n=2}^{\infty} a_n \left\{ \frac{(z+h)^n - z^n}{h} - n z^{n-1} \right\}.$$

NOW WE WANT TO ESTIMATE THE TERM IN BRACKETS TO SHOW THAT WE GET

0 AS $h \to 0$.

TO THIS END WE USE THE BINOMIAL THEOREM

$$(z+h)^n = \sum_{m=0}^{n} \binom{n}{m} z^{n-m} h^m = z^n + \binom{n}{1} z^{n-1} h + \sum_{m=2}^{n} \binom{n}{m} z^{n-m} h^m$$

$$\frac{(z+h)^n - z^n}{h} = \sum_{m=2}^{n} \binom{n}{m} z^{n-m} \frac{h^m}{h}$$

THEREFORE,

$$\frac{(z+h)^n - z^n}{h} - nz^{n-1} = \sum_{m=2}^{n} \binom{n}{m} z^{n-m} h^{m-1}$$

NOW $|z| = r$ AND $|h| < \frac{1}{2}[R-r]$ $\qquad \delta = \frac{1}{2}[R-r]$

THIS GIVES,

$$\left|\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right| \le h \sum_{m=2}^{n} \binom{n}{m} r^{n-m} \delta^{m-2} \le \frac{h}{\delta^2} \sum_{m=0}^{n} \binom{n}{m} r^{n-m} \delta^m$$

THEREFORE WE GET

$$\left|\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right| \le \frac{h}{\delta^2}(r+\delta)^n \qquad \delta = \frac{1}{2}[R-r]$$

$$r = R - 2\delta$$

THUS

$$\left|\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right| \le \frac{h}{\delta^2}(R-\delta)^n$$

NOW

$$\left|\frac{f(z+h) - f(z)}{h} - g(z)\right| \le \frac{h}{\delta^2} \sum_{n=2}^{\infty} |a_n| (R-\delta)^n$$

BUT NOW SINCE $\delta < R$ THE SERIES CONVERGES AND HENCE AS $h \to 0$ WE GET,

$$\left|\frac{f(z+h) - f(z)}{h} - g(z)\right| \to 0.$$

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = g(z)$$

f IS DIFFERENTIABLE AND ANALYTIC AT INSIDE $|z| < R$.

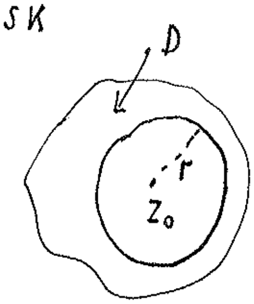
<u>THEOREM</u> IF $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ HAS RADIUS OF CONVERGENCE R, THEN

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, a_n (z-z_0)^{n-k} \qquad k = 1, 2, \ldots$$

IN PARTICULAR, $\dfrac{f^{(n)}(z_0)}{n!} = a_n$.

THEOREM (TAYLOR SERIES)

LET $f(z)$ BE ANALYTIC IN A DOMAIN D AND THAT THE DISK $\{z \mid |z-z_0| \le r\}$ LIES INSIDE D. THEN, WITHIN THE DISK, $f(z)$ IS GIVEN BY A CONVERGENT POWER SERIES



$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad |z-z_0| < r$$

WHERE $$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}\, d\zeta = \frac{f^{(n)}(z_0)}{n!}, \quad n = 0,1,2,3,\ldots$$

AND C IS A CIRCLE $\{\zeta : |\zeta - z_0| = r\}$ ORIENTED COUNTERCLOCKWISE.

PROOF FIX z WITH $|z-z_0| = s < r$. LET ζ BE ANY POINT WITH $|\zeta - z_0| = r$.

WE CALCULATE $$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z-z_0)} = \frac{1}{(\zeta - z_0)\left[1 - \left(\frac{z-z_0}{\zeta - z_0}\right)\right]} \quad (**)$$

NOW SINCE $\left|\frac{z-z_0}{\zeta - z_0}\right| = \frac{s}{r} < 1$ WE CAN REPRESENT $\left[1 - \left(\frac{z-z_0}{\zeta-z_0}\right)\right]^{-1}$

BY THE CONVERGENT GEOMETRIC SERIES $\left(\frac{1}{1-w} = 1 + w + w^2 + \ldots \text{ FOR } |w|<1\right)$

$$\frac{1}{1 - \left(\frac{z-z_0}{\zeta-z_0}\right)} = 1 + \left(\frac{z-z_0}{\zeta-z_0}\right) + \left(\frac{z-z_0}{\zeta-z_0}\right)^2 + \ldots = \sum_{k=0}^{\infty} \left(\frac{z-z_0}{\zeta-z_0}\right)^k. \quad (*)$$

SINCE THE SERIES CONVERGES THEN FOR ANY $\varepsilon > 0$ WE CAN CHOOSE N SUFFICIENTLY BIG SO THAT $\sum_{k=N}^{\infty} \left|\left(\frac{z-z_0}{\zeta-z_0}\right)\right|^k < \varepsilon$. HENCE FOR ALL ζ WITH $|\zeta - z_0| = r$

AND ALL $n \ge N$ WE HAVE

$$\left|\sum_{k=n}^{\infty} \left(\frac{z-z_0}{\zeta-z_0}\right)^k\right| \le \sum_{k=n}^{\infty} \left|\frac{z-z_0}{\zeta-z_0}\right|^k \le \sum_{k=N}^{\infty} \left|\frac{z-z_0}{\zeta-z_0}\right|^k < \varepsilon \quad (+).$$

NOW WE USE CAUCHY INTEGRAL FORMULA TOGETHER WITH $(*)$ AND $(**)$

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0} \left[\sum_{k=0}^{\infty} \left(\frac{z-z_0}{\zeta-z_0}\right)^k\right] d\zeta$$

SO $$f(z) = \frac{1}{2\pi i} \sum_{k=0}^{n} (z-z_0)^k \left(\int_C \frac{f(\zeta)}{(\zeta-z_0)^{k+1}}\, d\zeta\right) + \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0} \left(\sum_{k=n+1}^{\infty} \left(\frac{z-z_0}{\zeta-z_0}\right)^k\right) d\zeta$$

NOW WE HAVE

$$f(z) = \sum_{k=0}^{n} a_k (z-z_0)^k + E(z;z_0)$$

WHERE $\quad a_k = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta = \frac{f^{(k)}(z_0)}{k!}$

AND $\quad E(z;z_0) \equiv \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z_0} R_{n+1} \, d\zeta \qquad R_{n+1} = \sum_{k=n+1}^{\infty} \left(\frac{z-z_0}{\zeta - z_0} \right)^k .$

CHOOSE n SO LARGE THAT (†) HOLDS, I.E. $|R_{n+1}| < \varepsilon$.

WE ESTIMATE $\quad |E| \leq \frac{1}{2\pi} \max_{\zeta \text{ on } C} \left(\left| \frac{f(\zeta)}{(\zeta - z_0)} \varepsilon \right| \right) 2\pi r.$

BUT $|f(\zeta)| \leq M = \max_{\zeta \text{ on } C} [f]$ WE GET $\quad |E| \leq \varepsilon M.$

THUS THE INFINITE SERIES $\sum_{k=0}^{\infty} a_k (z-z_0)^k$ CONVERGES.

