Sharp estimates for fully bubbling solutions of a SU(3) Toda system

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Abstract

In this paper, we obtain sharp estimates of fully bubbling solutions of SU(3) Toda system in a compact Riemann surface. In geometry, the SU(n+1) Toda system is related to holomorphic curves, harmonic maps or harmonic sequences of the Riemann surface to CP^n. In order to compute the Leray-Schurder degree for the Toda system, we have to obtain accurate approximations of the bubbling solutions. Our main goals in this paper are (i) to obtain a sharp convergence rate, (ii) to completely determine the locations, and (iii) to derive the $\partial^2_i$ condition, an unexpected and important geometric constraint.

1 Introduction

Let $(M,g)$ be a compact Riemann surface. Consider the following system of equations:

\[
\begin{aligned}
\Delta u_1 + 2e^{u_1} - e^{u_2} &= 4\pi \sum_{j=1}^{m} \gamma_{j1} \delta_q^j \\
\Delta u_2 + 2e^{u_2} - e^{u_1} &= 4\pi \sum_{j=1}^{m} \gamma_{j2} \delta_q^j,
\end{aligned}
\]

where $\Delta = \Delta_g$ stands for the Laplace-Beltrami operator, $\gamma_{jk}$ are nonnegative integers, $q_j$ are distinct points in $M$ and $\delta_q^j$ are the Dirac measure at $q_j$. The system (1) is known as the SU(n + 1) Toda system when $n = 2$. This system of equations arises from many different research areas in geometry and physics. In physics, it is related to the relativistic version of non-abelian Chern-Simons

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models, see [9], [28], [36], [37] and references therein. In geometry, the SU(n + 1)
Toda system is closely related to holomorphic curves (or harmonic sequence)
of M into CP^n, see [3], [7], [11], [12]. When M = S^2, it was proved that
the solution space of the SU(n + 1) Toda system is identical to the space of
holomorphic curves of S^2 into CP^n. In particular, q_j are the ramified points
of the corresponding curve and γ_2 represents the total ramified index at q_j,
j = 1, 2, . . . , m. See [22]. However, when M is not S^2, the identity of the solution
space of PDE and holomorphic curves might not hold in general. Therefore it is
an interesting issue to clarify their relationship for Riemann surfaces with higher
genus. This is our initial motivation to study the Toda system in a compact
Riemann surface.

Integrating (1), we have
\begin{align}
\rho_1 & \doteq \int_M e^{u_1} = \frac{8\pi m_1 + 4\pi m_2}{3}, \\
\rho_2 & \doteq \int_M e^{u_2} = \frac{4\pi m_1 + 8\pi m_2}{3},
\end{align}
(2)
where m_1 = \sum_{j=1}^m \gamma_j and m_2 = \sum_{j=1}^m \gamma_j. Let |M| be the area of M as usual. By
introducing the Green function:
\begin{align}
\begin{cases}
\Delta G(x, p) = -\delta_p + \frac{1}{|M|} \\
\int_M G(x, p)dx = 0,
\end{cases}
\end{align}
(3)
and rewriting \(v_i\) by
\[ u_i = v_i + 4\pi \sum_{j=1}^m \gamma_j G(x, q_j), \quad i = 1, 2, \]
we have \(u_i\) satisfies the following system of equations:
\begin{align}
\begin{cases}
\Delta u_1 + 2\rho_1 \left( \frac{h_1 e^{u_1}}{h_1 e^{u_2}} - \frac{1}{|M|} \right) - \rho_2 \left( \frac{h_2 e^{u_2}}{h_1 e^{u_2}} - \frac{1}{|M|} \right) = 0, \\
\Delta u_2 + 2\rho_2 \left( \frac{h_2 e^{u_2}}{h_2 e^{u_1}} - \frac{1}{|M|} \right) - \rho_1 \left( \frac{h_1 e^{u_1}}{h_2 e^{u_1}} - \frac{1}{|M|} \right) = 0,
\end{cases}
\end{align}
(4)
where
\[ h_i(x) = \exp(-\sum_{j=1}^m 4\pi \gamma_j G(x, q_j)), \quad i = 1, 2. \]

We see that \(h_i(x) > 0\) in \(M \setminus \{q_1, \ldots, q_m\}\). It is easy to see that if \(u = (u_1, \ldots, u_n)\) is a solution of (4), \(u + c = (u_1 + c_1, \ldots, u_n + c_n)\) is still a solution.
Without loss of generality, we may assume each component \(u_i \in \bar{H}(M)\), where
\(\bar{H}(M) = \{u_i \in H(M) | \int_M u_i = 0\}\). Obviously, the equation (4) is the Euler-
Lagrange equation of the nonlinear functional \(\Phi_p\):
\[ \Phi_p(u) = \frac{1}{2} \int_M \sum_{i,j=1}^2 a_{ij} \nabla u_i \cdot \nabla u_j - \sum_{i=1}^2 \rho_i \log \int_M h_i e^{u_i}, \]
where \((a^{ij})\) is the inverse matrix of \(
abla^2 a + \rho \frac{h e^u}{\int h e^u} - \frac{1}{|M|} = 0 \quad \text{in } M. \) (5)

Equation (5) arises also in geometry and physics. In conformal geometry, it is related to the problem of prescribing Gaussian curvature with smooth metrics or metrics with conic singularity. For the past twenty years, the equation (5) has been extensively studied because it is closely related to the Abelian Chern-Simons theory. See [1, 2, 4, 5, 10], [15, 16], [19], [23], [27], [29], [30], [31], [33] and references therein.

For equation (4), the first main issue is to determine the set of critical parameters, i.e., those \( \rho = (\rho_1, \rho_2) \) such that the a-priori bounds for solutions of (4) fail. In [13], Jost-Lin-Wang proved the following a-priori estimates for equation (4). (We use \( \mathbb{N}^* \) to denote the set of all positive integers.)

**Theorem A.** Suppose \( h_i \) are positive smooth solutions, and \( \rho_i \notin 4\pi \mathbb{N}^*, \) \( i = 1, 2. \) Then there exists a positive constant \( c \) such that for any solution \( u \) of equation (4), there holds:

\[ |u_i(x)| \leq c \quad \forall \ x \in M, \quad i = 1, 2. \]

To prove Theorem A, the authors [13] considered a sequence of bubbling solutions \( u_k = (u_{1k}, u_{2k}) \) to the equation: for the simplicity, let \( |M| = 1 \) and \( u_k \) be a solution of

\[ \Delta u_k + \sum_{j=1}^2 a_{ij} \rho_j \left( \frac{h_{jk} e^{u_{jk}}}{\int_M h_{jk} e^{u_{jk}}} - 1 \right) = 0 \quad \text{in } M, \quad i = 1, 2 \] (6)

where \( \rho_j \to \rho_j, \ h_{jk} \to h_j \) in \( C^{2,\alpha}(M) \) for some \( \alpha > 0 \) as \( k \to +\infty, \) and \( S = \{p_1, \ldots, p_m\} \) is the blowup set of \( u_k. \) At each \( p_j, \) the local mass of \( u_k \) is assigned by the quantity \( \sigma: \)

\[ \sigma_i(p_j) = \lim_{r \to 0} \lim_{k \to +\infty} \frac{\int_{B_r(p_j)} \rho j h_{jk} e^{u_{jk}}}{\int_M h_{jk} e^{u_{jk}}}, \quad j = 1, 2, \ldots, m, \] (7)

where \( B_r(p_j) \) is the ball with center \( p_j \) and radius \( r. \) Jost-Lin-Wang [13] proved that for each \( p_j, \) there are only four possibilities for \( (\sigma_1, \sigma_2), \) i.e., \( (\sigma_1, \sigma_2) \) could be one of \( (4\pi, 0), (0, 4\pi), (8\pi, 4\pi), (4\pi, 8\pi) \) and \( (8\pi, 8\pi). \) It is easy to check any one of the couples could occur for global solutions in \( \mathbb{R}^2 \) with constant coefficients. Thus, it is a natural question to ask whether each of the couples could exist for a sequence of bubbling solutions in a compact Riemann surface \( M. \)

Obviously, \((8\pi, 8\pi)\) is the most interesting case among them. Suppose a sequence of bubbling solution \( u_k \) has the local masses \((8\pi, 8\pi)\) at \( p. \) The sequence of solutions \( u_k \) is called **fully bubbling** at \( P, \) if after a suitable scaling, the
sequence of solutions will converge to \((v_1, v_2)\) in \(C_{loc}^2(\mathbb{R}^2)\) satisfying:

\[
\begin{cases}
\Delta v_1 + 2e^{v_1} - e^{v_2} = 0 & \text{in } \mathbb{R}^2 \\
\Delta v_2 + 2e^{v_2} - e^{v_1} = 0 & \text{in } \mathbb{R}^2 \\
\int e^{v_1} < +\infty, \int e^{v_2} < +\infty
\end{cases}
\] (8)

More precisely, \(u_k\) is said to fully blow up at \(p\) if and only if \(u_k\) satisfies

\[
\left| u_{1k}(p_{k,1}) - \ln \int_M h_{1k} e^{u_{1k}} - u_{2k}(p_{k,2}) + \ln \int_M h_{2k} e^{u_{2k}} \right| \leq c
\] (9)

for some constant \(c\), where \(p_{k,i}\) are the local maxima of \(u_{ik}\) in \(B_r(p)\). We note that if \(v = (v_1, v_2)\) is an entire solution of (8), then

\[
\int_{\mathbb{R}^2} e^{v_1} = \int_{\mathbb{R}^2} e^{v_2} = 8\pi
\]

This quantization result was proved by Jost-Wang [14].

