INTEGRATION IN THE COMPLEX PLANE

DEFINITION LET t BE REAL-VALUED AND

$$w(t) = c(t) + i\,d(t)$$

WHERE $c(t)$, $d(t)$ ARE REAL-VALUED.

THEN WE DEFINE $\int_\alpha^\beta w(t)\,dt \equiv \int_\alpha^\beta c(t)\,dt + i\int_\alpha^\beta d(t)\,dt$

EX $\int_0^{\pi/2} (e^t + 2i\sin t)\,dt = \int_0^{\pi/2} e^t\,dt + 2i\int_0^{\pi/2}\sin t\,dt = e^{\pi/2} - 1 + 2i.$

KEY PROPERTIES IF $z(t) = a(t) + i\,b(t)$, $w(t) = c(t) + i\,d(t)$ AND χ IS A COMPLEX NUMBER, then

(i) $\frac{d}{dt}(z(t) + w(t)) = \dot{w}(t) + \dot{z}(t)$

(ii) $\frac{d}{dt}(w(t)z(t)) = w\dot{z} + \dot{w}z$

(iii) $\int_\alpha^\beta (w(t) + \chi z(t))\,dt = \int_\alpha^\beta w(t)\,dt + \chi\int_\alpha^\beta z(t)\,dt.$

(iv) $\int_\alpha^\beta \dot{w}(t)\,dt = w(\beta) - w(\alpha)$

(v) $\left|\int_\alpha^\beta w(t)\,dt\right| \le \int_\alpha^\beta |w(t)|\,dt \le \max_{\alpha\le t\le\beta}|w(t)|\,(\beta-\alpha).$

WHEN $w(t)$ IS CONTINUOUS ON $\alpha \le t \le \beta$.

THE PROOFS OF THESE ARE STRAIGHTFORWARD AND ARE OMITTED EXCEPT FOR THE IMPORTANT (V), WHICH WE NOW SHOW.

PROOF OF (V) SINCE $w(t)$ IS COMPLEX, THEN FOR SOME $\rho \ge 0$ AND φ WE HAVE $\int_\alpha^\beta w(t)\,dt = \rho e^{i\varphi}$.

WHERE $\rho = \left|\int_\alpha^\beta w(t)\,dt\right|$ MODULUS OF THIS COMPLEX NUMBER.

THUS $\quad \rho = \left| \int_\alpha^\beta w(t)\, dt \right| = e^{-i\varphi} \int_\alpha^\beta w(t)\, dt = \int_\alpha^\beta e^{-i\varphi} w(t)\, dt. \quad (*)$

NOW RECALL THAT FOR ANY COMPLEX NUMBER Z, THEN

$$\mathrm{RE}(z) \leq |\mathrm{RE}(z)| \leq |z|.$$

HENCE, SINCE RHS OF $(*)$ IS REAL (IT MUST BE ρ) THEN,

$$\rho = \int_\alpha^\beta \mathrm{RE}\left(e^{-i\varphi} w(t)\right) dt \leq \int_\alpha^\beta \left| \mathrm{RE}\left(e^{-i\varphi} w(t)\right)\right| dt$$

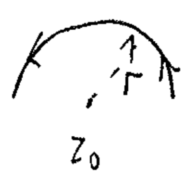
$$\leq \int_\alpha^\beta \left| e^{-i\varphi} w(t)\right| dt = \int_\alpha^\beta |w(t)|\, dt.$$

THUS $\quad \left| \int_\alpha^\beta w(t)\, dt \right| = \rho < \int_\alpha^\beta |w(t)|\, dt \quad \blacksquare.$

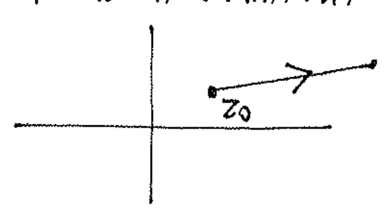
NEXT WE DEFINE A PATH OR CONTOUR IN THE COMPLEX PLANE.

<u>DEFINITION</u> A PATH OR CONTOUR C IN COMPLEX PLANE IS A PIECEWISE SMOOTH FUNCTION $z(t)$ WITH $a \leq t \leq b$ WITH Z COMPLEX. A CONTOUR HAS AN ORIENTATION OR DIRECTION AS $t \uparrow$.

