COMPUTATION OF AN INTEGRAL

LET $$I = \int_0^{2\pi} \cos^n \varphi \, d\varphi .$$

CLEARLY $I = 0$ IF $n = $ ODD. SO SUPPOSE $n = 0, 2, 4, \ldots$

FACT 1

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$$

1

1 2 1

1 3 3 1

1 4 6 4 1

SO $$(a+b)^n = \sum_{m=0}^{n} \binom{n}{m} a^{n-m} b^m .$$

$$\binom{n}{m} = \frac{n!}{(n-m)!\, m!} .$$

FACT 2

$$\int_0^{2\pi} e^{iK\varphi} \, d\varphi = \int_0^{2\pi} (\cos(K\varphi) + i \sin(K\varphi)) \, d\varphi = \begin{cases} 0 & \text{IF } K \neq 0 \\ 2\pi & \text{IF } K = 0. \end{cases}$$

FACT 3

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$

DEFINE $n = 2N$ $\quad N = 0, 1, 2, \ldots$

$$I_N = \int_0^{2\pi} (\cos\varphi)^{2N} \, d\varphi = \int_0^{2\pi} \frac{(e^{i\varphi} + e^{-i\varphi})^{2N}}{2^{2N}} \, d\varphi$$

$$I_N = \frac{1}{2^{2N}} \int_0^{2\pi} \sum_{m=0}^{2N} \binom{2N}{m} (e^{i\varphi})^{2N-m} (e^{-i\varphi})^m$$

$$I_N = \frac{1}{2^{2N}} \sum_{m=0}^{2N} \binom{2N}{N} \int_0^{2\pi} e^{i[2N-2m]} \, d\varphi .$$

WE ONLY get a NON-ZERO integral IF $m = N$.

THUS $$I_N = \frac{1}{2^{2N}} \binom{2N}{N} 2\pi = \frac{2\pi}{2^{2N}} \frac{(2N)!}{(N!)^2} .$$

FOR $I_2 = \int_0^{2\pi} \cos^4 \varphi \, d\varphi$ THEN

$$I_2 = \frac{1}{2^4} 2\pi \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \frac{2\pi}{2^4} \frac{4!}{(4-2)!\,2!} = \frac{3\pi}{4}.$$

FOR $I_1 = \int_0^{2\pi} \cos^2 \varphi \, d\varphi$ THEN $I_1 = \frac{2\pi}{2^2} \frac{2!}{1!} = \pi.$

REMARK $\cos^2\varphi = \frac{1+\cos(2\varphi)}{2}$ $\cos^4\varphi = \left(\frac{1+\cos(2\varphi)}{2}\right)^2 \ldots$

REMARKS ON MULTI-VALUEDNESS ETC.

$z = re^{i\varphi}$ $\varphi_0 = \operatorname{Arg} z$ WITH φ_0 IN $(-\pi, \pi]$.

THEN $\arg z = \varphi = \varphi_0 + 2k\pi$ $k = 0, \pm 1, \pm 2, \ldots$

(i) PROVE $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

set of all values of $\{\arg(z_1 z_2)\}$ AND $\{\arg(z_1)\} + \{\arg z_2\}$ ARE THE SAME

$\arg(z_1 z_2) = \varphi_1 + \varphi_2 + 2k\pi$ φ_1, φ_2 IN $(-\pi, \pi]$.

$\arg(z_1) = \varphi_1 + 2m\pi$

$\arg(z_2) = \varphi_2 + 2n\pi$

$$\{\varphi_1 + \varphi_2 + 2k\pi \mid k = 0, \pm 1, \ldots\} = \{\varphi_1 + 2m\pi \mid m = 0, \pm 1, \ldots\} + \{\varphi_2 + 2n\pi \mid n = 0, \pm 1, \ldots\}$$

i.e. is $k = m + n$

given k can we always find m, n? Yes.