In [13], Jost-Lin-Wang proved that any full bubble is simple, i.e., there exists a sequence of entire solutions \(v = (v_1, v_2)\) to (5) such that

\[
|u_{jk}(\varepsilon_k y) + 2\log \varepsilon_k - v_j(y)| \leq c, \quad \text{for } |y| \leq \delta_0 \varepsilon_k^{-1},
\]

where \(c, \delta_0\) are positive constant and

\[
\varepsilon_k = \max \left\{ u_{1k}(p_{k,1}) - \ln \int_M h_{1k} e^{u_{1k}}, u_{2k}(p_{k,2}) - \ln \int_M h_{2k} e^{u_{2k}} \right\}
\]

In this paper, we want to study the global behavior for a sequence of bubbling solution \(u_k\) to equation (6) and obtain some important information for this sequence of bubbling solutions. Those information will have very important applications when we come to construct bubbling solutions, to count the Morse index for each bubbling solutions and finally to compute the topological degree for solutions of equation (4). Throughout the paper, we assume that

\[
(H) \ u_k \text{ fully blows up at each } p_j.
\]

Under the assumption (H), it is proved (see [21]) that \(\rho_{jk} \to 8m\pi\ (m \in \mathbb{N}^*)\), and \(u_{1k}(x) \to \sum_{i=1}^m 8\pi G(x, p_i)\) and \(u_{2k}(x) \to \sum_{i=1}^m 8\pi G(x, p_i)\) in \(C^2(\mathcal{M} \setminus \{p_1, \ldots, p_m\})\) as \(k \to +\infty\). Choose small \(r_0 > 0\) such that \(B(p_i, 2r_0) \cap B(p_j, 2r_0) = \emptyset\) for \(i \neq j\), denote by \(p_{k,j}\) the local maxima of \(u_{1k}\) in \(B(p_j, r_0)\), \(1 \leq j \leq m\), and let \(\varepsilon_{k,j}, \varepsilon_k\) be defined by (53) and (54). Our main result is the following sharper estimates of \(u_k\).

**Theorem 1.1.** Let \(u_k \in \hat{H}(M)\) be a sequence of blowing up solutions to (6), such that (H) holds. Then it holds that
(i) **Convergence rate:**

\[
\rho_{1k} - 8m \pi = \sum \left( C_{1k,j} [\Delta \ln h_{1k}(p_{k,j}) + 8m \pi - 2K(p_{k,j})] \varepsilon_{k,j}^2 \ln \varepsilon_{k,j} + O(\varepsilon_k^2) \right),
\]

\[
\rho_{2k} - 8m \pi = \sum \left( C_{2k,j} [\Delta \ln h_{2k}(p_{k,j}) + 8m \pi - 2K(p_{k,j})] \varepsilon_{k,j}^2 \ln \varepsilon_{k,j} + O(\varepsilon_k^2) \right),
\]

where \( C_{ik} \) (\( i = 1, 2 \)) are constants satisfying \( 0 < C_1 < C_{ik,j} < C_2 < \infty \) and \( K \) denotes the Gauss curvature of \( M \). Furthermore we have

(ii) **Locations of \( p_j \):**

\[
8\pi \nabla_x H(p_{k,j}, p_{k,j}) + 8\pi \sum_{i \neq j} \nabla_x G(p_{k,j}, p_{k,i}) + \nabla \ln h_{1k}(p_{k,j}) = O(\varepsilon_k)
\]

\[
8\pi \nabla_x H(p_{k,j}, p_{k,j}) + 8\pi \sum_{i \neq j} \nabla_x G(p_{k,j}, p_{k,i}) + \nabla \ln h_{2k}(p_{k,j}) = O(\varepsilon_k)
\]

where \( H(x, p) \) is the regular part of \( G(x, p) \).

(iii) **The \( \partial^2_x \) condition:**

\[
6\pi (\partial_{11} - \partial_{22}) [\ln h_{2k}(p_{k,j}) - \ln h_{1k}(p_{k,j})] + \frac{T_{1k}^j}{4} [\Delta \ln h_{1k}(p_{k,j}) + 8m \pi - 2K(p_{k,j})] + \frac{T_{2k}^j}{4} [\Delta \ln h_{2k}(p_{k,j}) + 8m \pi - 2K(p_{k,j})] = O(\varepsilon_k^2)
\]

\[
12\pi \partial_{12} [\ln h_{2k}(p_{k,j}) - \ln h_{1k}(p_{k,j})] + \frac{T_{1k}^j}{4} [\Delta \ln h_{1k}(p_{k,j}) + 8m \pi - 2K(p_{k,j})] + \frac{T_{2k}^j}{4} [\Delta \ln h_{2k}(p_{k,j}) + 8m \pi - 2K(p_{k,j})] = O(\varepsilon_k^2),
\]

where \( T_{1k}^j, T_{2k}^j, T_{1k,1}^j, T_{2k,2}^j \) and \( T_{1k,2}^j \) are four constants defined in Proposition 7.1.

We note that for the mean field equation (5), an analogue theorem was proved by Chen and the first author [4]. However, this type of theorems is much harder for Toda system than for the mean field equation. In the case of scalar mean field equation, the local Pohozaev identity is a very powerful tool in the bubbling analysis since the number of Pohozaev identities equals to the number of free parameters (both are three) for the Liouville equation. For Toda system, the local Pohozaev identity only gives three equations, but there are eight free parameters in the solutions space of Toda system. See (30) and (31)
in section 2. We remark here that (10)-(15) are 8 scalar conditions. Thus, the Pochozaev identity is much less powerful for equation (6). The key technical part we use for Toda system is the non-degeneracy of the entire solutions of the SU(n) Toda system. This has been proved recently by Wei-Zhao-Zhou [35] for \( n = 3 \), and by Lin-Wei-Ye [22] for general \( n \).

The conclusion of Theorem 1.1 is surprising when comparing with other type of Liouville system. Suppose \( u_k = (u_{1k}, \ldots, u_{nk}) \) is a sequence of fully blowing up solutions to the following system:

\[
\Delta u_{ik} + \frac{\sum_{j=1}^{n} a_{ij} \rho_{jk} \left( \frac{h_{ij} e^{u_{jk}}}{h_{ij} e^{u_{jk}}} - \frac{1}{|M|} \right)}{= 0 \quad \text{in } M,}
\]

for \( 1 \leq i \leq n \), where \( h_j \) are positive smooth functions in \( M \), and the matrix \( A = (a_{ij}) \) is a symmetric, irreducible, nonnegative matrix and \( \det A \neq 0 \). This system of equations has been studied by Chanillo-Kiessling [6], Chipot-Shafir-Wolansky [8] and recently by Lin-Zhang [24], [25] and [26]. In [26], Lin-Zhang proved sharper estimates for \( u_k \). Suppose \( u_k \) has only one blowup point \( p \), and \( \rho_{ik} \to \rho_i \). Then they proved:

(i) location of the blow-up point \( p \):

\[
\sum_{i=1}^{n} \rho_i \nabla (\log h_i(x) + 2\pi H(x, p)) |_{x=p} = 0; \quad (17)
\]

(ii) the convergence rate:

\[
8\pi \sum_{i=1}^{n} \rho_{ik} - \sum_{i,j=1}^{n} a_{ij} \rho_{ik} \rho_{jk} = \sum_{i=1}^{n} c_i (\Delta \log h_i(p_k) - 2K(p_k) + 8\pi) |\varepsilon_k|^2 \log |\varepsilon_k|, \quad (18)
\]

where \( c_i \) are positive constants.

From (17) and (18), we see the obvious difference between (16) and Toda system.

The conditions (10)-(15) of Theorem 1.1 already contains a lot of informations related to the geometry of the flat torus \( M \). To explain it, let us consider the simplest case of (1),

\[
\begin{aligned}
\Delta v_1 + 2e^{v_1} - e^{v_2} &= \rho \delta_{q_1}, \\
\Delta v_2 + 2e^{v_2} - e^{v_1} &= \rho \delta_{q_2}
\end{aligned}
\]

in \( M \),

\[
\begin{aligned}
\Delta v_1 + 2e^{v_1} - e^{v_2} &= \rho \delta_{q_1}, \\
\Delta v_2 + 2e^{v_2} - e^{v_1} &= \rho \delta_{q_2}
\end{aligned}
\]

where \( M \) is a flat torus, \( q_1 \neq q_2 \). By (2), we see \( \rho = 8\pi \) is the first \( \rho \) where the fully blowing up may occur. In this case, there is only one blowup point \( p \). If \( p \in \{q_1, q_2\} \), i.e., if blow up occurs at one of the vortex points, then we can use the quantization result in [22] and show

\[
\int_M e^{v_1} \, dx = \int_M e^{v_2} \, dx = 16\pi,
\]

6
a contradiction to \( \rho = 8\pi \). Therefore we conclude \( p \notin \{q_1, q_2\} \).