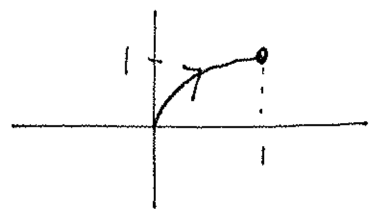
(i) $z(t) = z_0 + re^{it}$, $0 \leq t \leq \pi$ IS A SEMI-CIRCLE CENTERED AT z_0 ORIENTED COUNTERCLOCKWISE AND OF RADIUS r



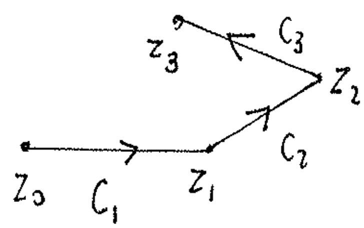
(ii) $z(t) = z_0(1-t) + z_1 t$ FOR $0 \leq t \leq 1$ IS A STRAIGHT LINE BETWEEN z_0 AND z_1



(iii) $z(t) = t^2 + it$ FOR $0 \le t \le 1$ PARAMETRIZES PARABOLA $x = y^2$.



(iv) THE PATH MAY BE THE UNION OF STRAIGHT LINE SEGMENTS



HERE $C = C_1 \sqcup C_2 \sqcup C_3$.

DEFINITION LET C BE A SMOOTH CONTOUR ON $a \le t \le b$.

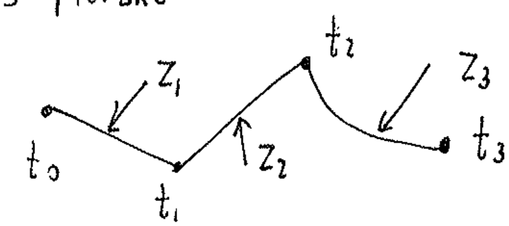
THEN $$\int_C f(z)\,dz \equiv \int_a^b f(z(t))\,z'(t)\,dt$$ WHEN f IS CONTINUOUS ON THE CONTOUR

IF C IS PIECEWISE SMOOTH AND CAN BE DECOMPOSED AS

$C = C_1 \sqcup C_2 \ldots \sqcup C_N$ WHERE $Z_j(t)$ PARAMETRIZES C_j, THEN

$$\int_C f(z)\,dz \equiv \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} f(z_j(t))\,z_j'(t)\,dt$$

$N=3$ picture



SOME SIMPLE PROPERTIES ARE:

(i) $\int_C (a_1 f_1 + a_2 f_2)\,dz = a_1 \int_C f_1\,dz + a_2 \int_C f_2\,dz$

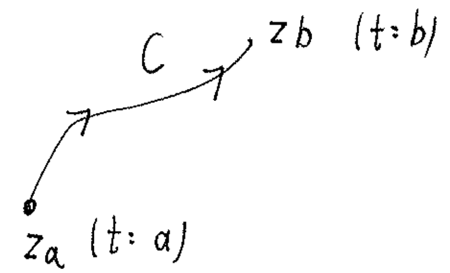
(ii) $\int_C f\,dz = -\int_{-C} f\,dz$ ⟵ changing orientation of path introduces − sign.

(iii) THE value of $\int_C f(z)\,dz$ IS independent OF how ONE parametrizes the path, provided that the re-parametrization IS 1-1, AND THE ORIENTATION OF THE PATH.

(iv) SUPPOSE $|f(z)|$ IS BOUNDED ON THE CONTOUR. THIS OCCURS, FOR INSTANCE, WHEN $f(z)$ IS ANALYTIC ON C.

THEN $\left| \int_C f(z)\,dz \right| \leq \max_C |f(z)| \cdot (\text{Length of } C)$

whenever (Length C) IS FINITE.



<u>PROOF</u>: DEFINE $I = \int_C f(z)\,dz$

THEN,

$$I = \int_a^b f(z(t))\,z'(t)\,dt.$$

RECALL THAT IF $w(t) \equiv f(z(t))\,z'(t)$ THEN BY (v) ON PAGE (I1),

$$|I| = \left| \int_a^b w(t)\,dt \right| \leq \int_a^b |w(t)|\,dt = \int_a^b |f(z(t))|\,|z'(t)|\,dt.$$

NOW $$|f(z(t))| \leq \max_{a \leq t \leq b} |f(z(t))| \equiv \max_C |f(z)|.$$

SO $$|I| = \left| \int_C f(z)\,dz \right| \leq \max_C |f(z)| \int_a^b |z'(t)|\,dt. \quad (*)$$

NOW WRITE $z(t) = x(t) + i\,y(t)$. THEN $|z'| = \sqrt{x'^2 + y'^2}$.

RECALL $\int_a^b \sqrt{x'^2 + y'^2}\,dt = \text{length}(C)$.

