# PROOF OF THEOREM 2 IN NOTES

**REMARK** IF $g(x)$ IS CONTINUOUS AND $g'(x)$ EXISTS THEN THE MVT (MEAN-VALUE THEOREM) SAYS

$$g(x_0+\Delta x) - g(x_0) = \Delta x \; g'(x^*) \qquad x^* \text{ BETWEEN } x_0,\ x_0+\Delta x$$

**PROOF OF THEOREM 2** (page (A7) OF NOTES)

WE CALCULATE $f'(z_0) = \lim\limits_{L\to 0} \dfrac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$

WITH $f = u + iv$.

NOW $u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0+\Delta y) = \Delta x \dfrac{\partial u}{\partial x}(x^*, y_0+\Delta y)$

BY THE MVT. NOW BY CONTINUITY OF $u_x, u_y$ AT $(x_0, y_0)$ WE HAVE

$$\frac{\partial u}{\partial x}(x^*, y_0+\Delta y) = \frac{\partial u}{\partial x}(x_0, y_0) + \varepsilon_1$$

WHERE $\varepsilon_1 \to 0$ AS $x^* \to x_0$ AND $\Delta y \to 0$ (i.e. AS $\Delta z \to 0$).

THUS $u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0+\Delta y) = \left(\dfrac{\partial u}{\partial x}(x_0, y_0) + \varepsilon_1\right)\Delta x$

SIMILARLY

$$u(x_0, y_0+\Delta y) - u(x_0, y_0) = \left(\frac{\partial u}{\partial y}(x_0, y_0) + \varepsilon_2\right)\Delta y$$

NOW

$$\begin{aligned} u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0) &= \Big(u(x_0+\Delta x, y_0+\Delta y) - u(x_0, y_0+\Delta y)\Big) + \Big(u(x_0, y_0+\Delta y) - u(x_0, y_0)\Big) \\ &= \left(\frac{\partial u}{\partial x}(x_0, y_0) + \varepsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y}(x_0, y_0) + \varepsilon_2\right)\Delta y \end{aligned}$$

NOW FOR CONVENIENCE LABEL $u_x^0 = \dfrac{\partial u}{\partial x}(x_0, y_0)$, $u_y^0 = \dfrac{\partial u}{\partial y}(x_0, y_0)$

SIMILARLY,

$$V(X_0+\Delta X,\ Y_0+\Delta y)-V(X_0,Y_0)=\left(V_X^0+\varepsilon_3\right)\Delta X+\left(V_y^0+\varepsilon_4\right)\Delta y.$$

THUS

$$(*)\quad \frac{F(z_0+\Delta z)-F(z_0)}{\Delta z}=\frac{(U_X^0+\varepsilon_1)\Delta X+(U_y^0+\varepsilon_2)\Delta y+i\left[(V_X^0+\varepsilon_3)\Delta X+(V_y^0+\varepsilon_4)\Delta y\right]}{\Delta X+i\Delta y}.$$

NOW ASSUME CR HOLD AT $(X_0, Y_0)$, i.e.

$$U_X^0=V_y^0$$
$$U_y^0=-V_X^0$$

ELIMINATE Y DERIVATIVES IN $(*)$

$$(+)\quad \left\{\begin{aligned}\frac{F(z_0+\Delta z)-F(z_0)}{\Delta z}&=\frac{(U_X^0+\varepsilon_1)\Delta X+(-V_X^0+\varepsilon_2)\Delta y+i\left[(V_X^0+\varepsilon_3)\Delta X+(U_X^0+\varepsilon_4)\Delta y\right]}{\Delta X+i\Delta y}\\ &=\frac{U_X^0(\Delta X+i\Delta y)+i\,V_X^0(\Delta X+i\Delta y)}{\Delta X+i\Delta y}+\frac{\lambda}{\Delta X+i\Delta y}\end{aligned}\right.$$

WHERE

$$\lambda\equiv\frac{\Delta X(\varepsilon_1+i\varepsilon_3)+\Delta y(\varepsilon_2+i\varepsilon_4)}{\Delta X+i\Delta y}$$

NOW BY $\Delta$-INEQUALITY,

$$\left|\frac{\lambda}{\Delta X+i\Delta y}\right|\leq\left|\frac{\Delta X}{\Delta X+i\Delta y}\right|\left|\varepsilon_1+i\varepsilon_3\right|+\left|\frac{\Delta y}{\Delta X+i\Delta y}\right|\left|\varepsilon_2+i\varepsilon_4\right|$$

$$\leq\left|\varepsilon_1+i\varepsilon_3\right|+\left|\varepsilon_2+i\varepsilon_4\right|\longrightarrow 0 \text{ AS } \Delta z\to 0$$

THUS FROM $(+)$ $\longrightarrow$ $L=\lim\limits_{\Delta z\to 0}\dfrac{F(z_0+\Delta z)-F(z_0)}{\Delta z}=U_X^0+iV_X^0$

$\to$ F is differentiable at $z_0$.