EXAMPLE 1 CALCULATE $\cos(\pi/5)$.

LET $Z = \cos(\pi/5) + i\sin(\pi/5) = e^{\pi i/5}$

THEN $Z^5 = -1$

$$\cos(\pi/5) = \frac{e^{i\pi/5} + e^{-i\pi/5}}{2} = \frac{Z + 1/Z}{2}$$

THIS HAS A ROOT $Z = -1$ SO

$$(Z+1)(Z^4 - Z^3 + Z^2 - Z + 1) = (Z^5 + 1) = 0.$$

SO SINCE $Z \neq -1$, THEN

$$Z^4 - Z^3 + Z^2 - Z + 1 = 0$$

$$Z^2 - Z + 1 - \frac{1}{Z} + \frac{1}{Z^2} = 0.$$

SO $$\left(Z^2 + \frac{1}{Z^2}\right) - \left(Z + \frac{1}{Z}\right) + 1 = 0.$$

BUT $Z + 1/Z = 2X \qquad X = \cos(\pi/5)$.

NOW $Z^2 + 1/Z^2 = (Z + 1/Z)^2 - 2 = 4X^2 - 2$.

THUS $$4X^2 - 2 - 2X + 1 = 0.$$

$$4X^2 - 2X - 1 = 0$$

SO $$X = \frac{2 \pm \sqrt{4 + 16}}{2(4)}$$

$$X = \frac{2 \pm \sqrt{20}}{4 \cdot 2}$$

NEED + ROOT $\longrightarrow$ $X = \cos(\pi/5) = (1 + \sqrt{5})/4$

EXAMPLE IN A SIMILAR WAY WE CAN SOLVE

$$\sin X + \sin(2X) + \sin(3X) = \cos(X) + \cos(2X) + \cos(3X).$$

REMARK IF X IS A SOLUTION, THEN SO IS $X + 2K\pi$, $K = 0, \pm 1, \ldots$

WE PUT $Z = \cos(X) + i\sin(X)$.

THEN

$$\sin X = \frac{Z - 1/Z}{2i} \qquad \cos X = \frac{Z + 1/Z}{2}$$

$$\sin(2X) = \frac{Z^2 - 1/Z^2}{2i} \qquad \cos(2X) = \frac{Z^2 + 1/Z^2}{2}$$

$$\sin(3X) = \frac{Z^3 - 1/Z^3}{2i} \qquad \cos(3X) = \frac{Z^3 + 1/Z^3}{2}.$$

THUS,

$$i\left(Z - 1/Z\right) + i\left(Z^2 - 1/Z^2\right) + i\left(Z^3 - 1/Z^3\right)$$
$$= \left(Z + 1/Z\right) + \left(Z^2 + 1/Z^2\right) + \left(Z^3 + 1/Z^3\right)$$

MULTIPLY BY Z^3 TO OBTAIN THAT Z IS A ROOT OF $P(Z) = 0$

WHERE

$$P(Z) \equiv Z^6 + Z^5 + Z^4 + Z^2 + Z + 1$$
$$- i\left[Z^6 + Z^5 + Z^4 - Z^2 - Z - 1\right]$$

THUS

$$P(Z) = (1-i)\left(Z^6 + Z^5 + Z^4 + Z^2 + Z + 1\right) = 0.$$

NOW THIS IMPLIES $Z^6 + Z^5 + Z^4 + Z^2 + Z + 1 = 0.$

$$p(z) = 0 \quad \longrightarrow \quad (z^6 + z^5 + z^4) + (z^2 + z + 1) = 0.$$

SO $$z^4(z^2 + z + 1) + (z^2 + z + 1) = 0.$$

SO $$(z^2 + z + 1)(1 + z^4) = 0.$$

WE CONCLUDE THAT EITHER

$$z^4 = -1 \qquad (i)$$

OR $$z^2 + z + 1 = 0. \qquad (ii)$$

<u>FOR POSSIBILITY (i)</u>

WE HAVE $z^4 + 1 = 0$.