<u>EXAMPLE</u> SOME COMMON TAYLOR SERIES ABOUT $z=0$ ARE

$$e^z = 1 + z + z^2/2! + z^3/3! + \dots = \sum_{n=0}^{\infty} z^n/n! \qquad \text{FOR ALL } z$$

$$\sin z = z - z^3/3! + z^5/5! - z^7/7! \dots = \sum_{n=0}^{\infty} (-1)^n z^{2n+1}/(2n+1)! \qquad \text{FOR ALL } z$$

$$\cos z = 1 - z^2/2! + z^4/4! - z^6/6! \dots = \sum_{n=0}^{\infty} (-1)^n z^{2n}/(2n)! \qquad \text{FOR ALL } z$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots = \sum_{n=0}^{\infty} z^n. \qquad \text{FOR } |z| < 1.$$

<u>REMARKS</u> (i) IF $f(z)$ IS ANALYTIC INSIDE C AND WE DEVELOP ITS TAYLOR SERIES CENTERED AT THE CENTER OF C, WE <u>DO NOT</u> NEED TO EXPLICITLY CHECK THE CONVERGENCE PROPERTY.

(ii) THE RADIUS OF CONVERGENCE OF THE TAYLOR SERIES OF $f(z)$ AT z_0 IS THE DISTANCE TO THE NEAREST SINGULARITY OF $f(z)$.

(iii) IN EXAMPLES BELOW WE MANIPULATE BASIC SERIES INSTEAD OF FINDING DERIVATIVES.

<u>EXAMPLE 0</u> RECALL THAT $\dfrac{1}{1-Z} = \sum_{n=0}^{\infty} Z^n$ IN $|Z|<1$.

WE INTEGRATE THE LHS, AND THE RHS TERM-BY-TERM WHICH IS PERMISSIBLE FOR POWER SERIES INSIDE THEIR RADIUS OF CONVERGENCE. THIS GIVES

$$\mathrm{LOG}(1-Z) = z + z^2/2 + z^3/3 + \ldots = \sum_{n=1}^{\infty} z^n/n \quad \text{FOR } |z|<1.$$

LOG(..) PRINCIPAL BRANCH

NOTICE THAT WE CHOOSE CONSTANT OF INTEGRATION CONSISTENT WITH LOG(1)=0

<u>EXAMPLE 1</u> FIND TAYLOR SERIES OF $f(z)=1/z^2$ ABOUT $Z=-1$. WHAT IS RADIUS OF CONVERGENCE?

<u>SOLUTION</u> $f(z)$ HAS A SINGULARITY AT $Z=0$. SO RADIUS OF CONVERGENCE IS DISTANCE FROM $Z=-1$ TO $Z=0$; I.E. RADIUS OF CONVERGENCE $=1$

NOW WRITE $f(z) = \dfrac{1}{[1-(z+1)]^2}$. NOTE: $\dfrac{1}{1-h} = \sum_{n=0}^{\infty} h^n$ FOR $|h|<1$

$$\rightarrow \frac{1}{(1-h)^2} = \sum_{n=1}^{\infty} n h^{n-1}$$

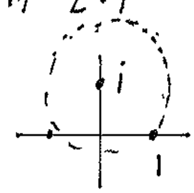
LET $h=(z+1)$. THUS

$$f(z) = \frac{1}{(1-(z+1))^2} = \sum_{n=1}^{\infty} n(z+1)^{n-1} = 1 + \sum_{n=1}^{\infty}(n+1)(z+1)^n,$$

FOR $|z+1|<1$.

<u>EXAMPLE 2</u> FIND TAYLOR SERIES OF $f(z) = \dfrac{1+z}{1-z}$ AROUND $z=i$. WHAT IS THE RADIUS OF CONVERGENCE?

<u>SOLUTION</u> $f(z)$ IS ANALYTIC EXCEPT AT $Z=1$. THE DISTANCE FROM $Z=i$ TO THE SINGULARITY AT $Z=1$ IS $\sqrt{2}$. SO THE SERIES WILL CONVERGE IN $|z-i|<\sqrt{2}$.



WRITE $f(z) = \dfrac{(1+i)+(z-i)}{(1-i)-(z-i)} = \dfrac{[(1+i)+(z-i)]}{(1-i)}\left[1-\left(\dfrac{z-i}{1-i}\right)\right]^{-1}$ LET $C = \dfrac{z-i}{1-i}$.

$$f(z) = \left[\frac{(1+i)}{(1-i)} + C\right]\left[1+C+C^2+\ldots\right] = \left[\frac{(1+i)(1+i)}{2} + C\right]\left[1+C+C^2+\ldots\right] = [i+C][1+C+\ldots+C^n+\ldots]$$

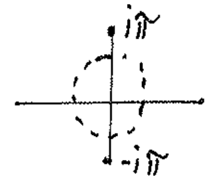
SO $f(z) = i(1+C+C^2+\ldots) + (C+C^2+\ldots) = i + (1+i)[C+C^2+C^3+\ldots]$

$$\text{SO} \quad f(z) = i + (1+i)\sum_{n=1}^{\infty}\left(\frac{z-i}{1-i}\right)^n \quad \text{FOR } |z-i| < |1-i| = \sqrt{2}.$$

EXAMPLE 3 FIND THE FIRST 3 NON-ZERO TERMS IN THE EXPANSION OF

(i) $\dfrac{1}{1+e^z}$ AROUND $z=0$ (ii) $\sec z$ AROUND $z=0$.

WHAT IS THE RADIUS OF CONVERGENCE OF EACH SERIES?



SOLUTION

(i) $f(z) = 1/_{1+e^z}$ IS NOT ANALYTIC WHEN $e^z = -1 \rightarrow z = \pm i\pi, \pm 3i\pi, \ldots$

THUS THE TAYLOR SERIES WILL CONVERGE FOR $|z| < \pi$.

NEXT, WE WRITE $1+e^z = 1+1+z+z^2/_2 + z^3/_6 + \ldots = 2\left[1+\left(z/_2 + z^2/_4 + z^3/_{12}\right)\right]$.

RECALL $\dfrac{1}{1+h} = 1-h+h^2+\ldots$ THUS $\dfrac{1}{2[1+h]} = \dfrac{1}{2}\left(1-h+h^2-h^3+\ldots\right)$

THIS GIVES
$$\frac{1}{1+e^z} = \frac{1}{2}\left[1-\left(z/_2+z^2/_4+z^3/_{12}\right)+\left(z/_2+z^2/_4+z^3/_{12}\right)^2-\left(z/_2+\ldots\right)^3\right]$$
$$= \frac{1}{2}\left[1-\frac{z}{2}-\frac{z^2}{4}-\frac{z^3}{12}+\left(z^2/_4+z^3/_4+\ldots\right)-z^3/_8+\ldots\right]$$

SO
$$\frac{1}{1+e^z} = \frac{1}{2}\left[1-\frac{z}{2}+0z^2+z^3\left(-\frac{1}{12}+\frac{1}{4}-\frac{1}{8}\right)+\ldots\right] = \frac{1}{2}\left[1-\frac{z}{2}+\frac{z^2}{24}\ldots\right]$$

(ii) $f(z) = \sec z = 1/_{\cos z}$. HAS SINGULARITIES WHEN $\cos z = 0 \rightarrow z = \pm\pi/_2, \pm 3\pi/_2$.

THUS THE TAYLOR SERIES WILL CONVERGE IN $|z| < \pi/_2$.