By applying Theorem 1.1, conditions (12) and (13) imply

\[
\nabla_x G(p, q_1) = \nabla_x G(p, q_2) = 0. \tag{20}
\]

Without loss of generality, one may assume \( p = 0 \) (by translation). Let \( G(x) \) denote the Green function with singularity at 0. Then (20) implies \( \nabla G(q_1) = \nabla G(q_2) = 0 \). Applying a result due to Lin-Wang [17], \( G(x) \) has either three critical points or five critical points. We claim:

*the Green function \( G \) has five critical points and \( q_1 = -q_2 \).*

Suppose \( G \) has three critical points only. Then these three critical points are all half periods. Hence both \( q_1 \) and \( q_2 \) are half periods. Let \( q_i = \frac{\omega_i}{2} \) and \( q_2 = \frac{\omega_j}{2} \) for some \( i \neq j \), where \( \omega_i, \omega_j \) are periods of \( M \). We can compute the second derivatives of \( G \) at \( \frac{\omega_i}{2} \) and \( \frac{\omega_j}{2} \) by using the Weierstrass \( \mathcal{P} \) function:

\[
2\pi G_{xx}(\frac{\omega_i}{2}) = \text{Re}(\mathcal{P}(\epsilon_i) + \eta_i)
\]

\[
2\pi G_{yy}(\frac{\omega_i}{2}) = -\text{Re}(\mathcal{P}(\epsilon_i) + \eta_i) + \frac{2\pi}{b}
\]

\[
2\pi G_{xy}(\frac{\omega_i}{2}) = -\text{Im}(\mathcal{P}(\epsilon_i) + \eta_i),
\]

where \( \eta_1 \) is one of quasi-period of \( \xi(z) = -\int \mathcal{P} \).

The \( \mathcal{P} \) condition implies

\[
G_{xx}(\frac{\omega_i}{2}) = G_{xx}(\frac{\omega_j}{2}), \quad G_{xy}(\frac{\omega_i}{2}) = G_{xy}(\frac{\omega_j}{2}) \text{ and } G_{yy}(\frac{\omega_i}{2}) = G_{yy}(\frac{\omega_j}{2}).
\]

By using the above formulas, we have \( \mathcal{P}(\frac{\omega_i}{2}) = \mathcal{P}(\frac{\omega_j}{2}) \), which implies \( q_i = q_2 \), a contradiction to our assumption. Therefore, the claim is proved, furthermore, by the same computation, we can prove that \( q_i \) are not half periods and \( q_1 = -q_2 \).

As we know, either the Louiville equation or the Toda system are closely related to holomorphic curves of \( M \) into \( \mathbb{CP}^n \), and are completely integrable systems. The integrability of Liouville equations allow us to define the developing map \( f \) defined in \( M \), and one of striking results in [17] is that if \( f \) is a developing map for a solution \( u \) of \( \Delta u + e^u = 8\pi\delta_0 \), then \( \lambda f \) is also a developing for another solution \( u_\lambda \), for any \( \lambda > 0 \). Thus, once a solution exists, there is a family of solutions \( u_\lambda \) and \( u_4 \) blows up at a non-half period critical point of \( G \) as \( \lambda \to +\infty \). Based on this phenomenon and the calculation above, we propose the following conjecture.

**Conjecture:** Suppose \( \rho = 8\pi \), \( M \) is a flat torus and \( q_i \in M \). Then equation (19) has one solution if and only if the Green function \( G(x) \) has five critical points.

We are also interested in studying equation (1) in a bounded domain \( \Omega \) in \( \mathbb{R}^2 \):

\[
\Delta u_{1k} + 2\rho_{1k} \frac{h_{1k} e^{x_{1k}}}{\int_0^1 h_{1k} e^{x_{1k}}} - \rho_{2k} \frac{h_{2k} e^{x_{2k}}}{\int_0^1 h_{2k} e^{x_{2k}}} = 0
\]

\[
\Delta u_{2k} + 2\rho_{2k} \frac{h_{2k} e^{x_{2k}}}{\int_0^1 h_{2k} e^{x_{2k}}} - \rho_{1k} \frac{h_{1k} e^{x_{1k}}}{\int_0^1 h_{1k} e^{x_{1k}}} = 0
\]

in \( \Omega \). \tag{21}
For the Dirichlet problem, it was proved that $u_k$ can not blow up on the boundary of $\Omega$, see [21] and related subjects in [18], [19], [20], [29], [32], [34]. Thus, we have the sharper estimates for $u_k$ similar to Theorem 1.1.

**Theorem 1.2.** Suppose $h_{ik}$ converges to positive functions $h_i$ in $C^2(\Omega)$, and $u_k$ is a sequence of blowup solutions to (21) with homogeneous Dirichlet boundary conditions and $S = \{p_1, \ldots, p_m\}$ is the blowup set. Assume $u_k$ fully blows at $p_j$, $j = 1, 2, \ldots, m$. Then it holds that

$$
\rho_{1k} - 8m\pi = \sum_{j=1}^{m} C_{1k,j} \Delta \ln h_{1k}(p_{k,j}) \varepsilon_{k,j}^2 \ln \varepsilon_{k,j} + O(\varepsilon_k^2),
$$

$$
\rho_{2k} - 8m\pi = \sum_{j=1}^{m} C_{2k,j} \Delta \ln h_{2k}(p_{k,j}) \varepsilon_{k,j}^2 \ln \varepsilon_{k,j} + O(\varepsilon_k^2),
$$

where $C_{ik,j}$ ($i = 1, 2$) are constants satisfying $0 < C_1 < C_{ik,j} < C_2 < \infty$. Furthermore, we have

$$
8\pi \nabla_x H(p_{k,j}, p_{k,j}) + 8\pi \sum_{i \neq j} \nabla_x G(p_{k,j}, p_{k,i}) + \nabla \ln h_{1k}(p_{k,j}) = O(\varepsilon_k),
$$

$$
8\pi \nabla_x H(p_{k,j}, p_{k,j}) + 8\pi \sum_{i \neq j} \nabla_x G(p_{k,j}, p_{k,i}) + \nabla \ln h_{2k}(p_{k,j}) = O(\varepsilon_k),
$$

and

$$
6\pi (\partial_{11} - \partial_{22}) \ln h_{2k}(p_{k,j}) - \ln h_{1k}(p_{k,j}) + \frac{T_{1k,1}^j}{4} \Delta \ln h_{1k}(p_{k,j})
$$

$$
+ \frac{T_{2k,1}^j}{4} \Delta \ln h_{2k}(p_{k,j}) = O(\varepsilon_k^2),
$$

$$
12\pi \partial_{12} \ln h_{2k}(p_{k,j}) - \ln h_{1k}(p_{k,j}) + \frac{T_{1k,2}^j}{4} \Delta \ln h_{1k}(p_{k,j})
$$

$$
+ \frac{T_{2k,2}^j}{4} \Delta \ln h_{2k}(p_{k,j}) = O(\varepsilon_k^2),
$$

where $T_{1k,1}^j$, $T_{2k,1}^j$, $T_{1k,2}^j$ and $T_{2k,2}^j$ are four constants defined in Proposition 7.1.

The paper is organized as follows. In section 2, we state and prove two important properties of entire solutions. For the simplicity of presentation, we will first prove Theorem 1.2. Hence, we will consider a sequence of blowing up solutions of (21) from section 3 to section 8. In section 3, we present two preliminary estimates of the blowing up solutions. Then we approximate the bubbles using the parameterized entire solutions and obtain inner estimates in Section 4. Here we need the non-degeneracy of entire solutions. Section 5 to Section 7 contain the computations of the bubbling rate and bubbling locations. Here we use the eight kernels to test the system locally. We combine the estimates to prove the main Theorem 1.2 in Section 8. In the final section, we give a brief account for the proof of Theorem 1.1. Finally the proof of Lemma 4.1 is presented at the appendix.
2 Properties of Entire Solutions

In this section, we collect several useful properties of the entire solution \((v_1, v_2)\) to (8). It is more convenient to consider the change of variables

\[
(w_1, w_2) = (2v_1 + v_2, v_1 + 2v_2)
\]

which satisfies

\[
\begin{align*}
\Delta w_1 + 3e^{\frac{w_1-w_2}{2}} &= 0 & \text{in } \mathbb{R}^2, \\
\Delta w_2 + 3e^{\frac{w_2-w_1}{2}} &= 0 & \text{in } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} e^{\frac{w_2-w_1}{2}} < \infty, \quad \int_{\mathbb{R}^2} e^{\frac{w_1-w_2}{2}} < \infty.
\end{align*}
\]

Explicit formula for \((w_1, w_2)\) is (see [14] and [35])

\[
w_1(y) = \ln \frac{256a_1^2a_2^2}{(a_1^2 + a_2^2|y|^2 + |y|^2 + c_1^2 + d_1^2)^3},
\]

\[
w_2(y) = \ln \frac{1024a_1^4a_2^4}{(a_1^4a_2^4 + a_1^2|2y| + a_2^2|y|^2 + 2by + b_1c_2 - d_1^2)^3}.
\]

Observe that \((w_1, w_2)\) depends on eight parameters \((a_1, a_2, b, c, d)\) \(\in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{C}^3\).

We first recall some results about the non-degeneracy of the solutions \((w_1, w_2)\) to the Toda system (29). The following theorem classifies the kernels of the linearized operator of (29) at \((w_1, w_2)\). Let \(\tau \in (0, 1)\) be any given number.