THUS, FROM $(*)$ $$|I| \leq \max_C |f(z)|\,(\text{Length of } C)$$

PROOF OF (iii) LET $z(t)$ BE A PARAMETRIZATION OF C ON $a \le t \le b$.

LET $t = \phi(s)$ WITH $\phi(s)$ AND s REAL SUCH THAT

$\phi(\alpha) = a$, $\phi(\beta) = b$. DEFINE $Z(s) = z[\phi(s)]$.

$$\int_\alpha^\beta f(Z(s))\, Z'(s)\, ds = \int_\alpha^\beta f(z(\phi(s)))\, z'(\phi(s))\, \phi'(s)\, ds.$$

NOW CHANGE VARIABLES: $t = \phi(s)$ SO THAT

$$\int_\alpha^\beta f(Z(s))\, Z'(s)\, ds = \int_\alpha^\beta f(z(\phi(s)))\, z'(\phi(s))\, \phi'(s)\, ds = \int_a^b f(z(t))\, z'(t)\, dt.$$

☒

THEOREM (FUNDAMENTAL THEOREM OF CALCULUS) SUPPOSE THAT $f(z)$ IS CONTINUOUS IN A DOMAIN D AND HAS AN ANTI-DERIVATIVE $\hat{f}(z)$ THROUGHOUT D (i.e. $d\hat{f}/dz = f(z)$ FOR EACH z IN D). THEN FOR ANY CONTOUR C IN D WITH INITIAL POINT z_i AND END POINT z_f WE GET

$$\int_C f(z)\, dz = \hat{f}(z_f) - \hat{f}(z_i).$$

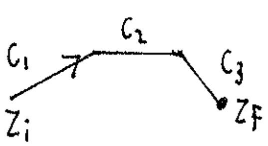
FTC = FUNDAMENTAL THEOREM CALCULUS

REMARK (i) THIS MEANS THAT $\hat{f}(z)$ IS ANALYTIC AND CONTINUOUS IN D.

PROOF SUPPOSE THAT C IS A CONTOUR IN D JOINING z_i TO z_f.

THEN IF C IS PIECEWISE SMOOTH,

$$\int_C f(z)\, dz = \sum_{j=1}^{n} \int_{C_j} f(z)\, dz = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} f(z(t))\, z'(t)\, dt \qquad (*)$$



$z_i \to t = t_0$

$z_f \to t = t_n$

NOW ON EACH SEPARATE INTERVAL dz/dt EXISTS AND IS CONTINUOUS.

THEREFORE, $\dfrac{d}{dt}\hat{f}(z(t)) = \hat{f}'(z(t))\, z'(t) = f(z(t))\, z'(t)$. FROM $(*)$ WE GET

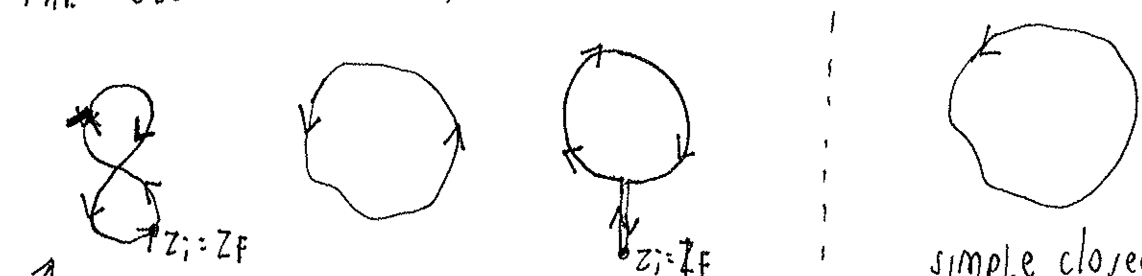
$$\int_C f(z)\, dz = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{d}{dt}\hat{f}(z(t))\, dt = \sum_{j=1}^{n} \left(\hat{f}(z(t_j)) - \hat{f}(z(t_{j-1}))\right)$$

telescoping sum

$$\longrightarrow \int_C f(z)\, dz = \hat{f}(z_f) - \hat{f}(z_i).$$

☐

<u>DEFINITION</u> C IS A CLOSED CONTOUR OR A LOOP IF ITS INITIAL AND TERMINAL POINTS COINCIDE. A SIMPLE CLOSED CONTOUR IS A CLOSED CONTOUR WITH NO MULTIPLE POINTS OTHER THAN ITS INITIAL-TERMINAL POINT; IN OTHER WORDS, IF $z(t)$ FOR $a \leq t \leq b$ IS A PARAMETRIZATION OF THE CLOSED CONTOUR, THEN $z(t)$ IS 1-1 ON $[a,b)$.