NOW $\cos z = 1 - z^2/_2 + z^4/_{24} + \ldots$ THUS $\dfrac{1}{\cos z} = \dfrac{1}{1-h}$ $h = z^2/_2 - z^4/_{24}\ldots$

HENCE $f(z) = \dfrac{1}{\cos z} = 1+h+h^2+\ldots = 1+\left(z^2/_2 - z^4/_{24}+\ldots\right)+\left(z^2/_2 - z^4/_{24}+\ldots\right)^2$

THUS $f(z) = 1+z^2/_2 - z^4/_{24} + z^4/_4 + \ldots = 1+z^2/_2 + 5z^4/_{24} + \ldots$

EXAMPLE 4 SUPPOSE $x>0$. CALCULATE $S = \displaystyle\sum_{n=1}^{\infty} \frac{e^{-nx}\cos(ny)}{n}$.

WE WRITE $S = \mathrm{RE}\left(\displaystyle\sum_{n=1}^{\infty}\frac{e^{-nx+iny}}{n}\right) = \mathrm{RE}\left(\displaystyle\sum_{n=1}^{\infty}\frac{z^n}{n}\right)$ WITH $z = e^{-x+iy}$.

FOR $x>0$, $|z| = e^{-x} < 1$. SO $S = \mathrm{RE}\left[\mathrm{LOG}(1-z)\right] = \ln|1-z| = \frac{1}{2}\ln\left(|1-z|^2\right)$

THUS $S = \dfrac{1}{2}\ln\left((1-e^{-x}\cos y)^2 + (e^{-x}\sin y)^2\right) = \dfrac{1}{2}\ln\left(1+e^{-2x}-2e^{-x}\cos y\right)$.

CLASSIFICATION OF SINGULARITIES

SUPPOSE THAT $f(z)$ IS ANALYTIC IN THE REGION $r_1 < |z-z_0| < r_2$ WITH $r_1 \geq 0$, $r_2 > r_1$ AND POSSIBLY $r_2 = \infty$. SUPPOSE THAT $f(z)$ HAS AN ISOLATED SINGULARITY AT $z = z_0$. THEN $f(z)$ HAS A LAURENT SERIES IN $r_1 < |z-z_0| < r_2$ OF THE FORM

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$$

$$(*) \quad f(z) = \left(a_0 + a_1(z-z_0) + \dots + a_n (z-z_0)^n + \dots \right) + \left(\frac{a_{-1}}{(z-z_0)} + \frac{a_{-2}}{(z-z_0)^2} + \dots + \frac{a_{-n}}{(z-z_0)^n} + \dots \right)$$

THE SERIES CONVERGES ON $r_1 < |z-z_0| < r_2$.

WE CLASSIFY THE SINGULARITY AT $z = z_0$ USING THE LAURENT SERIES $(*)$

(i) $f(z)$ HAS A <u>POLE OF ORDER $m > 0$</u> AT $z = z_0$ IF $a_{-m} \neq 0$ BUT $a_{-m-1} = a_{-m-2} = \dots = 0$. THE L-SERIES HAS THE FORM

terminates here

$$f(z) = \left(a_0 + a_1(z-z_0) + \dots \right) + \left(\frac{a_{-1}}{z-z_0} + \dots + \frac{a_{-m}}{(z-z_0)^m} \right)$$

(ii) $f(z)$ HAS AN <u>ESSENTIAL SINGULARITY</u> AT $z = z_0$ IF THE L-SERIES HAS AN <u>INFINITE NUMBER</u> OF TERMS IN POSITIVE POWERS OF $1/z-z_0$

(iii) $f(z)$ HAS A <u>REMOVABLE SINGULARITY</u> AT $z = z_0$ IF $f(z)$ IS NOT DEFINED AT $z = z_0$ BUT THAT $\lim_{z \to z_0} f(z)$ EXISTS.

IN THIS CASE $a_{-1} = a_{-2} = \dots = 0$.

<u>REMARK</u> (i) THE PROOF OF THE CONVERGENCE PROPERTY OF THE L-SERIES

(ii) A POLE OF ORDER 1 IS CALLED A SIMPLE POLE.

(iii) <u>IMPORTANT</u> $a_{-1} = \text{RESIDUE}\left[f(z) ; z_0 \right]$

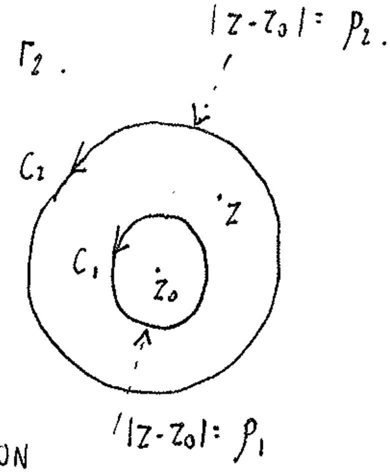
IS A CRITICAL TERM TO CALCULATE.

PROOF OF LAURENT EXPANSION

CHOOSE ρ_1 AND ρ_2 BY $\rho_1 < \rho_2$ WITH $\rho_1 > r_1$ AND $\rho_2 < r_2$.

LET z BE ANY POINT IN $r_1 < \rho_1 \leq |z - z_0| \leq \rho_2 < r_2$

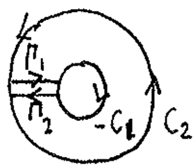
THEN BY CAUCHY INTEGRAL FORMULA WE HAVE



$$(*) \quad f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

THE PROOF OF THIS IS TO USE CAUCHY-INTEGRAL FORMULA ON

ON $C = C_1 \sqcup \Gamma_1 \sqcup (-C_2) \sqcup \Gamma_2$ AS SHOWN



NEXT WE CALCULATE THE TWO TERMS IN $(*)$.

<u>FIRST TERM</u> SINCE z INSIDE C_2: $\dfrac{1}{\zeta - z} = \dfrac{1}{(\zeta - z_0) - (z - z_0)}$

SO $$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\left[1 - \frac{(z - z_0)}{(\zeta - z_0)}\right]^{-1} = \frac{1}{\zeta - z_0}\sum_{j=0}^{\infty}\left(\frac{z - z_0}{\zeta - z_0}\right)^j$$ WHICH CONVERGES FOR $\left|\dfrac{z - z_0}{\zeta - z_0}\right| < 1$.

THUS SINCE $\left|\dfrac{z - z_0}{\zeta - z_0}\right| < 1$ FOR ζ ON C_2 WE HAVE

$$(+) \quad \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{j=0}^{\infty} (z - z_0)^j a_j \qquad a_j = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\, d\zeta.$$

<u>SECOND TERM</u> SINCE z OUTSIDE C_1: $\dfrac{1}{\zeta - z} = \dfrac{1}{(\zeta - z_0) - (z - z_0)}$

SO $$\frac{1}{\zeta - z} = -\frac{1}{(z - z_0)}\left[1 - \left(\frac{\zeta - z_0}{z - z_0}\right)\right]^{-1} = \frac{-1}{z - z_0}\sum_{j=0}^{\infty}\left(\frac{\zeta - z_0}{z - z_0}\right)^j$$ WHICH CONVERGES ON C_1 SINCE $\left|\dfrac{\zeta - z_0}{z - z_0}\right| < 1$ ON C_1

THUS $$-\frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{j=1}^{\infty} (z - z_0)^{-j} a_{-j} \quad (++)$$ WITH $a_{-j} = \dfrac{1}{2\pi i}\displaystyle\int_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{-j+1}}\, d\zeta$.