**Theorem 2.1.** ([35]) If \(\phi, \psi\) satisfy

\[
\begin{align*}
\Delta \phi + e^{\tau \phi} (2\phi - \psi) &= 0 & \text{in } \mathbb{R}^2, \\
\Delta \psi + e^{\tau \psi} (2\psi - \phi) &= 0, & \text{in } \mathbb{R}^2, \\
|\phi| &\leq C(1 + |y|^\tau), \quad |\psi| \leq C(1 + |y|^\tau),
\end{align*}
\]

then \(\phi, \psi\) belongs to the linear space

\[
\text{span} \left\{ \frac{\partial}{\partial a_1} w_1, \frac{\partial}{\partial a_2} w_1, \frac{\partial}{\partial b_1} w_1, \frac{\partial}{\partial b_2} w_1, \frac{\partial}{\partial c_1} w_1, \frac{\partial}{\partial c_2} w_1, \frac{\partial}{\partial d_1} w_1, \frac{\partial}{\partial d_2} w_1, \frac{\partial}{\partial a_1} w_2, \frac{\partial}{\partial a_2} w_2, \frac{\partial}{\partial b_1} w_2, \frac{\partial}{\partial b_2} w_2, \frac{\partial}{\partial c_1} w_2, \frac{\partial}{\partial c_2} w_2, \frac{\partial}{\partial d_1} w_2, \frac{\partial}{\partial d_2} w_2 \right\}.
\]

Our next lemma states that the eight parameters \((a_1, a_2, b, c, d)\) are uniquely determined by the initial values \((w_1(0), w_2(0), \partial_x w_1(0), \partial_x w_2(0), \partial_z w_1(0))\).

**Theorem 2.2.** Let \((\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)\) \(\in \mathbb{R} \times \mathbb{R} \times \mathbb{C}^3\) be given. Then there is a unique \((a_1, a_2, b, c, d)\) \(\in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{C}^3\) such that

\[
(w_1(0), w_2(0), \partial_z w_1(0), \partial_z w_2(0), \partial_{zz} w_1(0)) = (\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5).
\]
Proof. A direct computation shows that system (32) is equivalent to

\begin{align}
  a_1^2 + a_2^2 |b|^2 + |d|^2 &= A_1 a_1^\frac{4}{3} a_2^\frac{2}{3}, \\
  a_1^2 a_2^2 + a_1^2 |c|^2 + a_2^2 |bc - d|^2 &= A_2 a_1^\frac{4}{3} a_2^\frac{2}{3}, \\
  a_3^3 b + \bar{c} d &= A_3, \\
  a_3^3 c + a_3^2 |b|^2 - a_2^2 \overline{bd} &= A_4, \\
  \frac{(a_3^3 b + \bar{c} d)^2 - 2d(a_3^2 + a_3^2 |b|^2 + |d|^2)}{(a_3^2 + a_3^2 |b|^2 + |d|^2)^2} &= A_5,
\end{align}

where

\begin{align}
  A_1 &= e^{\frac{2\pi}{3}} 4^4, \\
  A_i &= e^{\frac{2\pi}{3}} 16, \\
  A_3 &= -\frac{\gamma_3}{6}, \\
  A_4 &= -\frac{\gamma_4}{12}, \\
  A_5 &= -\frac{\gamma_5}{12}.
\end{align}

We claim the existence of \(a_1, a_2, b, c, d\) with \(a_1, a_2\) uniformly bounded above and below from zero and \(b, c, d\) bounded. First, by the equations (33)-(37) it is not difficult to find out that

\begin{align}
  b &= A_1 A_3 a_1^\frac{4}{3} a_2^\frac{2}{3} - \left[\frac{1}{4} |A_3^3 - A_5|^2 A_1 A_3 + \frac{1}{2} (A_3^3 - A_5) A_2 A_4 \right] a_1^\frac{4}{3} a_2^\frac{2}{3}, \\
  c &= \left[\frac{1}{2} A_1 (A_3^3 - A_5) A_3 + \frac{A_2 A_4}{A_1} \right] a_1^\frac{4}{3} a_2^\frac{2}{3}, \\
  d &= \frac{1}{2} A_1 (A_3^3 - A_5) a_1^\frac{4}{3} a_2^\frac{2}{3}.
\end{align}

It remains to determine \(a_1\) and \(a_2\). To this end, we let \(t = a_1^\frac{4}{3} a_2^\frac{2}{3}\) and we solve in \(t\) first. From (33) and (35) we have

\begin{align}
  a_3^3 b + \bar{c} d &= A_1 A_3 t,
\end{align}

which implies that

\begin{align}
  a_3^3 |b|^2 + |c|^2 |d|^2 + a_3^2 |bc - d|^2 + a_3^2 |b| c d &= A_3^2 |A_3|^2 |t|^2.
\end{align}

Multiplying (42) with \(\bar{a}\), we have \(a_3^2 |bc - d|^2 + |c|^2 |d|^2 = A_1 A_3 \bar{t} d\). Adding it to its conjugate, we get \(a_3^2 |bc - d|^2 + a_3^2 |bcd - \bar{c} d|^2 = -2 |c|^2 |d|^2 + A_1 \bar{t} (A_3 c d + \bar{A}_3 \bar{c} \bar{d})\). Then (43) may be rewritten as

\begin{align}
  a_3^3 |b|^2 = |c|^2 |d|^2 + A_3^2 |A_3|^2 |t|^2 - A_1 \bar{t} (A_3 c d + \bar{A}_3 \bar{c} \bar{d}).
\end{align}

Expansion in (34) gives

\begin{align}
  a_1^2 a_2^2 + a_1^2 |c|^2 + a_2^2 |b|^2 |c|^2 + |d|^2 - \bar{b} c d - b \bar{c} d &= A_2 t^2.
\end{align}

Adding (43) and (45) yields

\begin{align}
  (a_3^2 + |c|^2) (a_3^2 + a_3^2 |b|^2 + |d|^2) = (A_2 + A_3^2 |A_3|^2) t^2.
\end{align}
and hence by (65) we have

\[ a_2^2 + |c|^2 = \left( \frac{A_2}{A_1} + A_3 |A_3|^2 \right) t. \]  

(47)

Multiplying (65) by \( a_2^2 \) we have

\[ t^3 + a_2^2 |b|^2 + a_2^2 |d|^2 = A_1 t a_2^2. \]  

(48)

Substituting (44) into (48), we obtain

\[ t^3 + |d|^2(a_2^2 + |c|^2) + A_1^2 |A_3|^2 t^2 - A_1 t(A_3 c\bar{d} + \bar{A}_3 \bar{c} d) = A_1 t a_2^2. \]  

(49)

Substituting (47) into (49), we get

\[ t^2 + \left( \frac{A_2}{A_1} |d|^2 + A_3 |dA_3 - c|^2 \right) = A_2 t, \]  

(50)

from which we can solve

\[ t = a_1^2 a_2^2 = \frac{4A_1 A_2}{4A_1 + A_1^2 A_2 |A_3|^2 - A_5 |d|^2 + 4A_2^2 |A_4|^2}. \]  

(51)

Obviously \( t \) is uniformly bounded above and also below from zero. Therefore \( b, c, d \) are all \( O(1) \)'s. Then by (47) \( a_1, a_2 \) can also be solved uniquely

\[ a_2^2 = \left( \frac{A_2}{A_1} + A_3 |A_3|^2 \right) t - |c|^2, \quad a_1^2 = \frac{t^2}{a_2^2}. \]  

(52)

It is also easy to see that \( a_2^2 > 0 \) and we can choose \( a_2 \) to be uniformly bounded below from zero. In fact,

\[ a_2^2 = \left[ \left( \frac{A_2}{A_1} + A_3 |A_3|^2 \right)(4A_1 + A_1^2 A_2 |A_3|^2 - A_5 |d|^2 + 4A_2^2 |A_4|^2) \right. \]

\[ - 4A_1 A_2 \left| \frac{1}{2} A_1 (A_3^2 - A_5) \bar{A}_3 + \frac{A_2 A_4}{A_1} \right|^2 \left( \frac{t^2}{4A_1 A_2} \right) \]

\[ \geq 4A_2 \geq C > 0, \]

where we should note that

\[ |A_3^2 - A_5|^2 + 4|A_3|^2 |A_4|^2 \geq 2A_3 \bar{A}_3 (A_3^2 - A_5) + 2A_3 A_4 (A_3^2 - A_5). \]

Finally we have that \( a_1 = t^{\frac{1}{2}}/a_2 \). Therefore the claim holds. \( \square \)

3 Preliminary Estimates on Blow-ups

In this section we derive two estimates of blowing-up solutions, one is near blow-up points and the other is far away from them. For simplicity of presentation, we will first prove Theorem 1.2. From now to section 8, we let \((u_{1_k}, u_{2_k})\)
be a blowing up sequence of solutions to (21) satisfying (H). For \(j = 1, \ldots, m\), let \(p_{k,j}\) be the local maxima of \(u_{1k}\) near \(p_j\), i.e., \(u_{1k}(p_{k,j}) = \max_{\Omega} u_{1k}\) where \(\delta\) is sufficiently small. (\(p_{k,j}\) may not be unique.)

Define
\[
-2 \ln \varepsilon_{k,j} \overset{\text{def}}{=} u_{1k}(p_{k,j}) - \ln \int_{\Omega} h_{1k}e^{u_{1k}} + \ln(\rho_{1k} h_{1k}(p_{k,j})),
\]
and
\[
\varepsilon_k = \max_{1 \leq j \leq m} \varepsilon_{k,j}.\tag{54}
\]

Set also
\[
e^{\alpha_{1k}} = \int_{\Omega} h_{1k}e^{u_{1k}}, \quad e^{\alpha_{2k}} = \int_{\Omega} h_{2k}e^{u_{2k}},
\]
\[
\tilde{u}_{1k}(x) = u_{1k}(x) - \alpha_{1k}, \quad \tilde{u}_{2k}(x) = u_{2k}(x) - \alpha_{2k}.
\]
Thus \(\tilde{u}_{1k}\) and \(\tilde{u}_{2k}\) satisfy
\[
\begin{cases}
\Delta \tilde{u}_{1k} + 2 \rho_{1k} h_{1k} \tilde{u}_{1k} - \rho_{2k} h_{2k} \tilde{u}_{2k} = 0 & \text{in } \Omega, \\
\Delta \tilde{u}_{2k} - \rho_{1k} h_{1k} \tilde{u}_{1k} + 2 \rho_{2k} h_{2k} \tilde{u}_{2k} = 0 & \text{in } \Omega.
\end{cases}
\]

The following sup + inf estimate plays an important role in later proofs. This follows from [13, Theorem 1.3].