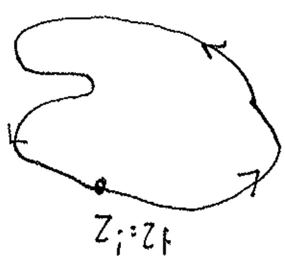
closed contour OR LOOP

closed contour

simple closed contour

<u>COROLLARY</u> IF $f(z)$ IS CONTINUOUS IN A DOMAIN D AND HAS AN ANTI-DERIVATIVE $\hat{f}(z)$ THROUGHOUT D, THEN $\int_C f(z)\,dz = 0$ FOR ALL LOOPS C LYING IN D. THUS, IN THIS CASE THE INTEGRAL IS INDEPENDENT OF THE SPECIFIC PATH.

THE PROOF IS IMMEDIATE BY THE THEOREM.

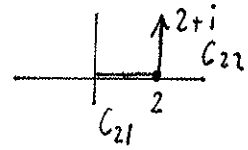


SINCE THE INTEGRAL ONLY DEPENDS ON INITIAL AND END STATES, WHICH ARE THE SAME, THEN $\int_C f(z)\,dz = \hat{f}(z_F) - \hat{f}(z_i) = \hat{f}(z_i) - \hat{f}(z_i) = 0.$

EXAMPLE 1 CALCULATE $I_1 = \int_{C_1} z^2\,dz$, $I_2 = \int_{C_2} z^2\,dz$

WHERE (i) C_1 IS STRAIGHT LINE FROM $z=0$ TO $z=2+i$

(ii) C_2 IS PATH FROM $z=0$ TO $z=2+i$ AS SHOWN



FOR C_1: LET $z=(2+i)t \rightarrow dz/dt = 2+i$ FOR $0 \le t \le 1$.

THEN $I_1 = \int_0^1 (2+i)^2 (2+i) t^3\,dt = \int_0^1 (z(t))^2 z'(t)\,dt$

$$I_1 = (2+i)^3 \int_0^1 t^2\,dt = (2+i)^3 \, t^3/3 \Big|_0^1 = (2+i)^3/3 = \frac{1}{3}(2+11i).$$

FOR C_2: FIRST LET $z_1 = 2t$. $dz_1 = 2\,dt$

THEN $I_{21} = \int_{C_{21}} z^2\,dz = \int_0^1 z_1^2(t)\, z_1'(t)\,dt = \int_0^1 (4t^2)\, 2t\,dt = 8\frac{t^3}{3}\Big|_0^1 = \frac{8}{3}$.

NOW $I_{22} = \int_{C_{22}} z^2\,dz$. LET $z_2 = 2+it$, $0 \le t \le 1$. $\frac{dz_2}{dt} = i$.

$$I_{22} = \int_0^1 (2+it)^2\, i\,dt = \int_0^1 [4+4it-t^2]\, i\,dt = 4i - 4\int_0^1 t\,dt - i\int_0^1 t^2\,dt$$

$$I_{22} = 4i - 2 - i/3 = -2 + 11i/3.$$

THUS ADDING TOGETHER, $\int_{C_2} = \int_{C_{21}} + \int_{C_{22}} = \frac{1}{3}(8-6+11i) = \frac{1}{3}(2+11i)$.

NOTICE THAT

$$\int_{C_1} = \int_{C_2} \longrightarrow \text{INDEPENDENCE OF PATH.}$$

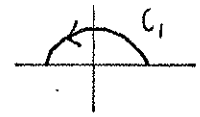
NOW BY THE THEOREM ON P. I5 (FTC), z^2 IS CONTINUOUS $\forall z$

AND THE ANTI-DERIVATIVE IS $F(z) = z^3/3$.

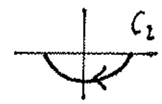
THEN, $\int_C z^2\,dz = F(2+i) - F(0) = (2+i)^3/3$.

EXAMPLE 2 CALCULATE $I_1 = \int_{C_1} \bar{z}\, dz$ AND $I_2 = \int_{C_2} \bar{z}\, dz$

WHERE (i) $C_1: \; z = e^{it} \quad 0 \le t \le \pi$



(ii) $C_2: \; z = e^{it} \quad 0 \ge t \ge -\pi$



NOTICE THAT BOTH PATHS HAVE SAME INITIAL AND FINAL STATE.

