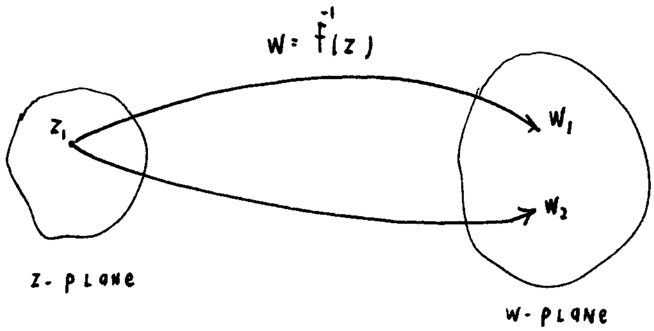
MULTI-VALUED FUNCTIONS

THERE ARE MANY FUNCTIONS WHOSE INVERSE FUNCTION IS MULTI-VALUED. FOR INSTANCE, CONSIDER

$$z = e^{w}, \quad z = w^{2}, \quad z = \cos w, \quad z = \sin w.$$

FOR EACH OF THESE FUNCTIONS A GIVEN VALUE OF z CORRESPONDS TO MORE THAN ONE VALUE OF w.



$w = f^{-1}(z)$ is multi-valued

$z = f(w)$ is single-valued

given w, there is a unique value of z.

THE GOALS ARE AS FOLLOWS:

(i) TO DETERMINE ALL POSSIBLE VALUES OF THE INVERSE FUNCTION w.

AND/OR (ii) TO CONSTRUCT AN INVERSE FUNCTION WHICH IS SINGLE-VALUED IN SOME REGION OF THE COMPLEX PLANE.

THE LOGARITHIM FUNCTION

WANT TO DEFINE THE INVERSE FUNCTION $f^{-1}(z)$ FOR

$$z = e^{w}.$$

let $z = re^{i\varphi}$ AND $w = u + iv$. THEN $re^{i\varphi} = e^{u}e^{iv}$.

HENCE $r = e^{u}$ AND $v = \varphi + 2k\pi \quad k = 0, \pm 1, \pm 2, \ldots$

$$w = \ln r + i(\varphi + 2k\pi) \qquad k = 0, \pm 1, \pm 2, \ldots$$

BUT $r = |z|$ AND WLOG (WITHOUT LOSS OF GENERALITY) WE CAN TAKE $\varphi \in (-\pi, \pi]$.

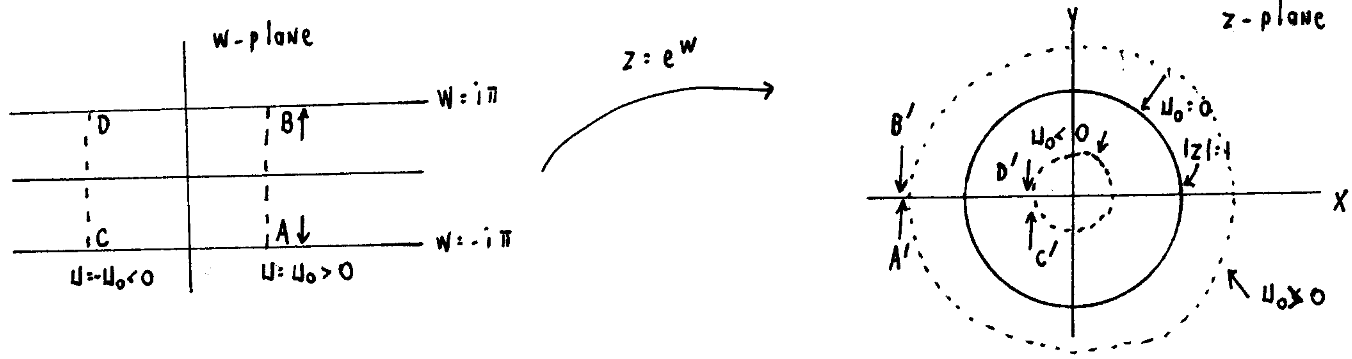
THUS $w = \ln|z| + i\left(\mathrm{Arg}(z) + 2k\pi\right) \qquad k = 0, \pm 1, \pm 2, \ldots$

THIS MOTIVATES THE DEFINITION OF THE INVERSE FUNCTION $f^{-1}(z)$ FOR $z = e^{w}$

$$① \quad w = \log z \equiv \ln|z| + i\left(\mathrm{Arg}(z) + 2k\pi\right) \qquad k = 0, \pm 1, \pm 2, \ldots$$

OR EQUIVALENTLY $② \quad \log z = \ln|z| + i \arg(z)$

NOW WE VIEW $z = e^w$ AS A MAPPING. CONSIDER THE STRIP $|\mathrm{Im}(w)| < \pi$



- NOW TAKE $u = u_0 > 0$ $v \in (-\pi, \pi)$ FOR LINE AB:

$$x + iy = e^{u_0}(\cos v + i \sin v)$$

$$\left.\begin{aligned} x &= e^{u_0}\cos v \\ y &= e^{u_0}\sin v \end{aligned}\right\} \rightarrow x^2 + y^2 = e^{2u_0} > 1$$

get FULL CIRCLE IN (x,y) PLANE OUTSIDE $|z| = 1$

NOW APPROACH A: $u = u_0 > 0$ $v = -\pi + \varepsilon$

$$x = e^{u_0}\cos(-\pi + \varepsilon) \rightarrow -e^{u_0} \quad \text{AS } \varepsilon \rightarrow 0^+ \qquad -e^{+u_0} < -1 \text{ WHEN } u_0 > 0.$$

$$y = e^{u_0}\sin(-\pi + \varepsilon) \rightarrow 0^- \quad \text{AS } \varepsilon \rightarrow 0^+$$