NOW PUTTING $(+)$ AND $(++)$ IN $(*)$ WE GET THE LAURENT EXPANSION THEOREM WHICH CONVERGES IN $r_1 < |z - z_0| < r_2$.

CLASSIFICATION OF SINGULARITIES: GENERAL REMARKS

REMARK 1 SUPPOSE THAT $f(z) = \dfrac{P(z)}{Q(z)}$

WHERE $P(z)$, $Q(z)$ ANALYTIC AT $z = z_0$ WITH

$$P(z_0) \neq 0, \quad Q(z_0) = 0, \quad Q'(z_0) \neq 0,$$

SO THAT z_0 IS A SIMPLE ZERO OF $Q(z) = 0$. THEN $f(z)$

HAS A SIMPLE POLE AT $z = z_0$ AND AS $z \to z_0$

$$f(z) \approx \frac{P(z_0) + (z - z_0) P'(z_0) + \cdots}{(z - z_0) Q'(z_0) + \cdots} = \frac{P(z_0)/Q'(z_0)}{z - z_0} + O(1)$$

THUS $a_{-1} = \mathrm{RES}[f; z_0] = P(z_0)/Q'(z_0)$.

THIS IS THE EASIEST METHOD TO CALCULATE THE RESIDUE AT A SIMPLE POLE.

REMARK 2 SUPPOSE THAT $f(z) = P(z)/Q(z)$

WITH $P(z)$, $Q(z)$ ANALYTIC AT $z = z_0$ WITH

$$P(z_0) \neq 0$$

$$Q(z_0) = 0, \quad Q'(z_0) = 0, \quad \ldots \quad Q^{(m)}(z_0) = 0, \quad Q^{(m+1)}(z_0) \neq 0$$

THEN $Q(z) \approx (z - z_0)^{m+1} \dfrac{Q^{(m+1)}(z_0)}{(m+1)!}$ AS $z \to z_0$.

THUS AS $z \to z_0$,

$$f(z) \approx \frac{\left(P(z_0)/Q^{(m)}(z_0) \right) (m+1)!}{(z - z_0)^{m+1}}$$

$\rightarrow$ pole at $z = z_0$ OF ORDER $m+1$.

REMARK 3 CALCULATION OF RESIDUE FOR A POLE OF ORDER m AT $z = z_0$.

THE LAURENT SERIES OF $f(z)$, WHICH CONVERGES IN $0 < |z - z_0| < r_0$ FOR SOME r_0, IS

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \dots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

TO CALCULATE a_{-1} MULTIPLY BY $(z-z_0)^m$:

$$(z-z_0)^m f(z) = a_{-m} + \dots + a_{-1}(z-z_0)^{m-1} + a_0(z-z_0)^m + \dots.$$

NOW DIFFERENTIATE $m-1$ TIMES:

$$\frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z) = a_{-1}(m-1)! + [m(m-1)\cdots\cdots 2]\, a_0 (z-z_0) + \dots$$

NOW LET $z \to z_0$, THEN
$$a_{-1} = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left((z-z_0)^m f(z) \right) \right]. \quad (*)$$

<u>EXAMPLE</u> CALCULATE THE RESIDUE OF $f(z) = \dfrac{z^2 - 2z}{[z+1]^2 (z^2+4)}$ AT $z = -1$.

DO THIS IN TWO DIFFERENT WAYS

<u>METHOD 1</u> USE THE FORMULA $(*)$. NOTICE THAT $z = -1$ IS A POLE OF ORDER 2. SO
$$a_{-1} = \lim_{z \to -1} \frac{d}{dz} \left[(z+1)^2 f(z) \right] = \lim_{z \to -1} \frac{d}{dz} \left[\frac{z^2 - 2z}{z^2 + 4} \right]$$

THIS GIVES
$$a_{-1} = \lim_{z \to -1} \left[\frac{(z^2+4)(2z-2) - (z^2-2z)2z}{(z^2+4)^2} \right] = \frac{5(-4) + (+1+2)2}{25} = -\frac{14}{25}.$$

<u>METHOD 2</u> WRITE $f(z) = \dfrac{P(z)}{Q(z)}$ WITH $Q(z) = (z+1)^2$, $P(z) = (z^2 - 2z)/(z^2+4)$.

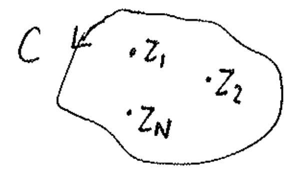
NOW AS $z \to -1$ $\quad P(z) \sim P(-1) + P'(-1)(z-1) + \dots$

THUS $f(z) \sim \dfrac{P(-1)}{(z+1)^2} + \dfrac{P'(-1)}{(z+1)} + \dots$ AS $z \to -1$. SO $a_{-1} = P'(-1) = -14/25$.

REMARK 4 IF $f(z)$ HAS AN <u>ESSENTIAL</u> SINGULARITY AT $z = z_0$, THE ONLY WAY TO CALCULATE THE RESIDUE IS TO CALCULATE THE LAURENT SERIES OF $f(z)$ AS $z \to z_0$ TO IDENTIFY THE COEFFICIENT $a_{-1}/(z-z_0)$.

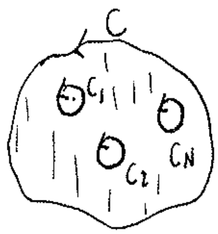
RESIDUE THEOREM LET C BE A SIMPLE CLOSED CURVE, ORIENTED COUNTERCLOCKWISE, AND SUPPOSE THAT $f(z)$ IS ANALYTIC INSIDE AND ON C EXCEPT AT THE ISOLATED SINGULARITIES $z_1, \dots, z_N$, WITH z_j INSIDE C FOR $j=1,\dots,N$. THEN

$$\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{N} \mathrm{Res}(f; z_j).$$



DERIVATION BY DEFORMING THE CONTOUR AS ON PAGE (I20) WE OBTAIN

$$\int_C f(z)\,dz = \sum_{j=1}^{N} \int_{C_j} f(z)\,dz \quad (*), \qquad C_j : |z-z_j| = \delta \text{ COUNTERCLOCKWISE.}$$



NOW SINCE δ IS SMALL, $f(z)$ HAS A LAURENT SERIES THAT CONVERGES IN $0 < |z-z_j| < \delta$ OF THE FORM

$$f(z) = \sum_{m=0}^{\infty} a_{m_j} (z-z_j)^m + \sum_{m=1}^{\infty} a_{-m_j} (z-z_j)^{-m}. \quad \text{FOR SOME } a_{m_j} \text{ AND } a_{-m_j}.$$

THEN $\displaystyle\int_{C_j} f(z)\,dz = 2\pi i\, a_{-1_j}$ SINCE $\displaystyle\int_{C_j} (z-z_j)^n\,dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n=-1 \end{cases}$.

FROM $(*)$ WE GET $\displaystyle\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{N} a_{-1_j} = 2\pi i \sum_{j=1}^{N} \mathrm{Res}[f; z_j].$

REMARKS TO CALCULATE $\mathrm{Res}[f; z_j]$ WE USE REMARKS 2, 3, 4 ON PAGE S11-S12.