**Lemma 3.1.** Under the assumptions of Theorem 1.1 or Theorem 1.2, there exists a small \(\delta > 0\) independent of \(k\) such that
\[
\tilde{u}_{jk} + 2 \ln \varepsilon_{k,j} - \left[ v_i \left( \frac{x - p_{k,j}}{\varepsilon_{k,j}} \right) - \ln(\rho_{1k} h_{1k}(p_{k,j})) \right] = O(1) \quad \text{in } B_\delta(p_j) \tag{55}
\]
for \(i = 1, 2\) and \(j = 1, \ldots, m\).

**Remark:** As in [13], the entire solutions \(v_i\) are chosen so that they equal to zero at the origin.

**Proof.** Letting
\[
\bar{u}_{1k}(x) = \tilde{u}_{1k}(x) + \ln(\rho_{1k} h_{1k}(p_{k,j})), \quad \bar{u}_{2k}(x) = \tilde{u}_{2k}(x) + \ln(\rho_{2k} h_{2k}(p_{k,j})),
\]
we have
\[
\begin{cases}
\Delta \bar{u}_{1k} + \frac{2 h_{1k}(x)}{h_{1k}(p_{k,j})} e^{\bar{u}_{1k}} - \frac{h_{2k}(x)}{h_{2k}(p_{k,j})} e^{\bar{u}_{2k}} = 0 & \text{in } B_\delta(p_j), \\
\Delta \bar{u}_{2k} - \frac{h_{1k}(x)}{h_{1k}(p_{k,j})} e^{\bar{u}_{1k}} + \frac{2 h_{2k}(x)}{h_{2k}(p_{k,j})} e^{\bar{u}_{2k}} = 0 & \text{in } B_\delta(p_j).
\end{cases}
\]

Then \(\bar{u}_{1k}\) and \(\bar{u}_{2k}\) satisfy the conditions of [13, Theorem 1.3]. So we conclude that there exist two constants \(\delta > 0\) and \(C > 0\) independent of \(k\) such that
\[
\bar{u}_{ik}(x) + 2 \ln \varepsilon_{k,j} - v_i \left( \frac{x - p_{k,j}}{\varepsilon_{k,j}} \right) \leq C \quad \text{in } B_\delta(p_j) \quad \text{for } i = 1, 2,
\]
which is equivalent to (55). \(\square\)
Remark 3.2. By considering \(a_{ik}(\varepsilon_k,j y + p_{k,j}) + 2 \ln \varepsilon_{k,j} (i = 1, 2)\), we have that the following holds:

\[
\begin{align*}
\varepsilon_{k,j} |\nabla \hat{u}_k(p_{k,j})| &\leq C, & \varepsilon_{k,j}^2 |\nabla^2 \hat{u}_k(p_{k,j})| &\leq C, \\
\varepsilon_{k,j} |\nabla u_{2k}(p_{k,j})| &\leq C, & \varepsilon_{k,j}^2 |\nabla^2 u_{2k}(p_{k,j})| &\leq C.
\end{align*}
\]

(56) (57)

From Lemma 3.1 we have the following important corollary.

Corollary 3.3. For any \(i = 1, 2\) and \(j = 1, \ldots, m\), it holds that

\[
\alpha_{ik} + 2 \ln \varepsilon_{k,j} = O(1),
\]

(58)

\[
C^{-1} \varepsilon_{k,j} \leq \varepsilon_{k,j} \leq C \varepsilon_{k,j} \quad \text{for any } \ell \neq j.
\]

(59)

Proof. Noting that \(v_i \left( \frac{x - p_{ik}}{\varepsilon_{k,j}} \right) \sim 4 \ln \varepsilon_{k,j}\) on \(\partial B_\delta(p_j)\), we get (58). (59) follows directly from (58).

For a fixed small \(\delta > 0\), we set the local mass \(\rho_{ik,j}\) to be

\[
\rho_{ik,j} = \rho_{ik} \int_{B_\delta(p_j)} h_{ik} e^{\hat{u}_{ik}}
\]

for \(i = 1, 2\).

By Lemma 3.1 we have

\[
\rho_{ik,j} = \rho_{ik} \int_{B_\varepsilon(p_{k,j})} h_{ik} e^{\hat{u}_{ik}} + O(\varepsilon_{k,j}^2).
\]

Observe that \(\rho_{ik} = \rho_{ik} \int_{\Omega} h_{ik} e^{\hat{u}_{ik}}\) and it is easy to see that

\[
\rho_{ik} = \sum_{j=1}^{m} \rho_{ik,j} + O(\varepsilon_{k}^2),
\]

where \(\varepsilon_{k}\) is defined in (54). Define again

\[
w_{1k}(x) = 2u_{1k} + u_{2k} - 3 \sum_{j=1}^{m} \rho_{1k,j} G(x, p_{k,j}),
\]

\[
w_{2k}(x) = u_{1k} + 2u_{2k} - 3 \sum_{j=1}^{m} \rho_{2k,j} G(x, p_{k,j}).
\]

Lemma 3.4. It holds that, for \(i = 1, 2\),

\[
|w_{ik}| + |\nabla w_{ik}| = O(\varepsilon_k) \quad \text{for } x \in \Omega \setminus \bigcup_{j=1}^{m} B_\delta(p_j).
\]

Proof. It is easy to see that

\[
\begin{cases}
-\Delta (2u_{1k} + u_{2k}) = 3\rho_{1k} h_{1k} e^{\hat{u}_{ik}} & \text{in } \Omega, \\
-\Delta (u_{1k} + 2u_{2k}) = 3\rho_{2k} h_{2k} e^{\hat{u}_{ik}} & \text{in } \Omega.
\end{cases}
\]

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This lemma follows from the Green’s representation formula. In fact, for \( \ell = 0, 1 \) and \( x \in \Omega \setminus \bigcup_{j = 1}^{m} B_{\varepsilon}(p_{j}) \)

\[
\partial^\ell(2u_{1k}(x) + u_{2k}(x)) = 3 \int_{\Omega} \partial^\ell G(x, z) \rho_{1k} h_{1k}(z) e^{\varepsilon_{1k}(z)} dz
\]

\[
= 3 \sum_{j=1}^{m} \int_{B_{\varepsilon}(p_{k,j})} \partial^\ell G(x, z) \rho_{1k} h_{1k}(z) e^{\varepsilon_{1k}(z)} dz + O(\varepsilon_{k}^{2})
\]

\[
= 3 \sum_{j=1}^{m} \int_{B_{\varepsilon}(p_{k,j})} \partial^\ell G(x, z) \rho_{1k} h_{1k}(z) e^{\varepsilon_{1k}(z)} dz + 3 \sum_{j=1}^{m} \rho_{1k,j} \partial^\ell G(x, p_{k,j}) + O(\varepsilon_{k})
\]

\[
= 3 \sum_{j=1}^{m} \rho_{1k,j} \partial^\ell G(x, p_{k,j}) + O(\varepsilon_{k}).
\]

The proof of other estimates is similar and thus omitted. \( \square \)

4 Sharp approximation of the bubbles

In this section, we give a sharp description of the bubbling behavior of \( 2u_{1k} + \bar{u}_{2k} \) and \( \bar{u}_{1k} + 2\bar{u}_{2k} \) in the ball \( B_{\varepsilon}(p_{k,j}) \), using the entire solutions of (29).

For simplicity, we set

\[
\tilde{G}_{1k,j}(x) = \rho_{1k,j} H(x, p_{k,j}) + \sum_{\ell \neq j} \rho_{1k,\ell} G(x, p_{k,\ell}),
\]

\[
\tilde{G}_{2k,j}(x) = \rho_{2k,j} H(x, p_{k,j}) + \sum_{\ell \neq j} \rho_{2k,\ell} G(x, p_{k,\ell}).
\]

Set also

\[
V_{1k,j}(y) = \ln \left\{ \frac{4 \left( a_{1h,j}^2 + a_{2h,j}^2 + |2y + c_{h,j}|^2 + a_{2h,j}^2 |y|^2 + 2b_{h,j}y + b_{h,j}c_{h,j} - d_{h,j}|^2 \right)}{a_{1h,j}^2 + a_{2h,j}^2 |y| + b_{h,j}|^2 + |y|^2 + c_{h,j}y + d_{h,j}|^2} \right\} \rho_{1k,h}(p_{k,j})
\]

\[
V_{2k,j}(y) = \ln \left\{ \frac{16a_{1h,j}^4 + a_{2h,j}^4 |2y + c_{h,j}|^2 + a_{2h,j}^2 |y|^2 + 2b_{h,j}y + b_{h,j}c_{h,j} - d_{h,j}|^2}{a_{1h,j}^2 + a_{2h,j}^2 |y| + b_{h,j}|^2 + |y|^2 + c_{h,j}y + d_{h,j}|^2} \right\} \rho_{2k,h}(p_{k,j})
\]

and

\[
U_{1k,j}(x) = \frac{V_{1k,j}}{\varepsilon_{k,j}} \left( \frac{x - p_{k,j}}{\varepsilon_{k,j}} \right), \quad U_{2k,j}(x) = \frac{V_{2k,j}}{\varepsilon_{k,j}} \left( \frac{x - p_{k,j}}{\varepsilon_{k,j}} \right),
\]