FOR C_1: $z = e^{it}, \; 0 \le t \le \pi \; \rightarrow \; dz/dt = ie^{it}$

SO $\int_{C_1} \bar{z}\, dz = \int_0^{\pi} \bar{z}(t)\, z'(t)\, dt = \int_0^{\pi} e^{-it}\, i e^{it}\, dt = \pi i.$

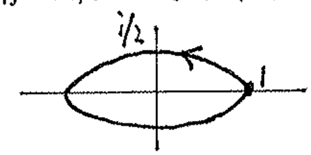
FOR C_2: $z = e^{it} \quad 0 \ge t \ge -\pi \; \rightarrow \; dz/dt = ie^{it}$

SO $\int_{C_2} \bar{z}\, dz = \int_0^{-\pi} \bar{z}(t)\, z'(t)\, dt = \int_0^{-\pi} e^{-it}\, i e^{it} = -i\pi.$

NOTICE THAT $\int_{C_1} \neq \int_{C_2}$ EVENTHOUGH INITIAL AND END STATES ARE IDENTICAL.

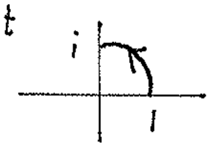
ALSO NOTICE THAT $\bar{z}$ DOES NOT HAVE AN ANTI-DERIVATIVE. HENCE THE THEOREM (FTC) CANNOT BE APPLIED.

EXAMPLE 3 CALCULATE $I = \int_C z^3\, dz$ WHERE C IS THE PORTION OF ELLIPSE $x^2 + 4y^2 = 1$ JOINING $z = 1$ TO $z = i/2$



SOLUTION $\mathcal{F}(z) = z^4/4$ IS THE ANTI-DERIVATIVE OF $f(z)$. THEN, FTC $\Rightarrow \int_C z^3\, dz = \mathcal{F}(i/2) - \mathcal{F}(1) = (i/2)^4/4 - 1/4 = 1/64 - 1/4 = -15/64.$

EXAMPLE CALCULATE $\int_C e^z\, dz$ WHERE C IS QUARTER-CIRCLE $z = e^{it}$ WITH $0 \le t \le \pi/2$



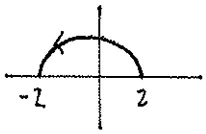
SOLUTION e^z HAS THE ANTI-DERIVATIVE $\mathcal{F}(z) = e^z$ DEFINED $\forall z$.

THUS BY FTC, $\int_C f(z)\, dz = \int_C e^z\, dz = \mathcal{F}(i) - \mathcal{F}(1) = e^i - e^1 = \cos(1) - e^1 + i\sin(1).$

NOTICE THAT $\int_C e^z\, dz = \int_0^1 e^{e^{it}}\, i e^{it}\, dt = \int_0^1 e^{\cos t + i \sin t}\, i\,[\cos t + i \sin t]\, dt.$

THUS $\text{RE}\left[\int_0^1 i\, e^{\cos t}\,[\cos(\sin t) + i \sin(\sin t)]\,[\cos t + i \sin t]\, dt \right] = \cos(1) - e^1.$

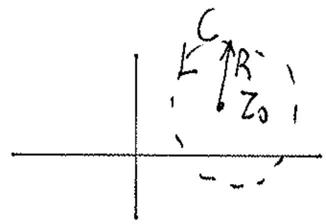
EXAMPLE CALCULATE $I = \int_C \cos z \, dz$ OVER THE ½ CIRCLE $z = 2e^{it}$ FROM $t=0$ TO $t = \pi/2$.



NOW THE ANTI-DERIVATIVE $F(z) = \sin z$ EXISTS $\forall z$.

HENCE, $\int_C \cos z \, dz = F(-2) - F(2) = +\sin(-2) - \sin(2) = -\sin(2) - \sin(2) = -2\sin(2)$.

AN IMPORTANT INTEGRAL IS TO CALCULATE



$$I = \int_C (z - z_0)^n \, dz \qquad n = \text{integer}$$

AND C IS A CIRCLE OF RADIUS R CENTERED AT z_0 ORIENTED COUNTERCLOCKWISE.

WE WILL FIRST CALCULATE DIRECTLY. LET $z = z_0 + Re^{it}$ WITH $0 \le t \le 2\pi$.