EXAMPLE 0 CALCULATE $\displaystyle I = \int_C \frac{dz}{z^2+z+1}$ $\quad C: |z|=2$ COUNTER-CLOCKWISE

$|z|=2$

METHOD 1 THE SIMPLE POLES ARE AT $z_\pm = \dfrac{-1 \pm \sqrt{3}\, i}{2}$ SO $|z_\pm| = 1$. THESE ARE INSIDE C. SO BY RESIDUE THEOREM AND REMARK 1 WE GET

$$I = 2\pi i \left[\mathrm{Res}(f; z_+) + \mathrm{Res}(f; z_-) \right] = 2\pi i \left[\frac{1}{2z_+ + 1} + \frac{1}{2z_- + 1} \right] = 2\pi i \left(\frac{2(z_- + z_+) + 2}{(2z_+ + 1)(2z_- + 1)} \right) = 0.$$

METHOD 2 WE CAN PROCEED BY PROB. 18 PAGE 203 OF THE BOOK SINCE THERE ARE NO SINGULARITIES OUTSIDE C AND $f(z) = O(|z|^{-2})$ AS $|z| \to \infty$. WE AGAIN CONCLUDE $I = 0$.

EXAMPLES OF THE USE OF RESIDUE THEOREM.

<u>EXAMPLE 1</u> CALCULATE $I = \int_C \frac{dz}{z^2 \sinh z}$ $C: |z| = 1$ counterclockwise

NOW THE SINGULARITIES OF $f(z) = 1/z^2 \sinh z$ ARE A POLE OF ORDER 3 AT $z = 0$ AND SIMPLE POLES WHERE $\sinh z = 0$ FOR $z \neq 0$. THESE ARE AT $z = n\pi i$, $n = \pm 1, \pm 2, \ldots$, BUT <u>ARE OUTSIDE</u> C.

SO $I = 2\pi i \,\mathrm{Res}\,[f; 0]$ (*)

NOW $\sinh z = \frac{e^z - e^{-z}}{2} = \frac{(1 + z + z^2/2 + z^3/3!) - (1 - z + z^2/2 - z^3/3!) + \cdots}{2} = z + z^3/3! + \cdots$

THUS NEAR $z = 0$ $\frac{1}{z^2 \sinh z} \hat{=} \frac{1}{z^2[z + z^3/6]} \approx \frac{1}{z^3}(1 - z^2/6 + \cdots) = \frac{1}{z^3} - \frac{1}{6z} + \cdots$

THUS $\mathrm{Res}\,[f; 0] = -1/6$ SO $I = 2\pi i(-1/6) = -\pi i/3$.

<u>EXAMPLE 2</u> CALCULATE $I = \int_C z^3 e^{1/z}\, dz$ WITH $C: |z| = 1$ counterclockwise.

NOTE: $z = 0$ IS AN ESSENTIAL SINGULARITY. THUS

$$I = 2\pi i \,\mathrm{Res}\,[f; 0].$$

WE CALCULATE LAURENT SERIES,

$$z^3 e^{1/z} = z^3\left(1 + 1/z + 1/2!z^2 + 1/3!z^3 + \cdots\right) = z^3 + z^2 + \frac{1}{2!}z + \frac{1}{3!} + \frac{1}{4!z} + \cdots$$

THUS $a_{-1} = 1/4!$. $\longrightarrow$ $I = 2\pi i/4! = \pi i/12$.

<u>EXAMPLE 3</u> CALCULATE $I = \int_C \frac{1}{z^2}\left(\frac{1+2z}{1+z}\right) dz$ $C: |z| = 1/2$, COUNTERCLOCKWISE.

NOW THE ONLY SINGULARITY INSIDE C IS AT $z = 0$, WHICH IS A POLE OF ORDER 2. WE CALCULATE THE RESIDUE

$$\frac{1}{z^2}\frac{(1+2z)}{(1+z)} = \frac{1}{z^2}(1+2z)(1 - z + z^2 + \cdots) = \frac{1}{z^2}(1 - z + z^2 + 2z - 2z^2 + \cdots)$$

$$= \frac{1}{z^2} + \frac{1}{z} + 1 \cdots \quad \text{SO} \quad a_{-1} = 1 \longrightarrow I = 2\pi i.$$

<u>EXAMPLE 4</u> CALCULATE $I = \int_C \frac{z+1}{z^2 - 2z}\, dz$ WHERE $C: |z| = 4$ COUNTERCLOCKWISE.

BOTH $z = 0$ AND $z = 2$ ARE SIMPLE POLES. IF WE WRITE $(z+1)/z^2 - 2z = P(z)/Q(z)$

THEN $\int_C \frac{z+1}{z^2 - 2z}\, dz = 2\pi i\left[\frac{P(0)}{Q'(0)} + \frac{P(2)}{Q'(2)}\right] = 2\pi i\left(\frac{1}{-2} + \frac{3}{2}\right) = 2\pi i$.

WE NOW GIVE TWO EXAMPLES SHOWING HOW AN INTEGRAL OF A PERIODIC FUNCTION CAN BE CONVERTED TO A CONTOUR INTEGRAL AND THEN EVALUATED BY RESIDUES:

RECALL
$$\cos\varphi = \frac{e^{i\varphi}+e^{-i\varphi}}{2} \longrightarrow \cos\varphi = \frac{z+1/z}{2} \quad \text{WHEN } z=e^{i\varphi} \to \frac{dz}{iz}=d\varphi$$
$$\sin\varphi = \frac{e^{i\varphi}-e^{-i\varphi}}{2i} \longrightarrow \sin\varphi = \frac{z-1/z}{2i} \quad \text{WHEN } z=e^{i\varphi} \to \frac{dz}{iz}=d\varphi.$$

EXAMPLE 1 CALCULATE $\int_0^{2\pi} \frac{d\varphi}{2+\sin\varphi} = I$.

NOW LET $z=e^{i\varphi} \to 2+\sin\varphi = 2+(z-1/z)/2i = \frac{1}{2}[4-iz+i/z]$.

THEN $I = \int_C \frac{2}{(4-iz+i/z)}\frac{dz}{iz} = 2\int_C \frac{dz}{z^2+4iz-1} \qquad C: |z|=1$

THE POLES (SIMPLE) ARE AT $z_\pm = \frac{-2i \pm \sqrt{-16+4}}{2} = -2i \pm \sqrt{3}\,i$.

THUS $z_+ = (-2+\sqrt{3})i$ IS INSIDE $|z_+|\le 1$, WHILE z_- IS OUTSIDE $|z_-| \ge 1$

THUS $I = 2\int_C \frac{dz}{z^2+4iz-1} = 4\pi i \,\mathrm{RES}\left(\frac{1}{z^2+4iz-1}; z_+\right) = 4\pi i \frac{1}{2z_+ + 4i} = \frac{4\pi i}{2\sqrt{3}\,i}$

THUS $I = 2\pi/\sqrt{3}$.

EXAMPLE 2 CALCULATE $I = \int_0^{2\pi} \frac{d\varphi}{1-2a\cos\varphi + a^2}$ FOR $0<a<1$.

WE WRITE $z=e^{i\varphi}$, $\cos\varphi = (z+1/z)/2$, $dz/iz = d\varphi$.

SO $I = \int_C \frac{dz}{iz\left[1+a^2-\frac{2a}{2}(z+1/z)\right]} = \int_C \frac{dz}{iz\left[1+a^2-az-a/z\right]} = \frac{1}{i}\int_C \frac{dz}{[-az^2+(1+a^2)z-a]}$

THE ZEROES OF THE DENOMINATOR ARE AT $z=a$ AND $z=1/a$. FOR $0<a<1$, THE ONLY POLE INSIDE $|z|=1$ IS AT $z=a$.