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where \(a_{1k,j}, a_{2k,j}, b_{k,j}, c_{k,j}\) and \(d_{k,j}\) are chosen such that

\[
(2U_{1k,j} + U_{2k,j})(p_{k,j}) = (2\bar{u}_{1k} + \bar{u}_{2k})(p_{k,j}) + 6\ln \varepsilon_{k,j},
\]

\[
(U_{1k,j} + 2U_{2k,j})(p_{k,j}) = (\bar{u}_{1k} + 2\bar{u}_{2k})(p_{k,j}) + 6\ln \varepsilon_{k,j},
\]

\[
\nabla \varepsilon(2U_{1k,j} + U_{2k,j})(p_{k,j}) = \nabla \varepsilon(2\bar{u}_{1k} + \bar{u}_{2k})(p_{k,j}) - 3\sqrt{\varepsilon}G_{1k,j}(p_{k,j}),
\]

\[
\nabla \varepsilon(U_{1k,j} + 2U_{2k,j})(p_{k,j}) = \nabla \varepsilon(\bar{u}_{1k} + 2\bar{u}_{2k})(p_{k,j}) - 3\sqrt{\varepsilon}G_{2k,j}(p_{k,j}),
\]

\[
\nabla \nabla \varepsilon(2U_{1k,j} + U_{2k,j})(p_{k,j}) = \nabla \nabla \varepsilon(2\bar{u}_{1k} + \bar{u}_{2k})(p_{k,j}) - 3\nabla \nabla \varepsilon G_{1k,j}(p_{k,j}).
\]

We remark that (60)-(64) can be solved in the coefficients \(a_{1k,j}, a_{2k,j}, b_{k,j}, c_{k,j}\) and \(d_{k,j}\). In fact, a direct computation shows that

\[
2V_{1k,j} + V_{2k,j} = \ln \frac{1}{\left(a_{1k,j}^2 + a_{2k,j}^2 |y + b_{k,j}|^2 + |y + c_{k,j}y + d_{k,j}|^2\right)^{\frac{3}{2}}} + \ln \frac{256a_{1k,j}^2a_{2k,j}}{\rho_{1k}h_{1k}(p_{k,j})\rho_{2k}h_{2k}(p_{k,j})},
\]

\[
V_{1k,j} + 2V_{2k,j} = \ln \frac{1}{\left(a_{1k,j}^2a_{2k,j}^2 + a_{1k,j}^2 |2y + c_{k,j}|^2 + a_{2k,j}^2 |y + b_{k,j}y + b_{k,j}c_{k,j} - d_{k,j}|^2\right)^{\frac{3}{2}}} + \ln \frac{1024a_{1k,j}^4a_{2k,j}^4}{\rho_{1k}h_{1k}(p_{k,j})\rho_{2k}h_{2k}(p_{k,j})}.
\]

We omit the subscript \(j\) for simplicity. System (60)-(64) is then rewritten as

\[
a_{1k}^2 + a_{2k}^2 |b_k|^2 + |d_k|^2 = A_{1k}a_{1k}^2 a_{2k}^2, \tag{65}
\]

\[
a_{1k}^2 a_{2k}^2 + a_{1k}^2 |c_k|^2 + a_{2k}^2 |b_k c_k - d_k|^2 = A_{2k}a_{1k}^2 a_{2k}^2, \tag{66}
\]

\[
a_{2k}^2 b_k + c_k d_k \overline{a_{1k}^2 + a_{2k}^2 |b_k|^2 + |d_k|^2} = A_{3k}, \tag{67}
\]

\[
a_{2k}^2 c_k + a_{2k}^2 |b_k|^2 c_k - a_{2k}^2 b_k d_k \overline{a_{1k}^2 a_{2k}^2 + a_{1k}^2 |c_k|^2 + a_{2k}^2 |b_k c_k - d_k|^2} = A_{4k}, \tag{68}
\]

\[
\frac{(a_{2k}^2 b_k + c_k d_k)^2 - 2d_k(a_{1k}^2 + a_{2k}^2 |b_k|^2 + |d_k|^2)}{(a_{1k}^2 + a_{2k}^2 |b_k|^2 + |d_k|^2)^2} = A_{5k}, \tag{69}
\]

where \(0 < C < A_{1k} \in \mathbb{R}, 0 < C < A_{2k} \in \mathbb{R}, A_{3k}, A_{4k}, A_{5k} \in \mathbb{C}\) are uniquely decided by the terms on the right hand side of (60)-(64). Because of the definition of \(\varepsilon_{k,j}\), the assumption in the main theorem and (56), (57), all of \(A_{ik}\) \((i = 1, \ldots, 5)\) are uniformly of order \(O(1)\).

By the same proof as in Theorem 2.2, we obtain the existences of \(a_{1k}, a_{2k}, b_k, c_k, d_k\) with \(a_{1k}, a_{2k}\) uniformly bounded away from zero and \(b_k, c_k, d_k\) bounded.
In what follows, we define for \( x \in B_z(p_{k,j}) \)
\[
\eta_{k,j}(x) = 2\bar{U}_{1,k,j} + \bar{U}_{2,k,j} + 6\ln \| x \|_{k,j} - 2U_{1,k,j} - U_{2,k,j} - 3\tilde{G}_{1,k,j}(x) + 3\tilde{G}_{1,k,j}(p_{k,j}),
\]
\[
\eta_{2k,j}(x) = \bar{u}_{1,k} + \bar{u}_{2,k} + 6\ln \| x \|_{k,j} - \bar{U}_{1,k,j} - 2U_{2,k,j} - 3\tilde{G}_{2,k,j}(x) + 3\tilde{G}_{2,k,j}(p_{k,j}).
\]
In \( B_{\delta}(p_{k,j}) \setminus B_{\frac{\delta}{2}}(p_{k,j}) \), by a Taylor expansion, it is easy to see that
\[
2U_{1,k,j} + U_{2,k,j} - 6\ln \| x \|_{k,j}
\]
\[
= -3\ln \left[ a_{1,k,j}^2 + a_{2,k,j}^2 \| x \|_{k,j}^2 \right] + \varepsilon_{k,j} \| x \|_{k,j}^2 + \varepsilon_{k,j} d_{k,j}^2
\]
\[
= -12\ln \| x \|_{k,j} + 6\ln \| x \|_{k,j} + \frac{256a_{1,k,j}^2 a_{2,k,j}^2 \| x \|_{k,j}^6}{\rho_{1,k} h_{1,k}(p_{k,j}) \rho_{2,k} h_{2,k}(p_{k,j})} + O(\varepsilon_{k,j}),
\]
and
\[
U_{1,k,j} + 2U_{2,k,j} - 6\ln \| x \|_{k,j}
\]
\[
= -3\ln \left[ a_{1,k,j}^2 + a_{2,k,j}^2 \| x \|_{k,j}^2 \right] + \varepsilon_{k,j} \| x \|_{k,j}^2 + \varepsilon_{k,j} d_{k,j}^2
\]
\[
= -12\ln \| x \|_{k,j} + 6\ln \| x \|_{k,j} + \frac{1024a_{1,k,j}^4 a_{2,k,j}^4 \| x \|_{k,j}^6}{\rho_{1,k} h_{1,k}(p_{k,j}) \rho_{2,k} h_{2,k}(p_{k,j})} + O(\varepsilon_{k,j}).
\]
Thus we have, in \( B_{\delta}(p_{k,j}) \setminus B_{\frac{\delta}{2}}(p_{k,j}) \), that
\[
\eta_{1,k,j} = u_{1,k} + 3\rho_{1,k,j} G(x, p_{k,j}) - 2\alpha_{1,k} - \alpha_{2,k} - 3\rho_{1,k,j} H(x, p_{k,j}) + 3\tilde{G}_{1,k,j}(p_{k,j})
\]
\[
+ 12\ln \| x \|_{k,j} - 6\ln \| x \|_{k,j} - \ln \frac{256a_{1,k,j}^2 a_{2,k,j}^2 \| x \|_{k,j}^6}{\rho_{1,k} h_{1,k}(p_{k,j}) \rho_{2,k} h_{2,k}(p_{k,j})} + O(\varepsilon_{k,j})
\]
\[
= \frac{3}{2\pi} \left( 8\pi - \rho_{1,k,j} \right) \ln \| x \|_{k,j} + A_{1,k,j} + O(\varepsilon_{k,j}),
\]
where \( A_{1,k,j} \) is a constant given by
\[
A_{1,k,j} = -2\alpha_{1,k} - \alpha_{2,k} - 6\ln \| x \|_{k,j} + 3\tilde{G}_{1,k,j}(p_{k,j}) - \ln \frac{256a_{1,k,j}^2 a_{2,k,j}^2 \| x \|_{k,j}^6}{\rho_{1,k} h_{1,k}(p_{k,j}) \rho_{2,k} h_{2,k}(p_{k,j})}.
\]
From Corollary 3.3 we derive that \( A_{1,k} = O(1) \). Moreover Lemma 3.4 also indicates that (70) holds for \( \nabla \eta_{k,j} \). Analogously, in \( B_{\delta}(p_{k,j}) \setminus B_{\frac{\delta}{2}}(p_{k,j}) \),
\[
\eta_{2k,j} = u_{2,k} + 3\rho_{2,k,j} G(x, p_{k,j}) - \alpha_{1,k} - 2\alpha_{2,k} - 3\rho_{2,k,j} H(x, p_{k,j}) + 3\tilde{G}_{2,k,j}(p_{k,j})
\]
\[
= \frac{3}{2\pi} \left( 8\pi - \rho_{2,k,j} \right) \ln \| x \|_{k,j} + A_{2,k,j} + O(\varepsilon_{k,j}),
\]
where \( A_{2,k,j} \) is a constant given by
\[
A_{2,k,j} = -2\alpha_{1,k} - \alpha_{2,k} - 6\ln \| x \|_{k,j} + 3\tilde{G}_{2,k,j}(p_{k,j}) - \ln \frac{256a_{1,k,j}^2 a_{2,k,j}^2 \| x \|_{k,j}^6}{\rho_{1,k} h_{1,k}(p_{k,j}) \rho_{2,k} h_{2,k}(p_{k,j})}.
\]
\[ + 12 \ln |x - p_k,j| - 6 \ln \varepsilon_{k,j} - \ln \frac{1024 a_{1k,j} a_{2k,j}^2}{\rho_{1k}^2 h_{1k}^2 (p_{k,j}) \rho_{2k} h_{2k}^2 (p_{k,j})} + O(\varepsilon_k) \]