THEN $I = \int_C (z - z_0)^n \, dz = \int_0^{2\pi} (z(t) - z_0)^n \frac{dz}{dt} \, dt = \int_0^{2\pi} R^n e^{int} \, i R e^{it} \, dt$.

$$(*) \qquad I = i R^{n+1} \int_0^{2\pi} e^{i(n+1)t} \, dt.$$

THERE ARE TWO CASES:

- IF $n \neq -1$ THEN $I = i R^{n+1} \int_0^{2\pi} [\cos((n+1)t) + i \sin((n+1)t)] \, dt = 0$

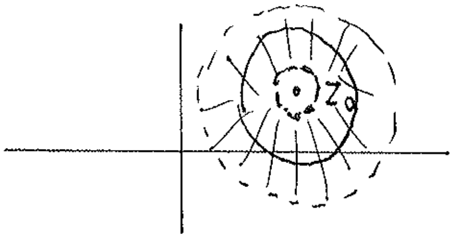
SINCE $\int_0^{2\pi} \cos(m\varphi) \, d\varphi = \int_0^{2\pi} \sin(m\varphi) \, d\varphi = 0$ FOR $m \neq 0$ $m = \text{integer}$

THUS $I = 0$ IF $n \neq -1$ (ALSO NOTE: IF $n \neq -1$ $I = iR^{n+1} \frac{1}{i(n+1)} e^{i(n+1)t} \Big|_0^{2\pi} = 0$)

- IF $n = -1$ THEN $(*)$ GIVES $I = i \int_0^{2\pi} 1 \, dt = 2\pi i$.

THUS
$$\int_C (z - z_0)^n \, dz = \begin{cases} 0 & \text{IF } n \neq -1 \\ 2\pi i & \text{IF } n = -1. \end{cases}$$

WE NOW CALCULATE INDIRECTLY. SUPPOSE $n \neq -1$ THEN LET D BE ANY DOMAIN EXCLUDING THE POINT z_0 WHICH CONTAINS C (SEE THE FIGURE)



THE DOMAIN D IS THE ANNULUS BETWEEN DOTTED LINES.

IN D, THE ANTI-DERIVATIVE FOR $n \neq -1$ IS $\hat{f}(z) = \frac{1}{(n+1)}(z-z_0)^{n+1}$. BY FTC, SINCE INITIAL AND FINAL STATES ARE SAME, $\int_C (z-z_0)^n \, dz = 0$, $n \neq -1$.

NOTICE, IF $n=-1$ THEN THERE IS <u>NO</u> ANTIDERIVATIVE DEFINED IN ANY ANNULAR DOMAIN AROUND z_0, WHICH CONTAINS C, SINCE $\log(z-z_0)$ IS NOT CONTINUOUS ACROSS THE BRANCH CUT $\operatorname{Im}(z-z_0)=0$, $\operatorname{Re}(z-z_0)<0$.

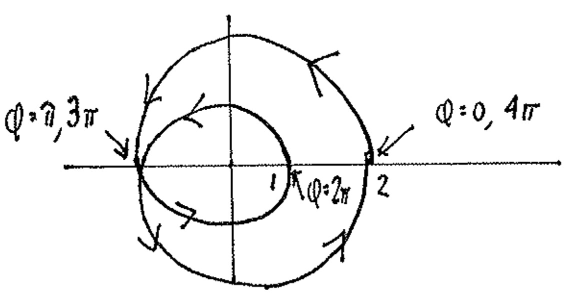
$$\text{i.e.} \quad \frac{d}{dz} \log(z-z_0) = \frac{1}{z-z_0} \quad \text{IN} \quad \mathbb{C} \setminus \{\operatorname{Im}(z-z_0)=0,\ \operatorname{Re}(z-z_0)<0\}.$$

THUS FOR $n=-1$, FTC <u>CANNOT</u> BE INVOKED.

<u>EXAMPLE</u> CALCULATE $I = \int_C \frac{1}{z} \, dz$

WHEN C IS THE CLOSED CONTOUR DEFINED BY THE POLAR EQUATION $z = re^{i\varphi}$ WITH $r = 2 - \sin^2(\varphi/4)$ WITH $0 \leq \varphi \leq 4\pi$.

WE NOTE THAT THIS PATH IS AS SHOWN.



THE PATH CIRCLES THE ORIGIN TWICE.

WE CALCULATE

$$dz/d\varphi = r' e^{i\varphi} + i r e^{i\varphi}$$

THUS
$$I = \int_C \frac{1}{re^{i\varphi}} \frac{dz}{d\varphi} \, d\varphi = \int_0^{4\pi} \frac{1}{re^{i\varphi}} [r'e^{i\varphi} + ire^{i\varphi}] \, d\varphi$$

SO
$$I = \int_0^{4\pi} \left(\frac{r'}{r} + i\right) d\varphi = \ln(r(\varphi)) \Big|_0^{4\pi} + 4\pi i \quad (\text{SINCE } r \neq 0 \text{ FOR ANY } \varphi)$$