NOW APPROACH B: $u = u_0 > 0$, $v = \pi - \varepsilon$

$$x = e^{u_0}\cos(\pi - \varepsilon) \rightarrow -e^{u_0} \quad \text{AS } \varepsilon \rightarrow 0^+$$

$$y = e^{u_0}\sin(\pi - \varepsilon) \rightarrow 0^+ \quad \text{AS } \varepsilon \rightarrow 0^+$$

- NOW TAKE $u = -u_0 < 0$ $v \in (-\pi, \pi)$ FOR LINE CD:

$$\left.\begin{aligned} x &= e^{-u_0}\cos v \\ y &= e^{-u_0}\sin v \end{aligned}\right\} \quad x^2 + y^2 = e^{-2u_0} < 1$$

get FULL CIRCLE IN (x,y) PLANE INSIDE $|z| = 1$

NOW APPROACH C: $u = -u_0 < 0$, $v = -\pi + \varepsilon$

$$x = e^{-u_0}\cos(-\pi + \varepsilon) \rightarrow -e^{-u_0} > -1 \quad \text{AS } \varepsilon \rightarrow 0^+$$

$$y = e^{-u_0}\sin(-\pi + \varepsilon) \rightarrow 0^- \quad \text{AS } \varepsilon \rightarrow 0^+$$

NOW APPROACH D: $u = -u_0 < 0$ $v = \pi - \varepsilon$

$$x = e^{-u_0}\cos(\pi - \varepsilon) \rightarrow -e^{-u_0} > -1 \quad \text{AS } \varepsilon \rightarrow 0^+$$

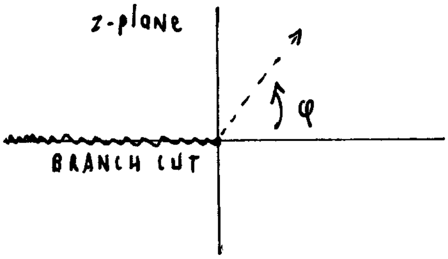
$$y = e^{-u_0}\sin(\pi - \varepsilon) \rightarrow 0^+ \quad \text{AS } \varepsilon \rightarrow 0^+$$

<u>CONCLUSION</u>: POINTS ALONG THE NEGATIVE REAL AXIS IN Z-PLANE YIELD MULTIPLE W VALUES. THUS IN OBTAINING A SINGLE VALUED INVERSE FUNCTION FOR THE FUNDAMENTAL STRIP $|\mathrm{Im}(w)| < \pi$ WE NEED A CUT IN Z-PLANE ALONG $\mathrm{Re}(z) < 0$.

THE MAPPING $z = e^w$ AND $w = f^{-1}(z)$ WILL BE SINGLE VALUED IN $|\mathrm{Im}(w)| < \pi$ AND $z \in \mathbb{C} \setminus (-\infty, 0)$. NOTICE ALSO THAT $w = 0 \rightarrow z = 1$. CLEARLY THE INVERSE FUNCTION SHOULD BE THE <u>PRINCIPAL VALUE</u> OF $\log z$ DEFINED BY

$$w = \mathrm{Log}(z) = \ln|z| + i(\mathrm{Arg}\, z) \qquad -\pi < \mathrm{Arg}(z) \leq \pi.$$

$w = \mathrm{Log}(z)$



discontinuous as we approach the branch cut from top and bottom.

- Using Cauchy-Riemann equations $\mathrm{Log}(z)$ is analytic in the cut-plane and $d/dz\, \mathrm{Log}(z) = 1/z$ $\quad z \neq 0$, $z \notin \{-\infty,$

THE POINT $z = 0$ IS CALLED A BRANCH POINT OF $\mathrm{Log}(z)$ SINCE IF WE ENCIRCLE THE ORIGIN $z = 0$ BY A CLOSED CONTOUR THEN $\mathrm{Log}(z)$ CHANGES BY AN AMOUNT PROPORTIONAL TO $2\pi i$.

PROPERTIES OF $\log z$

(i) $\log(z_1 z_2) = \log z_1 + \log z_2$ (means that the set of all values of $\log z_1 + \log z_2$ is the same as the set of all values of $\log(z_1 z_2)$.)

(ii) $z = e^{\log z}$ BUT $\log(e^z) = z + 2K\pi i \quad K = 0, \pm 1, \pm 2, \ldots$

PROOF: $z = x + iy$

$$\log(e^{x+iy}) = \ln(e^x) + i\left(\tan^{-1}\left(\frac{\sin y}{\cos y}\right) + 2K\pi\right) = x + iy + 2K\pi i$$
$$= z + 2K\pi i$$

(iii) SHOW $\log(z^n) \neq n \log(z)$ IN general.

MUST SHOW THAT THE TWO SETS OF VALUES DO NOT COINCIDE.