DIVIDE BY z^2: $$z^2 + 1/z^2 = 2\cos(2x) = 0.$$

SO $$\cos(2x) = 0 \longrightarrow 2x = \frac{\pi}{2} + K\pi \qquad K = 0, \pm 1, \pm 2, \ldots$$

THUS $$x_K = \frac{\pi}{4} + \frac{K\pi}{2}, \qquad K = 0, \pm 1, \pm 2, \ldots \qquad (*)$$

<u>NOW POSSIBILITY (ii)</u> $z^2 + z + 1 = 0$.

DIVIDE BY z: $$z + 1/z + 1 = 0$$

BUT $$z + 1/z = 2\cos x \longrightarrow 2\cos x = -1$$

$$\text{SO} \quad \cos(x) = -1/2 \longrightarrow x = 2\pi/3,\ 4\pi/3, \ldots$$

SO $$x_K = \begin{cases} 2\pi/3 + 2K\pi & K = 0, \pm 1, \ldots \\ 4\pi/3 + 2K\pi & K = 0, \pm 1, \pm 2, \ldots \end{cases} \qquad (**)$$

<u>SUMMARY</u>:
THE ROOTS ARE EITHER (*) OR (**)

BOUNDS FIND AN UPPER\LOWER BOUND FOR

$|z-3|$ IF $|z-i| \leq 1$.

(i) $|z-3| = |z-i+(-3+i)| \leq |z-i| + |3-i| \leq 1 + \sqrt{10}$ TRIANGLE INEQUALITY.

SO $|z-3| \leq 1 + \sqrt{10}$

(ii) $|z-3| = |z-i+(-3+i)| \geq \big| |z-i| - |3-i| \big| = \sqrt{10} - |z-i|$ (Reverse Δ-inequality)

SINCE $|z-i| \leq 1$, THEN $|z-3| \geq \sqrt{10} - 1$. FOR $|z-i| \leq 1$

QUESTION (i) FIND AN UPPER BOUND FOR $\dfrac{1}{|1-z|}$ IF $|z-i| \leq 1/2$.

(ii) FIND AN UPPER BOUND FOR $|z^2+z+1|$ IF $|z| \leq 1/2$.

SOLUTION 1 NEED REVERSE Δ-INEQUALITY.

$$|1-z| = |z-1| = |(z-i)+(-1+i)| \geq \big| |z-i| - |-1+i| \big|$$

SO $|1-z| \geq \big| |z-i| - \sqrt{2} \big| = \sqrt{2} - |z-i|$ SINCE $|z-i| \leq 1/2$.

THEN SINCE $|z-i| \leq 1/2 \rightarrow -|z-i| \geq -1/2$

AND SO $|1-z| \geq \sqrt{2} - 1/2$.

THUS $\dfrac{1}{|1-z|} \leq \dfrac{1}{(\sqrt{2}-1/2)}$.

SOLUTION 2 NEED REVERSE Δ INEQUALITY $|z_1 + \dots + z_N| \geq \big| |z_1 + \dots z_{N-1}| - |z_N| \big|$

SO $|z^2+z+1| \geq \big| |z+1| - |z^2| \big| = \big| |z+1| - |z|^2 \big|$

BUT IF $|z| < 1/2$ THEN $|z+1| \geq 1/2 \rightarrow |z^2+z+1| \geq |z+1| - |z|^2$

AND $|z^2+z+1| \geq \big| |z| - 1 \big| - |z|^2 = 1 - |z| - |z|^2 \geq 1 - 1/2 - 1/4 = 1/4$.

THEREFORE $\quad |z^2+z+1| \geq 1/4 \quad$ IF $\quad |z| \leq 1/2.$

$$\longrightarrow \quad \frac{1}{|z^2+z+1|} \leq 4 \quad \text{IF} \quad |z| \leq 1/2.$$