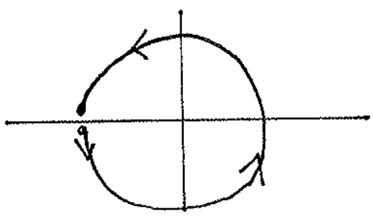
BUT $r(4\pi) = r(0)$ SO $\ln(r(\varphi))\Big|_0^{4\pi} = 0$ AND $I = 4\pi i$

$\square$

EXAMPLE SHOW BY A LIMITING PROCEDURE THAT

$$\int_C \frac{1}{z}\,dz = 2\pi i$$ WHERE C IS A CIRCLE OF RADIUS R CENTERED AT $z=0$ ORIENTED COUNTERCLOCKWISE.

AS REMARKED EARLIER, WE CANNOT USE FTC. INSTEAD WE CONSIDER THE PICTURE



WE DEFINE $\text{LOG } z$ TO BE THE PRINCIPAL BRANCH OF $\log z$ WITH

$$\text{LOG } z = \ln|z| + i\varphi \qquad -\pi < \varphi < \pi.$$

$$\frac{d}{dz}\text{LOG } z = 1/z \qquad \text{FOR } z \text{ IN } \mathbb{C}\setminus(-\infty,0].$$

WE PARAMETRIZE $z = Re^{it}$ WITH $-\pi+\varepsilon < t < \pi-\varepsilon$ FOR $\varepsilon > 0$. CALL THIS C_ε.

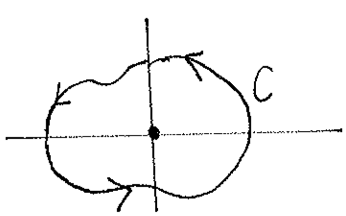
THEN WE CAN USE FTC FOR ANY $\varepsilon > 0$ TO GET

$$\int_{C_\varepsilon} \frac{1}{z}\,dz = \text{LOG } z_F - \text{LOG } z_I \qquad z_I = Re^{i(-\pi+\varepsilon)}, \quad z_F = Re^{i(\pi-\varepsilon)}$$

$$\int_{C_\varepsilon} \frac{1}{z}\,dz = \big(\ln R + i(+\pi-\varepsilon)\big) - \big(\ln R + i(-\pi+\varepsilon)\big)$$
$$= 2i\pi - 2i\varepsilon$$

NOW LET $\varepsilon \to 0^+ \longrightarrow \displaystyle\int_C \frac{1}{z}\,dz = 2i\pi.$

REMARK • THIS CAN CLEARLY BE EXTENDED TO PROVE THAT



$$\int_C \frac{1}{z}\,dz = 2\pi i$$ FOR ANY SIMPLE CLOSED CONTOUR CONTAINING ORIGIN.

• IN ADDITION $$\int_C \frac{1}{z^n}\,dz = 0 \qquad \text{FOR } n \neq 1$$ WHERE

C IS ANY CLOSED SIMPLE CONTOUR CONTAINING $z=0$.

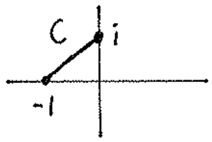
(IT FOLLOWS BY existence OF ANTI-derivative).

NEXT WE GIVE A FEW EXAMPLES TO ILLUSTRATE BOUNDS ON INTEGRALS

WE RECALL THAT IF $f(z)$ IS BOUNDED ON C THEN

$$\left| \int_C f(z)\,dz \right| \leq \max_C |f(z)| \text{ Length}(C).$$

<u>EXAMPLE 1</u> ESTIMATE $\left| \int_C \frac{1}{z^4}\,dz \right|$ WHERE C IS LINE JOINING $z=-1$ TO $z=i$.



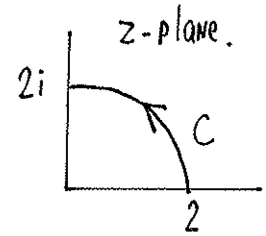
NOW LET $z=-1+t(i+1)$ WHEN $t:0 \to z=-1$
$t:1 \to z=i$.

NOW $$\max_C \left| \frac{1}{z^4} \right| = \frac{1}{\min_C |z^4|} = \frac{1}{\min_C |z|^4} = \frac{1}{\left[\left| \frac{1}{2}(-1+i) \right|^4 \right]} = \frac{1}{\left[\sqrt{2}/2 \right]^4} = 4.$$

Length $C = \sqrt{2}$.