THUS $I = \frac{1}{i}\int_C \frac{dz}{[-az^2+(1+a^2)z-a^2]} = \frac{1}{i}(2\pi i)\,\mathrm{RES}\left[\frac{1}{-az^2+(1+a^2)z-a^2}, a\right]$

THUS $I = 2\pi\left(\frac{1}{-2az+(1+a^2)}\right)\Big|_{z=a} = \frac{2\pi}{1-a^2}, \quad 0<a<1$

REMARK FOR AN INTEGRAL OF THE FORM

$I = \int_0^{2\pi} F(\cos\varphi, \sin\varphi)\,d\varphi$ LET $\cos\varphi = \frac{z+1/z}{2}$, $\sin\varphi = \frac{z-1/z}{2i}$, $d\varphi = dz/iz$

TO GET $I = \int_C F\left((z+1/z)/2, (z-1/z)/2i\right)\frac{dz}{iz}$. THEN CALCULATE BY RESIDUES.

EXAMPLES

(i) $f(z) = \dfrac{z - \sin z}{z^5}$

NEAR $z = 0$, $f(z) = \dfrac{z - [z - z^3/3! + z^5/5! \dots]}{z^5} = \dfrac{z^3/3! - z^5/5! + z^7/7!}{z^5}$

$$f(z) = \frac{z^{-2}}{3!} - \frac{z^0}{5!} + \frac{z^2}{7!} \dots = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4}$$

- SINCE $f(z) \sim z^{-2}/3!$ THEN $z = 0$ IS A POLE OF ORDER 2
- COEFFICIENT a_{-1} VANISHES $\longrightarrow$ $\text{RES}[f(z); 0] = 0$.
- LAURENT SERIES CONVERGES FOR ALL $|z| > 0$ SINCE $f(z)$ IS ANALYTIC EXCEPT AT $z = 0$.

(ii) $f(z) = e^{1/z}$

$f(z)$ IS ANALYTIC EXCEPT AT $z = 0$.

WE WRITE $e^{1/z} = 1 + 1/z + 1/2!z^2 + \dots + 1/n!z^n + \dots$

THUS THE L-SERIES IS

$$e^{1/z} = 1 + \underbrace{1/z}_{\text{RES}[f;0]} + 1/2!z^2 + \dots + 1/n!z^n + \dots$$

HENCE NO HIGHEST POWER OF $1/z$.

- SO $z = 0$ IS AN <u>ESSENTIAL SINGULARITY</u>
- $\text{RES}[f(z); 0] = 1$ (coefficient of $1/z$ TERM)

(iii) $f(z) = \dfrac{z^2 - (\sin z)^2}{z^4}$

THE ONLY POSSIBLE SINGULAR POINT IS AT $z = 0$.

USE $(\sin z)^2 \doteq (z - z^3/6)^2 \approx z^2 - 2z^4/6 + O(z^6)$ AS $z \to 0$.

THUS $f(z) \approx \dfrac{2z^4/6 + O(z^6)}{z^4} \approx 2/3 + O(z^2)$ AS $z \to 0$.

THIS MEANS THAT $\lim_{z \to 0} f(z)$ EXISTS $\longrightarrow$ $z = 0$ IS A REMOVABLE SINGULARITY

THEN, $\mathrm{RES}(f; 0) = 0$.

EXAMPLE FIND THE LAURENT SERIES OF $f(z) = \dfrac{z}{z^2+1}$

CENTERED AT $z_0 = i$ THAT CONVERGES IN $0 < |z-i| < 2$.

SOLUTION A PARTIAL FRACTION DECOMPOSITION GIVES

$$\frac{z}{(z+i)(z-i)} = \frac{A}{z-i} + \frac{B}{z+i} \longrightarrow z = A(z+i) + B(z-i)$$

IF $z = -i \longrightarrow B = 1/2$

$z = i \longrightarrow A = 1/2$

THUS
$$\frac{z}{(z+i)(z-i)} = \frac{1}{2(z-i)} + \frac{1}{2(z+i)} \quad (*)$$

NOW EXPAND FOR $z \approx -i$. WE WILL USE $\dfrac{1}{1+w} = \sum_{n=0}^{\infty} (-1)^n w^n$ FOR $|w|<1$.

$$\frac{1}{z+i} = \frac{1}{2i+(z-i)} = \frac{1}{2i\left[1 + \frac{(z-i)}{2i}\right]}$$

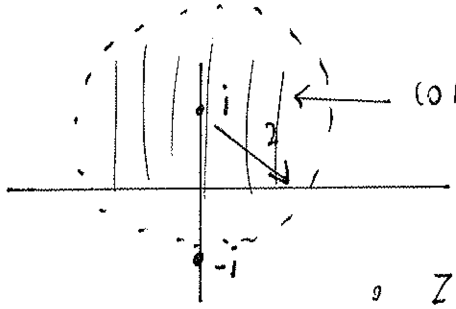
$$\frac{1}{z+i} = \frac{1}{2i}\left[\sum_{n=0}^{\infty} (-1)^n \left(\frac{z-i}{2i}\right)^n\right] \quad \text{FOR } \left|\frac{z-i}{2i}\right|<1$$

$$= \sum_{n=0}^{\infty} (-1)^n 2^{-n-1} i^{-n-1} (z-i)^n$$

THUS $(*)$ GIVES

$$f(z) = \underbrace{\frac{1}{2(z-i)}}_{\text{RESIDUE TERM}} + \sum_{n=0}^{\infty} (-1)^n 2^{-n-1} i^{-n-1} (z-i)^n \quad \text{FOR } |z-i|<2$$

WHICH CONVERGES FOR $|z-i|<2$. NOTICE THAT THE DISTANCE FROM $z = i$ TO THE OTHER SINGULARITY OF $f(z)$ AT $z = -i$ IS EXACTLY 2.



- $z = i$ IS A SIMPLE POLE (POLE OF ORDER 1)
- $\mathrm{RES}\,[\,f\,;\,i\,] = 1/2$.

<u>EXAMPLE</u> FIND THE LAURENT SERIES FOR $f(z) = \dfrac{1}{z(z-1)(z-2)}$

THAT CONVERGES IN

(i) $0 < |z| < 1$

(ii) $1 < |z| < 2$

<u>SOLUTION</u> WE DO PARTIAL FRACTIONS:

$$\frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2} \longrightarrow A(z-1)(z-2) + Bz(z-2) + Cz(z-1) = 1$$

SO $z = 1 \rightarrow B = -1$

$z = 0 \rightarrow 2A = 1 \rightarrow A = 1/2$

$z = 2 \rightarrow C = 1/2$.

$$\text{SO} \quad f(z) = \frac{1}{2z} + \frac{1}{1-z} + \frac{1}{2(z-2)}$$

$$= \frac{1}{2z} + \frac{1}{1-z} - \frac{1}{4(1-z/2)}$$

(i) $$f(z) = \frac{1}{2z} + \sum_{n=0}^{\infty} z^n - \frac{1}{4}\sum_{n=0}^{\infty} (z/2)^n \quad \text{CONVERGES IN } 0 < |z| < 1.$$

(ii) NOW FOR $|z| > 1$ WITH $|z| < 2$

$$f(z) = \frac{1}{2z} + \frac{1}{z(-1+1/z)} - \frac{1}{4(1-z/2)} = \frac{1}{2z} - \frac{1}{z(1-1/z)} - \frac{1}{4(1-z/2)}$$

$$\text{THUS} \quad f(z) = \frac{1}{2z} - \frac{1}{z}\sum_{n=0}^{\infty} (1/z)^n - \frac{1}{4}\sum_{n=0}^{\infty} (z/2)^n$$

CONVERGES FOR $1 < |z| < 2$.