LET $z = re^{i\varphi}$. $\quad \log(z^n) = n \ln r + i(n\varphi + 2K\pi) \quad K = 0, \pm 1, \pm 2, \ldots$

$n \log(z) = n \ln r + in(\varphi + 2m\pi) \quad m = 0, \pm 1, \pm 2, \ldots$

LET n BE FIXED. THE SET OF VALUES OF $\{K\}$ $K = 0, \pm 1, \pm 2, \ldots$ DO NOT COINCIDE WITH THE SET OF VALUES OF $\{mn\}$ $m = 0, \pm 1, \pm 2, \ldots$

$\longrightarrow \log(z^n) \neq n \log(z)$

(iv) SHOW $\log(z^{1/n}) = \frac{1}{n} \log(z)$ (sets of values are the same) n positive integer.

NOW $z = re^{i\varphi}$. $\quad z^{1/n} = r^{1/n} e^{i[\Theta + 2K\pi]/n} \quad K = 0, 1, \ldots, n-1 \quad \Theta = \mathrm{Arg}(z)$

NOW $\log(z^{1/n}) = \frac{1}{n} \ln r + i\left[\frac{\Theta + 2K\pi}{n} + 2p\pi\right] \quad K = 0, 1, \ldots, n-1 \quad p = 0, \pm 1, \pm 2, \ldots$

$\frac{1}{n} \log z = \frac{1}{n} \ln r + i\left[\frac{\Theta}{n} + \frac{2q\pi}{n}\right] \quad q = 0, \pm 1, \pm 2, \ldots$

THE SET OF VALUES OF $\log(z^{1/n})$ AND $\frac{1}{n} \log(z)$ ARE THE SAME IF THE SETS $\{K + pn\}$ $K = 0, 1, \ldots, n-1$; $p = 0, \pm 1, \pm 2, \ldots$ coincide with the set $\{q\}$ $q = 0, \pm 1, \pm 2, \ldots$

$q \stackrel{?}{=} K + pn$ ✓✓ $\Rightarrow$

- FOR ANY K AND p WE CAN GET A q.
- DIVIDING q BY n gives an integer plus a remainder K in $\{0, \ldots, n-1\}$

COMPLEX EXPONENTS

<u>DEFINITION</u> IF α IS COMPLEX AND $z \neq 0$ THEN

$$z^{\alpha} \equiv e^{\alpha \log z} \quad \text{MULTI-VALUED.} \qquad z^{\alpha} = e^{\alpha[\ln|z| + i(\operatorname{Arg} z + 2k\pi)]} \quad k = 0, \pm 1, \pm 2$$

- AGREES WITH OUR PREVIOUS RESULTS IF $\alpha = m$, $\alpha = 1/m$ m = integer
- SINCE $z^{\alpha} = |z|^{\alpha} e^{i(\alpha \operatorname{Arg}(z) + 2\alpha k\pi)}$ $k = 0, \pm 1, \pm 2, \ldots$

  THERE WILL BE A FINITE # OF VALUES OF z^{α} ONLY IF α IS THE RATIO OF TWO INTEGERS (I.E. RATIONAL). (I.E. WANT αk = integer FOR some k)
- z^{α} TAKES INFINITELY MANY VALUES WHEN α IS NOT REAL AND RATIONAL

<u>EXAMPLE</u> FIND ALL VALUES OF i^{-2i}.

$$i^{-2i} = e^{-2i \log(i)} = e^{-2i[\ln 1 + i(\pi/2 + 2k\pi)]} = e^{(4k+1)\pi} \qquad k = 0, \pm 1, \pm 2, \ldots$$

INFINITY OF VALUES.

<u>EXAMPLE</u> FIND ALL SOLUTIONS OF $z^{1+i} = 4$

$$e^{(1+i)\log z} = e^{\ln 4} \longrightarrow (1+i)\log z = \ln 4 + 2\pi n i \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$2 \log z = (1-i)[\ln 4 + 2\pi n i] \qquad n = 0, \pm 1, \pm 2, \ldots$$

$$\log z = (1-i)[\ln 2 + \pi n i] = \ln 2 + \pi n + i(\pi n - \ln 2)$$

THUS $z = e^{\ln 2 + \pi n + i(\pi n - \ln 2)}$ (EXPONENTIATE)

$$z = 2e^{\pi n}\left(\cos(\pi n - \ln 2) + i \sin(\pi n - \ln 2)\right)$$

$$z = 2(-1)^n e^{\pi n}\left(\cos(\ln 2) - i \sin(\ln 2)\right) \qquad n = 0, \pm 1, \pm 2, \ldots$$

THE <u>PRINCIPAL VALUE</u> OF z^{α} IS DEFINED BY

$$z^{\alpha} = e^{\alpha \operatorname{Log}(z)} = e^{\alpha[\ln|z| + i \operatorname{Arg}(z)]}$$

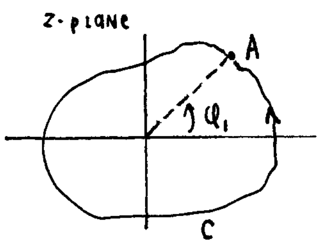
SINCE $\operatorname{Log}(z)$ IS ANALYTIC IN THE SLIT DOMAIN $\mathbb{C} \setminus (-\infty, 0]$ AND e^z IS ENTIRE $\longrightarrow$ z^{α} IS ANALYTIC IN $\mathbb{C} \setminus (-\infty, 0]$.

ALSO $\dfrac{d}{dz} z^{\alpha} = \alpha z^{\alpha} \dfrac{1}{z}$ IN $\mathbb{C} \setminus (-\infty, 0]$.