\[ = \frac{3}{2\pi} (8\pi - \rho_{2k,j}) \ln |x - p_{k,j}| + A_{2k,j} + O(\varepsilon_k), \]  

(71)

where \( A_{2k,j} = O(1) \) and

\[ A_{2k,j} = -2a_{2k} - 6 \ln \varepsilon_{k,j} + 3G_{2k,j} (p_{k,j}) - \ln \frac{1024 a_{1k,j} a_{2k,j}^2}{\rho_{1k}^2 h_{1k}^2 (p_{k,j}) \rho_{2k} h_{2k}^2 (p_{k,j})}. \]

Also we have that (71) holds for \( \nabla \eta_{2k,j} \).

In order to estimate \( \eta_{k,j} \) in the whole \( B_0 (p_{k,j}) \), let us define, for \( |y| \leq \frac{\delta}{\varepsilon_{k,j}} \),

\[ \tilde{\eta}_{1k,j} (y) = \eta_{1k,j} (p_{k,j} + \varepsilon_{k,j} y), \quad \tilde{\eta}_{2k,j} (y) = \eta_{2k,j} (p_{k,j} + \varepsilon_{k,j} y). \]  

(72)

By the definition of \( \tilde{\eta}_{1k,j} \) and \( \tilde{\eta}_{2k,j} \), it is easy to see that they satisfy

\[
\begin{cases}
-\Delta \tilde{\eta}_{1k,j} = 3 \rho_{1k} h_{1k} (p_{k,j}) e^{V_{1k,j} D_{1k,j} (y)} & \text{in } B_{\frac{\delta}{\varepsilon_{k,j}}}, \\
-\Delta \tilde{\eta}_{2k,j} = 3 \rho_{2k} h_{2k} (p_{k,j}) e^{V_{2k,j} D_{2k,j} (y)} & \text{in } B_{\frac{\delta}{\varepsilon_{k,j}}}, \\
\tilde{\eta}_{1k,j} = O(1), \quad \tilde{\eta}_{2k,j} = O(1) & \text{on } \partial B_{\frac{\delta}{\varepsilon_{k,j}}},
\end{cases}
\]

where

\[
D_{1k,j} (y) = \exp \left[ \frac{1}{3} (2\tilde{\eta}_{1k,j} - \tilde{\eta}_{2k,j}) + Q_{1k,j} (\varepsilon_{k,j} y + p_{k,j}) - Q_{1k,j} (p_{k,j}) \right] - 1,
\]

(73)

\[
D_{2k,j} (y) = \exp \left[ \frac{1}{3} (2\tilde{\eta}_{2k,j} - \tilde{\eta}_{1k,j}) + Q_{2k,j} (\varepsilon_{k,j} y + p_{k,j}) - Q_{2k,j} (p_{k,j}) \right] - 1.
\]

(74)

In (73) and (74), \( Q_{1k,j} (x) \) and \( Q_{2k,j} (x) \) denote

\[
Q_{1k,j} = 2 \tilde{G}_{1k,j} - \tilde{G}_{2k,j} + \ln h_{1k}, \quad (75)
\]

\[
Q_{2k,j} = 2 \tilde{G}_{2k,j} - \tilde{G}_{1k,j} + \ln h_{2k}. \quad (76)
\]

Since \( Q_{1k,j} (\varepsilon_{k,j} y + p_{k,j}) - Q_{1k,j} (p_{k,j}) = \nabla Q_{1k,j} (p_{k,j}) \varepsilon_{k,j} y + O(\varepsilon_{k,j}^2 |y|^2) \), we have in \( B_{\frac{\delta}{\varepsilon_{k,j}}} \) that

\[
\begin{cases}
-\Delta \tilde{\eta}_{1k,j} = \rho_{1k} h_{1k} (p_{k,j}) e^{V_{1k,j}} \frac{2(n_{1k,j} - n_{2k,j}) - 1}{2(n_{1k,j} - n_{2k,j})} (2\tilde{\eta}_{1k,j} - \tilde{\eta}_{2k,j}) \\
+ O \left( \frac{\varepsilon_{k,j}^2}{1 + |y|^2} \right) + O \left( \frac{\varepsilon_{k,j}^2}{1 + |y|^2} \right), \\
-\Delta \tilde{\eta}_{2k,j} = \rho_{2k} h_{2k} (p_{k,j}) e^{V_{2k,j}} \frac{2(n_{1k,j} - n_{2k,j}) - 1}{2(n_{1k,j} - n_{2k,j})} (2\tilde{\eta}_{2k,j} - \tilde{\eta}_{1k,j}) \\
+ O \left( \frac{\varepsilon_{k,j}^2}{1 + |y|^2} \right) + O \left( \frac{\varepsilon_{k,j}^2}{1 + |y|^2} \right), \\
\tilde{\eta}_{1k} = O(1), \tilde{\eta}_{2k} = O(1), \quad \text{on } \partial B_{\frac{\delta}{\varepsilon_{k,j}}},
\end{cases}
\]

(77)

The following lemma plays an important role in all the subsequent estimates. The proof is lengthy and we delay it to the appendix.
Lemma 4.1. Suppose $|\nabla Q(x,y_j(p_{k,j})| = O(\varepsilon_{k,j}^n)$ for some $0 \leq \sigma_0 \leq 1$. Then for any $\tau \in (0,1)$ and $\tau \leq \tau_0 = \frac{1+\sigma_0}{2}$, in $B_{\frac{d}{x_{k,j}}}$ there holds that

$$|\bar{\eta}_{1k,j}| \leq C_{\tau}(1 + |y|^\tau) \left( \varepsilon_k^2 + \varepsilon_{k,j}^2 \sup_{B_{\frac{d}{x_{k,j}}} \leq |y| \leq B_{\frac{d}{x_{k,j}}}} |\bar{\eta}_{1k,j}| \right),$$

$$|\bar{\eta}_{2k,j}| \leq C_{\tau}(1 + |y|^\tau) \left( \varepsilon_k^2 + \varepsilon_{k,j}^2 \sup_{B_{\frac{d}{x_{k,j}}} \leq |y| \leq B_{\frac{d}{x_{k,j}}}} |\bar{\eta}_{2k,j}| \right).$$

Remark 4.2. We will prove $\sigma_0 = 1$ later. Hence Lemma 4.1 holds for any $\tau \in (0,1)$.

Lemma 4.3. For any $\frac{1}{2} \leq \tau \leq \tau_0$, we have

$$A_{1k,j} = O \left( \varepsilon_k^2 + \varepsilon_{k,j}^2 \sup_{B_{\frac{d}{x_{k,j}}} \leq |y| \leq B_{\frac{d}{x_{k,j}}}} (|\bar{\eta}_{1k,j}| + |\bar{\eta}_{2k,j}|) \right),$$

$$A_{2k,j} = O \left( \varepsilon_k^2 + \varepsilon_{k,j}^2 \sup_{B_{\frac{d}{x_{k,j}}} \leq |y| \leq B_{\frac{d}{x_{k,j}}}} (|\bar{\eta}_{1k,j}| + |\bar{\eta}_{2k,j}|) \right).$$

Proof. By Green’s formula,

$$2u_{1k}(p_{k,j}) + u_{2k}(p_{k,j})$$

$$= 3 \sum_{i=1}^{m} \int_{B_{\frac{d}{x_{p_{k,i}}}}} \rho_{1k} h_{1k}(x) e^{\tilde{a}_{1k}(x)} G(p_{k,j}, x) \, dx + O(\varepsilon_k^2)$$

$$= 3 \sum_{i=1}^{m} \int_{B_{\frac{d}{x_{p_{k,i}}}}} \rho_{1k} h_{1k}(x) e^{\tilde{a}_{1k}(x)} \left[ \frac{1}{2\pi} \ln \frac{1}{|p_{k,j} - x|} \right] + H(p_{k,j}, x) \, dx + O(\varepsilon_k^2)$$

$$= 3 \tilde{G}_{1k,j}(p_{k,j}) + \frac{3}{2\pi} \int_{B_{\frac{d}{x_{p_{k,j}}}}} \rho_{1k} h_{1k}(x) e^{\tilde{a}_{1k}(x)} \ln \frac{1}{|p_{k,j} - x|} + O(\varepsilon_k). \quad (78)$$