EXAMPLE FIND LAURENT SERIES FOR

$$f(z) = \frac{z}{z^2-9} \quad \text{IN } |z|<3, \text{ THEN IN } |z|>3.$$

SOLUTION WE HAVE SIMPLE POLES OF $f(z)$ AT $z = \pm 3$ SINCE z^2-9 HAS SIMPLE ZEROS AT $z = \pm 3$.

- IN $|z|<3$ WE WRITE $f(z) = \dfrac{-z}{9-z^2} = \dfrac{-z}{9\left(1-(z/3)^2\right)}$

BUT $\dfrac{1}{1-w} = \sum\limits_{n=0}^{\infty} w^n$ IF $|w|<1$. THUS

$$f(z) = -\frac{z}{9}\sum_{n=0}^{\infty}\left(z/3\right)^{2n} \quad \text{IN } |z|<3.$$

- NOW IN $|z|>3$ WE WRITE $f(z) = \dfrac{z}{z^2(1-9/z^2)} = \dfrac{-z}{z^2\left(1-(3/z)^2\right)}$

THUS $f(z) = -\dfrac{1}{z}\sum\limits_{n=0}^{\infty}\left(3/z\right)^{2n}$ VALID IN $|z|>3$.

EXAMPLE CLASSIFY THE SINGULARITIES OF THE FOLLOWING FUNCTIONS

(i) $f(z) = z^2 \sin(1/z)$ (ii) $f(z) = \dfrac{1-\cos(z^2)}{z^6}$ (iii) $f(z) = \dfrac{1}{z\tan z}$

(iv) $f(z) = \dfrac{z^2(z-1)}{\sin^2(\pi z)}$ (v) $f(z) = \dfrac{z}{1-\cos(z^2)}$ (vi) $f(z) = \dfrac{\sin(\sqrt{z})}{\sqrt{z}}$

SOLUTION

(i) $z=0$ IS AN ESSENTIAL SINGULARITY. SINCE $\sin w = \sum\limits_{n=0}^{\infty}(-1)^n w^{2n+1}/(2n+1)!$ FOR ALL w, THEN

$$f(z) = \sum_{n=0}^{\infty}(-1)^n\frac{z^2 z^{-2n-1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n z^{-2n+1}}{(2n+1)!} = z - \frac{1}{3!\,z} + \cdots$$

THERE ARE AN INFINITE # TERMS IN POWERS OF $1/z \rightarrow z=0$ ESSENTIAL SINGULARITY

$$\text{RES}[f;0] = -1/3!$$

(ii) $z=0$ is a pole of order 4.

$\cos(z^2) = 1 - z^2/2! + z^4/4! \cdots$

Thus $\dfrac{-\cos(z^2)+1}{z^6} = \dfrac{z^2/2! - z^4/4! \cdots}{z^6} = \dfrac{1}{2!\,z^4} - \dfrac{1}{4!\,z^2} + O(1)$ as $z \to 0$.

So $f(z)$ has a pole of order 4 at $z=0$ and $\text{Res}[f(z);0] = 0$.

We can write $1-\cos(z^2) = z^2/2! - z^4/4! + z^6/6! = \sum_{n=1}^{\infty} (-1)^{n-1} z^{2n}/(2n)!$

Thus $\dfrac{1-\cos(z^2)}{z^6} = \sum_{n=1}^{\infty} (-1)^{n-1} z^{2n-6}/(2n)!$

(iii) $f(z) = \dfrac{1}{z\tan z}$

- $z=0$ is a pole of order 2 since

$\tan z = \dfrac{\sin z}{\cos z} = \dfrac{z - z^3/6 + \cdots}{1-z^2/2} = (z-z^3/6)(1+z^2/2) + \cdots = z + z^3/3 + \cdots$

So near $z=0$ we have $z\tan z = z(z+z^3/3+\cdots) = z^2(1+z^2/3+\cdots)$

So $f(z) \approx \dfrac{1}{z^2(1+z^2/3)} \approx \dfrac{1}{z^2}(1-z^2/3+\cdots) \sim \dfrac{1}{z^2} - \dfrac{1}{3} + \cdots$

Thus $\text{Res}[f,0] = 0$.

- Now $z_n = n\pi$, $n = \pm 1, \pm 2, \ldots$ are simple poles of $f(z)$ since they are simple zeroes of $Q(z) = z\tan z$.

Note: $Q(n\pi) = n\pi \tan(n\pi)$

But $Q'(z) = \tan z + z\sec^2 z$

$Q'(n\pi) = 0 + (n\pi)\sec^2(n\pi) \neq 0$.

(iv) $f(z) = \dfrac{z^2(z-1)}{\sin^2(\pi z)}$

- $z=0$ IS A REMOVABLE SINGULARITY SINCE $\sin(\pi z) \simeq \pi z$ FOR $|z| \ll 1$ AND SO

$$f(z) \simeq \frac{z^2(z-1)}{(\pi z)^2} \approx \frac{1}{\pi}(z-1) \quad \text{AS } z \to 0.$$

THUS $\lim\limits_{z\to 0} f(z) = \text{FINITE} = -1/\pi$.

- $z=1$ IS A POLE OF ORDER 1.

NEAR $z=1$ WE DEFINE $Q(z) = \sin^2(\pi z)$.

$$Q(1) = 0, \quad Q'(z) = 2\pi \sin(\pi z)\cos(\pi z) \rightarrow Q'(1) = 0$$

$$Q''(z) = 2\pi^2 \cos^2(\pi z) - 2\pi^2 \sin^2(\pi z) \rightarrow Q''(1) = 2\pi^2 \neq 0.$$

THUS $Q(z) \approx \dfrac{Q''(1)}{2}(z-1)^2$ FOR $z \simeq 1$.

THUS, $f(z) \sim \dfrac{z^2(z-1)}{Q''(1)(z-1)^2/2} \simeq \dfrac{2}{Q''(1)} \dfrac{1}{z-1}$ FOR $z \simeq 1$.

SO $z=1$ IS A POLE OF ORDER 1 AND

$$\text{RES}[f;1] = 2/Q''(1) = 1/\pi^2.$$

- $z = -1, \pm 2, \pm 3, \pm 4, \ldots$ ARE POLES OF ORDER 2.

THIS FOLLOWS SINCE $Q'(z_k) = 0$, $Q''(z_k) \neq 0$ WHILE $P(z_k) \neq 0$ WHERE $z_k \in \{-1, \pm 2, \pm 3, \pm 4, \ldots\}$ AND

$$f(z) = P(z)/Q(z) \qquad P(z) = z^2(z-1)$$
$$Q(z) = \sin^2(\pi z).$$

(v) $f(z) = \dfrac{z}{1-\cos(z^2)}$

SINGULARITIES ARE WHERE $\cos(z^2) = 1$.