DEFINITION $F(z)$ is a **branch** of the multi-valued function $f(z)$ in a domain D if $F(z)$ is single-valued and continuous in D and has the property that for each z in D the value $F(z)$ is one of the values of $f(z)$.

To determine $F(z)$ we introduce a line emanating from a point (called a branch point) to ensure that F is single-valued in cut plane. A branch point is one for which if we enclose it with a curve the function changes discontinuously.

For instance consider $w = z^{1/2}$: let (w_1, z_1) be coordinates of A when $\varphi = \varphi_1$



i.e. let $z_1 = r_1 e^{i\varphi_1}$ $\quad 0 < \varphi_1 < 2\pi$.

Then at A: $w = w_1 \equiv r_1^{1/2} e^{i\varphi_1/2}$

Now enclose the origin with the loop C as shown.

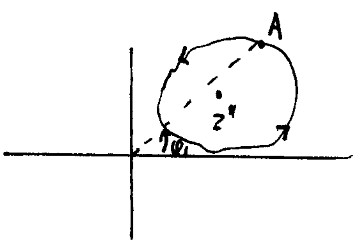
Upon returning to A we have $\varphi = \varphi_1 + 2\pi$

$\rightarrow$ $w = -r_1^{1/2} e^{i\varphi_1/2}$ at point A.

$\rightarrow$ $w = -w_1$ at A.

$\rightarrow$ w has changed discontinuously after performing a loop about $z = 0$ $\rightarrow$ $z = 0$ is a branch point.

Now consider $w = z^{1/2}$ and perform a closed loop around some point $z = z^* \neq 0$ as shown below.



Then $z = r e^{i\varphi}$ $\rightarrow$ $w = r^{1/2} e^{i\varphi/2}$.

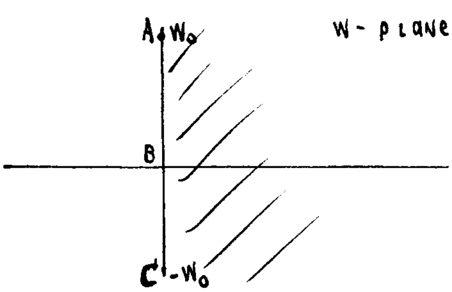
Upon returning to A we have $\varphi = \varphi_1$ again.

Hence w is continuous after performing the loop

$z = z^*$ is not a branch point.

Now show how to construct branch cuts for power law functions.

EX: 1 let $z = w^2$ and consider $\mathrm{Re}(w) > 0$



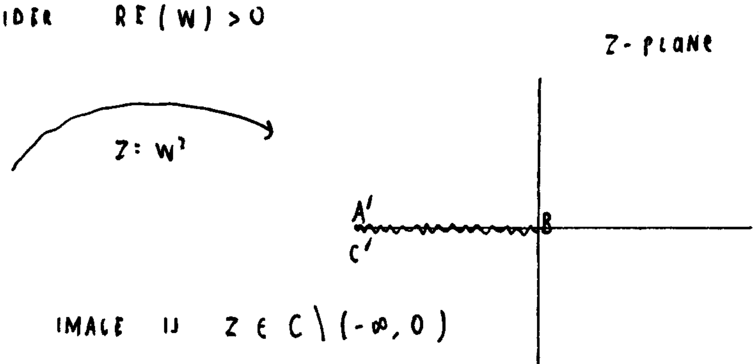


Image is $z \in \mathbb{C} \setminus (-\infty, 0)$

Note: 1-1 mapping if $\mathrm{Re}(w) > 0$ and $z \in \mathbb{C} \setminus (-\infty, 0)$

M6

NEED A BRANCH CUT ALONG negative real-axis IN Z-PLANE. HENCE

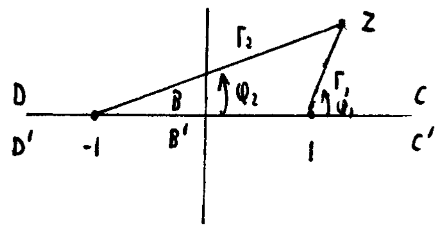
$$W = z^{1/2} \qquad z = re^{i\varphi} \qquad -\pi < \varphi \le \pi .$$

THIS ALSO ENSURES THAT $RE(W) > 0$. NOTICE THAT POINTS ON THE CUT GO EITHER TO A OR C. HENCE WE HAVE CHOSEN THE CUT TO BE $\varphi = \pi \rightarrow$ GOES TO A. (this IS ARBITRARY).