THUS, $$\left| \int_C \frac{1}{z^4}\,dz \right| \leq 4\sqrt{2}.$$

<u>EXAMPLE 2</u> ESTIMATE $\left| \int_C \frac{1}{z^2+1}\,dz \right|$ WHERE C IS THE



QUARTER CIRCLE $z=2e^{i\varphi}$ WITH $0 \leq \varphi \leq \pi/2$.

WE RECALL THE Δ-INEQUALITY $|z_1+z_2| \geq \big| |z_1|-|z_2| \big|$.

THUS $$\frac{1}{|z^2+1|} \leq \frac{1}{\big| |z|^2-1 \big|}$$ (<u>NOTE</u>: $|z^2-(-1)| \geq \big| |z^2|-|-1| \big| = \big| |z|^2-1 \big|$

BUT SINCE $|z|^2=4$ ON C WE GET

$$\frac{1}{|z^2+1|} \leq \frac{1}{|4-1|} = \frac{1}{3}$$

THUS $$\max_C \frac{1}{|z^2+1|} \leq \frac{1}{3}$$ LENGTH $C = \frac{\pi}{2}(2) = \pi$.

THUS $$\left| \int_C \frac{1}{z^2+1}\,dz \right| \leq \pi/3.$$

<u>EXAMPLE 3</u> ESTIMATE $\left| \int_C (e^z - \bar{z})\,dz \right|$ WHERE C IS THE CIRCLE $|z|=2$.

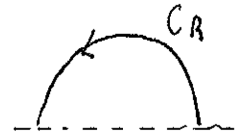
WE USE $$|e^z - \bar{z}| \leq |e^z| + |\bar{z}| = |e^{RE\,z + i\,IM(z)}| + |\bar{z}| = |e^{RE(z)}| + |\bar{z}|$$

CONTINUING ON $$|e^z - \bar{z}| \leq e^{RE(z)} + |\bar{z}| \leq e^{|RE(z)|} + |z| \leq e^2 + 2.$$

NOW LENGTH $(C) = 2\pi(2)$. THUS, $$\left| \int_C (e^z - \bar{z})\,dz \right| \leq (e^2+2)4\pi.$$

LET C_R DENOTE THE SEMI-CIRCLE $z = Re^{i\varphi}$ WITH $0 \le \varphi \le \pi$.

SHOW THAT



(i) $\left| \int_{C_R} \frac{z}{z^3+1} \, dz \right| \to 0$ AS $R \to \infty$

(ii) $\left| \int_{C_R} \frac{\text{LOG } z}{(z^2+1)} \, dz \right| \to 0$ AS $R \to \infty$ LOG Z IS THE P.V. OF $\log z$.

(iii) $\left| \int_{C_R} \frac{e^{ikz}}{z^2+1} \, dz \right| \to 0$ AS $R \to \infty$ FOR ANY $k > 0$ REAL.

<u>PROOF</u> (i) $|z^3+1| \ge \big| |z|^3 - 1 \big| = R^3 - 1$ FOR $R > 1$.

THUS $\left| \frac{z}{z^3+1} \right| \le \frac{|z|}{R^3-1} = \frac{R}{R^3-1}$ ON C_R.

NOW $\text{length}(C_R) = \pi R$.

THUS $\left| \int_{C_R} \frac{z}{z^3+1} \, dz \right| \le \left(\frac{R}{R^3-1} \right) \pi R = \frac{\pi R^2}{R^3-1} \to 0$ AS $R \to \infty$.

(ii) $|\text{LOG } z| = \big| \ln|z| + i\varphi \big| \le \big| \ln R + i\pi \big| = \sqrt{(\ln R)^2 + \pi^2}$ ON C_R.

$|z^2+1| \ge \big| |z|^2 - 1 \big| = R^2 - 1$ FOR $R > 1$.

THUS $\left| \int_{C_R} \frac{\text{LOG } z}{z^2+1} \, dz \right| \le \frac{\sqrt{(\ln R)^2 + \pi^2}}{R^2-1} \pi R = O\left(\frac{R \ln R}{R^2} \right)$ FOR $R \gg 1$

$\longrightarrow 0$ AS $R \to \infty$.

(iii) LET $z = x + iy$. $|e^{ikz}| = |e^{ikx - ky}| = e^{-ky} \le 1$ FOR $y \ge 0$ AND $k > 0$.

THUS $|e^{ikz}| \le 1$ ON C_R.

$|z^2+1| \ge \big| |z|^2 - 1 \big| = R^2 - 1$ ON C_R.

HENCE $\left| \int_{C_R} \frac{e^{ikz}}{z^2+1} \, dz \right| \le \frac{1}{R^2-1} (\pi R) \to 0$ AS $R \to \infty$.