THUS $z^2 = 2n\pi, \quad n = 0, \pm 1, \pm 2, \dots$

NOTICE $z_0 = 0$

$z_k = \pm\sqrt{2\pi k}$ IF $k = 1, 2, \dots$

$z_m = \pm i\sqrt{2\pi |m|}$ IF $m = -1, -2, -3, \dots$

- FOR $z \to 0$ $\quad \cos(z^2) \approx 1 - z^4/2! + z^8/4! \dots$

SO $f(z) = \dfrac{z}{1-\cos z^2} \approx \dfrac{z}{z^4/2! - z^8/4!} \approx \dfrac{2z}{z^4(1 - z^4/12 + \dots)} \approx \dfrac{2}{z^3}$

AS $z \to 0$.

THUS RES $[f; 0] = 2$ $\quad z=0$ IS SIMPLE POLE.

- NOW $z_k = \pm\sqrt{2\pi k}$ IF $k = 1, 2, 3, \dots$

$z_m = \pm i\sqrt{2\pi |m|}$ IF $m = -1, -2, \dots$

NOW IF $Q(z) = 1 - \cos(z^2)$

THEN $Q(z_k) = 0$ BUT $Q'(z_k) = +2z_k \sin(z_k^2) = 0$

$Q''(z_k) = 2z_k(2k\cos(z_k^2)) \neq 0.$

SO z_k, z_m IS A ZERO OF ORDER 2 OF $Q(z)$.

HENCE THEY ARE POLES OF ORDER 2 OF $f(z)$.

(vi) $\quad f(z) = \dfrac{\sin(\sqrt{z})}{\sqrt{z}}$. $\quad$ (NOTE: $f(re^{i\theta}) = \dfrac{\sin(\sqrt{r}e^{i\theta/2})}{\sqrt{r}e^{i\theta/2}}$ LETTING θ INCREASE BY 2π, THEN f HAS SAME VALUE.

WE MIGHT EXPECT THAT $z=0$ IS A BRANCH POINT,

WHICH WOULD NOT BE AN ISOLATED SINGULARITY $\longrightarrow$ WRONG!

METHOD 1 $\quad$ WE WRITE $\sin(\sqrt{z}) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(\sqrt{z})^{2n+1}}{(2n+1)!}$

NOW $\dfrac{\sin(\sqrt{z})}{\sqrt{z}} = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{z}} \frac{z^n\sqrt{z}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n+1)!}$

SO $z=0$ IS REMOVABLE SINGULARITY

EXAMPLE CALCULATE THE RESIDUE OF THE FOLLOWING FUNCTIONS AT THE INDICATED POINT

(i) $f(z) = 1/(z^2-1)$ AT $z = 1$ (ii) $f(z) = (e^z-1)/z^2$ AT $z = 0$

(iii) $f(z) = z^2 e^{1/z}$ AT $z = 0$ (iv) $f(z) = \dfrac{z^2+1}{\sin(\pi z)}$ AT $z = 1$

(vi) $f(z) = \dfrac{z^2+4}{z(z-1)^2}$ AT $z = 1$ (v) $f(z) = \dfrac{\log z}{(z-i)^3}$ AT $z = i$

SOLUTION

(i) METHOD 1 $f(z) = \dfrac{1}{(z-1)(z+1)} \hat{=} \dfrac{1}{2(z-1)} + \ldots$ AS $z \to 1 \Rightarrow \text{RES}[f, 1] = \dfrac{1}{2}$

METHOD 2 : $z = 1$ IS A SIMPLE ZERO OF $Q(z) = z^2 - 1$.

SO $f(z) = \dfrac{P(z)}{Q(z)} \longrightarrow \text{RES}[f, 1] = \dfrac{P(1)}{Q'(1)} = \dfrac{1}{2}$.

(ii) $e^z = 1 + z + z^2/2! + z^3/3! + \ldots$ SO $\dfrac{e^z-1}{z^2} = (z + z^2/2! + \ldots)/z^2 \hat{=} \dfrac{1}{z} + \dfrac{1}{2!} + \ldots$

THUS $\text{RES}[f; 0] = 1$.

(iii) $f(z) = z^2 e^{1/z} = z^2\left(1 + 1/z + 1/z^2 2! + 1/z^3 3! + \ldots\right)$

SO $f(z) = z^2 + z + 1/2! + 1/3!z + 1/4!z^2 + \ldots$

THUS $\text{RES}[f; 0] = 1/3! = 1/6$.

(iv) $f(z) = \dfrac{z^2+1}{\sin(\pi z)}$ AT $z = 1$.

THUS $f(z) = \dfrac{P(z)}{Q(z)}$ $P(1) = 2,\ Q(1) = 0,\ Q'(z) = \cos(\pi z)\pi$

SO $Q(1) = 0,\ Q'(1) = \cos(\pi)\pi = -\pi$

SO $\text{RES}[f, 1] = \dfrac{P(1)}{Q'(1)} = \dfrac{2}{-\pi}$.

(vi) $f(z) = \dfrac{z^2+4}{z(z-1)^2}$ AT $z=1$.

NOTICE $z=1$ IS A POLE OF ORDER 2. TO DETERMINE RESIDUE WE CAN USE:

<u>METHOD 1</u> FROM THE FORMULA FOR A POLE OF ORDER 2

$$a_{-1} = \mathrm{RES}[f,1] = \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 f(z)\right]$$
$$= \lim_{z\to 1} \frac{d}{dz}\left[(z^2+4)/z\right] = \lim_{z\to 1}\left[1-4/z^2\right] = -3.$$

THUS $\mathrm{RES}[f,1] = -3$.

<u>METHOD 2</u> NEED TO WRITE IN TERMS OF $z-1$.

$$f(z) = \frac{z+4/z}{(z-1)^2} = \frac{(z-1)+1+\frac{4}{(z-1)+1}}{(z-1)^2} = \frac{(z-1)+1+4[1-(z-1)+(z-1)^2..]}{(z-1)^2}$$

SO $$f(z) = \frac{(z-1)+4-4(z-1)+4(z-1)^2+\dots}{(z-1)^2} = \frac{4}{(z-1)^2} - \frac{3}{(z-1)} + 4\dots$$

SO $\mathrm{RES}[f,1] = -3$

(v) $f(z) = \dfrac{\mathrm{LOG}\, z}{(z-i)^3}$ AT $z=i$. $\mathrm{LOG}\, z$ IS PRINCIPAL BRANCH OF $\log z$.

NOTICE THAT $f(z)$ HAS A POLE OF ORDER 3 AT $z=i$

SIMPLEST METHOD HERE IS TO DEFINE $g(z) = \mathrm{LOG}\, z$.

THEN NEAR $z=i$ $\quad g(z) = g(i) + (z-i)g'(i) + (z-i)^2 g''(i)/2! + \dots$

THUS $\mathrm{RES}[f;i] = g''(i)/2! = \frac{1}{2}\frac{d}{dz}(1/z) = -\frac{1}{2z^2}\Big|_{z=i} = -\frac{1}{2i^2} = \frac{1}{2}$

(<u>NOTE</u>: $f(z) = \dfrac{g(i)+(z-i)g'(i)+(z-i)^2 g''(i)/2 + }{(z-i)^3} = \dfrac{g(i)}{(z-i)^3} + \dfrac{g'(i)}{(z-i)^2} + \dfrac{g''(i)/2}{(z-i)}$

THUS $\mathrm{RES}[f;i] = 1/2$.