EX: 2 FIND A BRANCH ~~CUT~~ OF $(z^2-1)^{1/2}$ THAT IS ANALYTIC IN THE EXTERIOR OF THE UNIT CIRCLE. BRANCH POINTS AT $z = \pm 1$:

$$W = (z^2-1)^{1/2} = (z+1)^{1/2}(z-1)^{1/2}$$

$$W = (r_1 r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2}$$

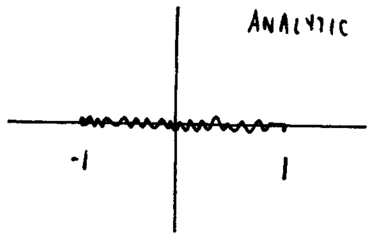


TRY φ_1 IN $(-\pi, \pi)$

φ_2 IN $(-\pi, \pi)$

	φ_1	φ_2	$e^{i(\varphi_1+\varphi_2)/2}$	
B	π	0	$e^{i\pi/2} = i$	DISCONTINUOUS
B'	$-\pi$	0	$e^{-i\pi/2} = -i$	
C	0	0	1	CONTINUOUS
C'	0	0	1	
D	π	π	$e^{i\pi} = -1$	CONTINUOUS
D'	$-\pi$	$-\pi$	$e^{-i\pi} = -1$	

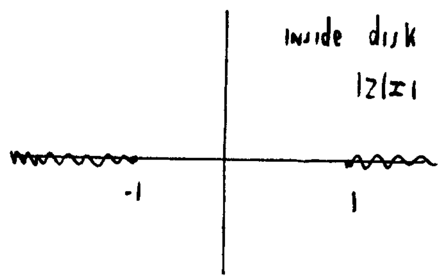
$(z^2-1)^{1/2}$ ANALYTIC OUTSIDE DISK $|z|=1$



NOW TRY $\varphi_1 \in (0, 2\pi)$ $\quad \varphi_2 \in (-\pi, \pi)$

	φ_1	φ_2	$e^{i(\varphi_1+\varphi_2)/2}$	
B	π	0	$e^{i\pi/2}$	CONTINUOUS
B'	π	0	$e^{i\pi/2}$	
C	0	0	e^{i0}	DISCONTINUOUS
C'	2π	0	$e^{i\pi}$	
D	π	π	$e^{i\pi}$	DISCONTINUOUS
D'	π	$-\pi$	e^{i0}	

$(z^2-1)^{1/2}$ ANALYTIC inside disk $|z| \le 1$



INVERSION OF TRIGONOMETRIC FUNCTIONS

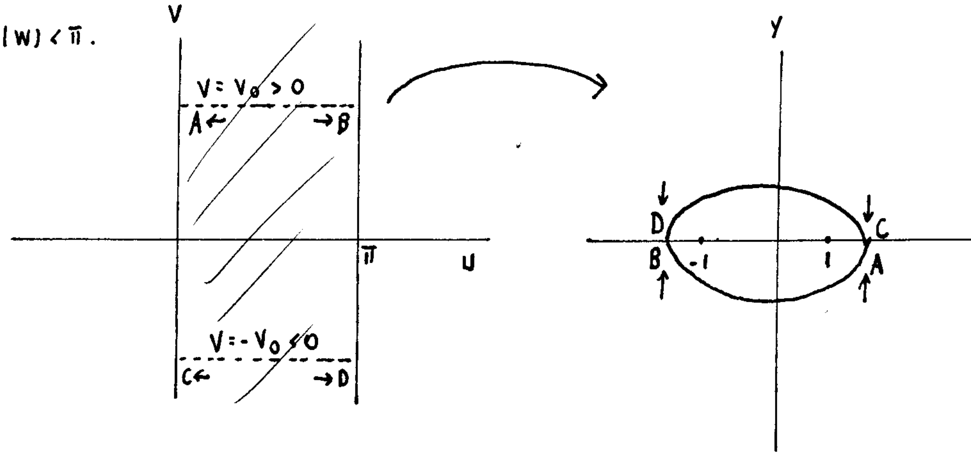
CONSIDER $Z = \cos W$ $\quad 0 < \mathrm{Re}(W) < \pi$.

$X = \cos u \cosh v$

$y = -\sin u \sinh v$

let v be fixed:

$$\frac{X^2}{\cosh^2 v} + \frac{y^2}{\sinh^2 v} = 1 \rightarrow \text{ellipse}$$



. NOW TAKE $v = v_0 > 0$ $\quad u \in (0,\pi)$:

$$\left.\begin{aligned} X &= \cosh v_0 \cos u \\ y &= -\sinh v_0 \sin u < 0 \end{aligned}\right\} \longrightarrow \text{get } 1/2 \text{ ellipse (lower } 1/2 \text{ plane)}$$

NOTE WHEN $u = 0 \rightarrow X = \cosh v_0 > 1$

$u = \pi \rightarrow X = -\cosh v_0 < -1$

. NOW TAKE $v = -v_0 < 0$ $\quad u \in (0,\pi)$

$$\left.\begin{aligned} X &= \cosh v_0 \cos u \\ y &= \sinh v_0 \sin u > 0 \end{aligned}\right\} \longrightarrow \text{get } 1/2 \text{ ellipse (upper } 1/2 \text{ plane)}$$

WHEN $u = 0 \longrightarrow X = \cosh v_0 > 1$

$u = \pi \longrightarrow X = -\cosh v_0 < -1$.

. IF $v_0 = 0 \longrightarrow X = \cos u \longrightarrow$ get segment on real axis between $-1 < X < 1$.

. TO SEE THAT EVERYTHING IS MAPPED CORRECTLY CONSIDER POINT A: $u = \varepsilon$, $v = v_0$ $\quad \varepsilon \to 0^+$.

$$X = \cosh v_0 \cos \varepsilon \longrightarrow \cosh v_0 > 1 \text{ AS } \varepsilon \to 0^+.$$

$$y = -\sinh v_0 \sin \varepsilon \rightarrow 0^- \text{ AS } \varepsilon \to 0^+$$

<u>NOTE</u>: POINTS ALONG THE REAL AXIS IN X, y PLANE ON $|X| > 1$ GET MAPPED TO 2 DIFFERENT POINTS IN W- PLANE.

THUS $Z = \cos W$ IS A 1-1 MAPPING OF $\underset{\uparrow \text{ FUNDAMENTAL STRIP}}{0 < \mathrm{Re}(W) < \pi}$ TO $\underset{\uparrow \text{ CUT Z- PLANE}}{Z \in \mathbb{C} \setminus \{(-\infty,-1),(1,\infty)\}}$

IN THIS CUT PLANE WE CAN UNIQUELY DEFINE AN INVERSE FUNCTION $W = \cos^{-1}(Z)$.

WE ALSO WANT $W = \pi/2 \longleftrightarrow Z = 0$

<u>ANALYSIS</u>

$$Z = \cos W = \frac{e^{iW} + e^{-iW}}{2}$$

$$\longrightarrow e^{2iW} - 2Z e^{iW} + 1 = 0$$

$$\longrightarrow e^{iW} = \frac{2Z \pm 2(Z^2-1)^{1/2}}{2}$$

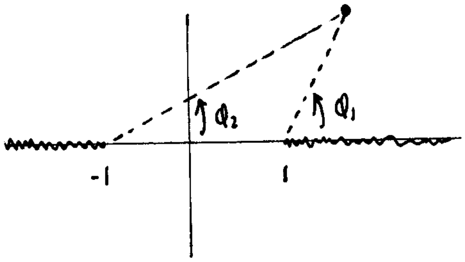
$(Z^2-1)^{1/2}$ IS MULTI-VALUED AND DIFFERENT VALUES DIFFER BY $\pm$ SIGN. HENCE TAKE + SIGN WITHOUT LOSS OF GENERALITY.

$$e^{iw} = z + (z^2-1)^{1/2}$$

$$w = -i \log (z + (z^2-1)^{1/2})$$

NEED TO CHOOSE A BRANCH OF $\log$ AND MAKE $(z^2-1)^{1/2}$ SINGLE-VALUED BY INTRODUCING BRANCH CUTS ALONG $|RE(z)| > 1$. NOTICE $z + (z^2-1)^{1/2} \neq 0$ FOR ANY z. (i.e. $z^2 = z^2-1 \to -1 \neq 0$).

NOW TAKE



$$(z^2-1)^{1/2} = (r_1 r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2}$$

$$-\pi < \varphi_2 < \pi \;,\quad 0 < \varphi_1 < 2\pi$$

this gives us branch cuts as shown (see earlier page)

NOW WHEN $z=0 \longrightarrow \varphi_1 = \pi,\ \varphi_2 = 0 \to (-1)^{1/2} = e^{i\pi/2} = i$.
$r_1 = r_2 = 1$

THUS $w = -i \log(i)$ WHEN $z=0$.

$$\log(i) = \ln 1 + i(\pi/2 + 2k\pi) \qquad k=0,\pm1,\ldots$$

$$w = \pi/2 + 2k\pi \qquad k = 0,\pm1,\ldots$$ Need $k=0$ to get $w = \pi/2$.

$\longrightarrow$ PRINCIPAL BRANCH OF $\log$

<u>DEFINITION</u>: THE PRINCIPAL VALUE OF THE FUNCTION $\cos^{-1} z$ IS DEFINED BY

$$w = -i \operatorname{Log}(z + (z^2-1)^{1/2})$$

Log : principal branch
$(z^2-1)^{1/2}$ defined above.

FOR $z \in \mathbb{C} \setminus \{(-\infty,-1),(1,\infty)\}$ IT TAKES VALUES IN THE FUNDAMENTAL STRIP $0 < RE(w) < \pi$.

<u>REMARK</u> IF WE WANT TO FIND <u>ALL</u> POSSIBLE VALUES w FOR $\cos w = 2i$ WE USE $w = -i\log(z + (z^2-1)^{1/2})$ WITH $z = 2i$ AND TAKE ALL BRANCHES OF $\log$ AND THE TWO VALUES OF $(z^2-1)^{1/2}$

LET $z = 2i$. $(z^2-1)^{1/2} = (-5)^{1/2} = \{i\sqrt5, -i\sqrt5\}$.

<u>+ sign</u>

$$w_+ = -i\log[2i + i\sqrt5] = -i\left[\ln(\sqrt5+2) + i\left(\frac{\pi}{2} + 2k\pi\right)\right] \qquad k=0,\pm1,\pm2,\ldots$$

$$(1) \qquad w_+ = \frac{\pi}{2} + 2k\pi - i\ln(\sqrt5+2) \qquad k=0,\pm1,\pm2,\ldots$$

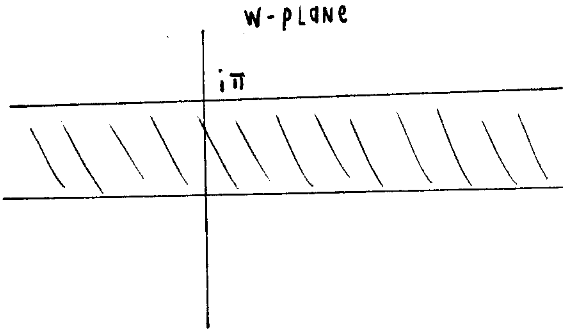
<u>- sign</u>

$$w_- = -i\log(2i - i\sqrt5) = -i\left[\ln(\sqrt5-2) + i\left(\frac{-\pi}{2} + 2k\pi\right)\right] \qquad k=0,\pm1,\pm2,\ldots$$

$$(2) \qquad w_- = \frac{-\pi}{2} + 2k\pi - i\ln(\sqrt5-2) \qquad k=0,\pm1,\pm2,\ldots$$

NOTE: IF w_0 IS A ROOT OF $\cos w_0 = 2i$ THEN ALSO $\cos(-w_0) = 2i$. THUS IS HOW (1) AND (2) ARE RELATED SINCE $\ln(\sqrt5-2) = \ln\left((\sqrt5-2)(\sqrt5+2)/(\sqrt5+2)\right) = -\ln(\sqrt5+2)$.

NOW CONSIDER $z = \cosh w$. DEFINE THE INVERSE FUNCTION UNIQUELY ONTO THE FUNDAMENTAL STRIP $0 < \mathrm{Im}(w) < \pi$.



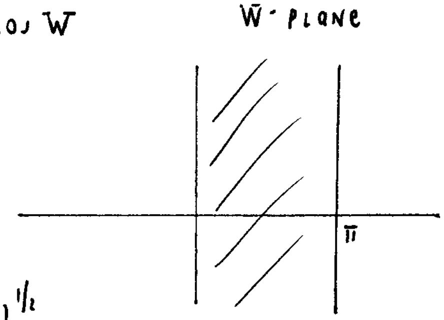
NOW LET $w = i\bar{W}$ $\quad \bar{W} = -iw$ ↗ rotation by $-\pi/2$ in w-plane

$\cosh(i\bar{W}) = \cos\bar{W}$

$z = \cos(\bar{W})$

$\bar{W} = -i\log\left(z + (z^2-1)^{1/2}\right)$

uniquely defined inverse with branches for $(z^2-1)^{1/2}$ as discussed on previous page.



this maps to cut z-plane on previous page

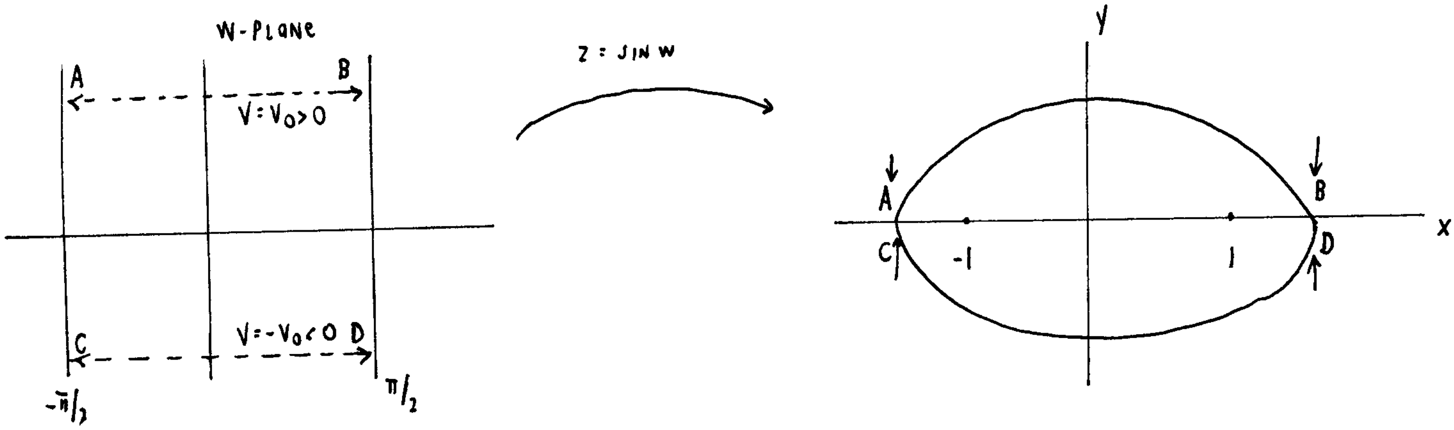
but $\bar{W} = -iw$.

$\longrightarrow$ THE PRINCIPAL VALUE FOR THE INVERSE OF $z = \cosh w$

IS $\quad w = \mathrm{Log}\left(z + (z^2-1)^{1/2}\right)$ $\quad$ principal value for Log.

this takes values in $0 < \mathrm{Im}(w) < \pi$.

NOW CONSTRUCT THE INVERSE FUNCTION FOR $z = \sin w$.



$x + iy = \sin u \cosh v + i\cos u \sinh v$

$$\left.\begin{array}{l} x = \sin u \cosh v \\ y = \cos u \sinh v \end{array}\right\} \rightarrow \frac{x^2}{\cosh^2 v} + \frac{y^2}{\sinh^2 v} = 1 \quad \text{FOR } v \text{ FIXED}$$

ellipse

- let $v = v_0 > 0$ (line AB) $\quad u \in (-\pi/2, \pi/2) \longrightarrow y > 0$ AND (get ½ ellipse) $\quad x = \cosh v_0 > 1$ WHEN $u = \pi/2 \to B$; $\quad x = -\cosh v_0 < -1$ WHEN $u = -\pi/2 \to A$
- let $v = -v_0 < 0$ (line CD) $\quad u \in (-\pi/2, \pi/2) \longrightarrow y < 0$ AND (get ½ ellipse) $\quad x = \cosh v_0 > 1$ WHEN $u = \pi/2 \to D$; $\quad x = -\cosh v_0 < -1$ WHEN $u = -\pi/2 \to C$
- NOTICE $v_0 = 0 \longrightarrow y = 0 \quad x = \sin u$ WITH $u \in (-\pi/2, \pi/2) \longrightarrow |x| < 1$.

<u>REMARK</u>: POINTS ON REAL AXIS IN $|x| > 1$ COME FROM 2 DIFFERENT POINTS IN THE W-PLANE $\longrightarrow$ NEED BRANCH CUTS ALONG THESE SEGMENTS.

(T4)

WE ALSO WANT $z=0 \longleftrightarrow w=0$. SO $z=\sin w$ IS 1-1 IN $|\mathrm{Re}(w)|<\pi/2$

AND $z \in \mathbb{C}\setminus\{(-\infty,-1),(1,\infty)\}$.

NOW CONSTRUCT INVERSE ANALYTICALLY. $z = \dfrac{e^{iw}-e^{-iw}}{2i}$. $e^{2iw} - 2ize^{iw} - 1 = 0$

$$e^{iw} = iz \pm (-z^2+1)^{1/2} \quad \text{TAKE + SIGN WITHOUT LOSS OF generality.}$$

$$e^{iw} = iz + (1-z^2)^{1/2}.$$

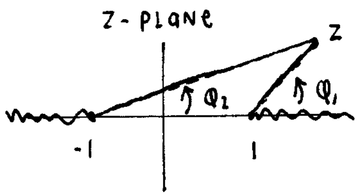
$$(*) \quad w = -i\log\left(iz + (1-z^2)^{1/2}\right).$$

NOW $1-z^2 = e^{i\pi}(z+1)(z-1)$ $\quad (1-z^2)^{1/2} = i(z+1)^{1/2}(z-1)^{1/2}$.

$$(1-z^2)^{1/2} = i(r_1 r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2} \qquad r_1 = |z+1| \qquad r_2 = |z-1|.$$

WANT BRANCH CUTS ALONG $|\mathrm{Re}(z)|>1$ BUT ALSO WANT $(1)^{1/2} = 1$ SINCE WE WILL TAKE Log FOR log AND WHEN $z=0$ WE NEED $(*)$ TO GIVE US $w=0$.

NOW TAKE $\varphi_2 \in (-\pi,\pi)$ AND $\varphi_1 \in (2\pi, 4\pi)$



WHEN $z=0$ $\quad \varphi_1 = 3\pi$, $\varphi_2 = 0$ $\quad r_1 = r_2 = 1$

$$(1)^{1/2} = ie^{3\pi i/2} = 1.$$

there are branch cuts as shown.

IN SUMMARY THE PRINCIPAL BRANCH OF $\sin^{-1} z$ IS DEFINED BY

$$w = -i\,\mathrm{Log}\left(iz + (1-z^2)^{1/2}\right) \quad \text{WHERE THE VALUES FOR } (1-z^2)^{1/2} \text{ IS}$$

$$(1-z^2)^{1/2} = i(r_1r_2)^{1/2} e^{i(\varphi_1+\varphi_2)/2} \qquad \varphi_1 \in (2\pi,$$

$$\varphi_2 \in (-\pi,$$

<u>EXAMPLE</u> FIND ALL POSSIBLE VALUES OF $\sin w = i$

$$w = -i\log\left(iz + (1-z^2)^{1/2}\right) \quad \text{WITH } z = i \qquad 2^{1/2} = \pm\sqrt{2}$$

<u>+ sign</u>

$$w_+ = -i\log\left(-1+\sqrt{2}\right) = -i\left[\ln(\sqrt{2}-1) + 2k\pi i\right]$$

$$① \quad w_+ = 2k\pi - i\ln(\sqrt{2}-1) \qquad k = 0, \pm1, \pm2, \ldots$$

<u>- sign</u>

$$w_- = -i\log\left(-1-\sqrt{2}\right) = -i\left[\ln(\sqrt{2}+1) + i(\pi + 2m\pi)\right] \qquad m = 0, \pm1, \pm2, \ldots$$

$$② \quad w_- = \pi + 2m\pi - i\ln(\sqrt{2}+1) \qquad m = 0, \pm1, \pm2, \ldots$$

<u>NOTE:</u> $\ln(\sqrt{2}+1) = -\ln(\sqrt{2}-1)$ SO THAT ① AND ② ARE RELATED BY $w \mapsto \pi - w$.

NOTE THAT IF w_0 IS A SOLUTION TO $\sin w_0 = z$ THEN SO IS $\pi - w_0$.

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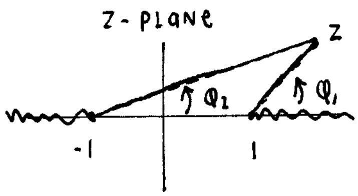

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① $\quad w_+ = 2k\pi - i\ln(\sqrt{2}-1)$ $\quad k = 0, \pm1, \pm2, \ldots$

- sign $\quad w_- = -i\log(-1-\sqrt{2}) = -i[\ln(\sqrt{2}+1) + i(\pi + 2m\pi)]$ $\quad m = 0, \pm1, \pm2, \ldots$

② $\quad w_- = \pi + 2m\pi - i\ln(\sqrt{2}+1)$ $\quad m = 0, \pm1, \pm2, \ldots$

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