Note that

$$\rho_{1k} h_{1k} e^{\tilde{a}_{1k}} = \rho_{1k} h_{1k}(p_{k,j}) e^{-2V_{1k,j}} e^{\tilde{a}_{1k}(p_{k,j})} \left[ 1 + D_{1k,j}(\frac{x - p_{k,j}}{\varepsilon}) \right]$$

where $D_{1k,j}(y)$ is given by (73). Thus we obtain, recalling (72) and using the fact that $\frac{1}{2} \leq \tau \leq \tau_0$,

$$- \frac{3}{2\pi} \int_{B_{\frac{d}{x_{p_{k,j}}}}} \rho_{1k} h_{1k} e^{\tilde{a}_{1k}} \ln |p_{k,j} - x| \, dx$$
\[
= - \frac{3}{2\pi} \ln \varepsilon_{k,j} \int_{B_{\frac{1}{T_{n,j}}}} \rho_{1k} h_{1k}(p_{k,j}) e^{V_{1,k,j}} (1 + D_{1,k,j}) \\
- \frac{3}{2\pi} \int_{B_{\frac{1}{T_{n,j}}}} \rho_{1k} h_{1k}(p_{k,j}) e^{V_{1,k,j}} (1 + D_{1,k,j}) \ln |y| \\
= -12 \ln \varepsilon_{k,j} + \ln \left( \frac{1}{(a_{1,k,j}^2 + a_{2,k,j}^2 |b_{k,j}|^2 + |d_{k,j}|^2)^3} \right) \\
+ O \left( \varepsilon_{k,j} + \varepsilon_{k,j}^7 \sup_{B_{\frac{1}{T_{n,j}}}} \left( |\tilde{\eta}_{1,k,j}| + |\tilde{\eta}_{2,k,j}| \right) \right). \tag{79}
\]

In fact, a direct computation shows that, for \( \frac{1}{T} \leq \tau \leq \tau_0, \)

\[
\int_{B_{\frac{1}{T_{n,j}}}} \rho_{1k} h_{1k}(p_{k,j}) e^{V_{1,k,j}} = 8\pi + O(\varepsilon_{k,j}^2),
\]

\[
\int_{B_{\frac{1}{T_{n,j}}}} \rho_{1k} h_{1k}(p_{k,j}) e^{V_{1,k,j}} D_{1,k,j}
\]

\[
= \int_{B_{\frac{1}{T_{n,j}}}} \rho_{1k} h_{1k}(p_{k,j}) e^{V_{1,k,j}} [O(|\tilde{\eta}_{1,k,j}| + |\tilde{\eta}_{2,k,j}|) + O(\varepsilon_{k,j} |y|)]
\]

\[
= O \left( \varepsilon_{k,j} + \varepsilon_{k,j}^7 \sup_{B_{\frac{1}{T_{n,j}}}} \left( |\tilde{\eta}_{1,k,j}| + |\tilde{\eta}_{2,k,j}| \right) \right),
\]

and

\[
- \frac{3}{2\pi} \int_{B_{\frac{1}{T_{n,j}}}} \rho_{1k} h_{1k}(p_{k,j}) e^{V_{1,k,j}} \ln |y| = \frac{1}{2\pi} \int_{B_{\frac{1}{T_{n,j}}}} \Delta (2V_{1,k,j} + V_{2,k,j}) \ln |y|
\]

\[
= 2V_{1,k,j}(0) + V_{2,k,j}(0)
\]

\[
+ \frac{1}{2\pi} \int_{\partial B_{\frac{1}{T_{n,j}}}} \left[ \ln |y| \frac{\partial}{\partial \nu} (2V_{1,k,j} + V_{2,k,j}) - (2V_{1,k,j} + V_{2,k,j}) \frac{\partial \ln |y|}{\partial \nu} \right]
\]

\[
= \frac{1}{2\pi} \int_{\partial B_{\frac{1}{T_{n,j}}}} \left[ - \ln |y| \frac{12}{|y|} \left( \ln \frac{256a_{1,k,j}^2 a_{2,k,j}^2}{\rho_{1k}^2 h_{1k}(p_{k,j}) \rho_{2k} h_{2k}(p_{k,j})} - 12 \ln |y| \right) \frac{1}{|y|} \right]
\]

\[
+ \ln \left( \frac{256a_{1,k,j}^2 a_{2,k,j}^2}{(a_{1,k,j}^2 + a_{2,k,j}^2 |b_{k,j}|^2 + |d_{k,j}|^2)^3 \rho_{1k}^2 h_{1k}(p_{k,j}) \rho_{2k} h_{2k}(p_{k,j})} \right) + O(\varepsilon_{k,j})
\]

\[
= \ln \frac{1}{(a_{1,k,j}^2 + a_{2,k,j}^2 |b_{k,j}|^2 + |d_{k,j}|^2)^3} + O(\varepsilon_{k,j}).
\]

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In the above estimate, we used the fact that on $\partial B_{\ell_{h,j}}$

$$2V_{1,k,j} + V_{2,k,j} = \ln \frac{256a_1^2 \rho_{1,k}^2 \rho_{2,k}^2 (p_{k,j})}{\rho_{1,k}^2 h_{1,k}^2 (p_{k,j})} - 12 \ln |y| + O\left(\frac{1}{|y|}\right),$$

$$\frac{\partial (2V_{1,k,j} + V_{2,k,j})}{\partial \nu} = -\frac{12}{|y|} + \frac{3(c_{k,j} \bar{y} + c_{k,j} y)}{|y|^3} + O\left(\frac{1}{|y|^3}\right).$$

On the other hand, by our choice we have

$$2u_{1,k} (p_{k,j}) + u_{2,k} (p_{k,j}) = 2\bar{u}_{1,k} (p_{k,j}) + \bar{u}_{2,k} (p_{k,j}) + 2\alpha_{1,k} + \alpha_{2,k}$$

$$= 2\alpha_{1,k} + \alpha_{2,k} - 6 \ln \varepsilon_{k,j} + \ln \frac{256a_1^2 \rho_{1,k}^2 \rho_{2,k}^2 (p_{k,j})}{\rho_{1,k}^2 h_{1,k}^2 (p_{k,j})}$$

$$+ \ln \left(\frac{1}{a_{1,k,j}^2 + a_{2,k,j}^2 |b_{k,j}|^2 + |d_{k,j}|^2}\right)^3. \quad (80)$$

Combining (78), (79) and (80), we get the estimate of $A_{1,k,j}, A_{2,k,j}$ can be dealt with similarly. The proof is complete. \(\square\)

Using Lemma 4.3, we have from (70) and (71) that in $B_{\ell} (p_{k,j}) \setminus B_{\ell_{h,j}} (p_{k,j})$,

$$\eta_{k,j} = \frac{3}{2\pi} (8\pi - \rho_{1,k,j}) \ln |x - p_{k,j}| + O \left(\varepsilon_k + \varepsilon_{k,j}^{\tau} \sup_{B_{\ell_{h,j}}} \left(|\tilde{\eta}_{k,j}| + |\tilde{\eta}_{2,k,j}|\right)\right),$$

$$\eta_{2,k,j} = \frac{3}{2\pi} (8\pi - \rho_{2,k,j}) \ln |x - p_{k,j}| + O \left(\varepsilon_k + \varepsilon_{k,j}^{\tau} \sup_{B_{\ell_{h,j}}} \left(|\tilde{\eta}_{k,j}| + |\tilde{\eta}_{2,k,j}|\right)\right).$$

We then have

$$\sup_{B_{\ell_{h,j}}} \left(|\tilde{\eta}_{k,j}| + |\tilde{\eta}_{2,k,j}|\right) \leq C \left(|\rho_{1,k,j} - 8\pi| + |\rho_{2,k,j} - 8\pi| + O(\varepsilon_k)\right).$$

Hence Lemma 4.1 can be refined as follows.

**Proposition 4.4.** It holds in $B_{\ell_{h,j}} (x_{k,j})$ that

$$|\tilde{\eta}_{k,j}| + |\tilde{\eta}_{2,k,j}| \leq C (1 + |y|) \left[\varepsilon_{k,j}^{2\tau} + \varepsilon_{k,j}^{\tau} (|\rho_{1,k,j} - 8\pi| + |\rho_{2,k,j} - 8\pi|)\right],$$

where $\frac{1}{2} \leq \tau \leq \tau_0$.

5 **Estimates of $\nabla Q_{1,k,j}$ and $\nabla Q_{2,k,j}$**

In this section we estimate the gradients of the functions $Q_{1,k,j}$ and $Q_{2,k,j}$ defined in (75) and (76).
Proposition 5.1. For \( \frac{1}{2} \leq \tau \leq \tau_0 \) and any \( j = 1, \ldots, m \), we have

\[
\nabla Q_{1k,j} = O(\varepsilon_k | \ln \varepsilon_k | (| \rho_{1k,j} - 8\pi | + | \rho_{2k,j} - 8\pi |)
+ O(\varepsilon_k^{2\tau-1})(| \rho_{1k,j} - 8\pi |^2 + | \rho_{2k,j} - 8\pi |^2) + O(\varepsilon_k),
\]

\[
\nabla Q_{2k,j} = O(\varepsilon_k | \ln \varepsilon_k | (| \rho_{1k,j} - 8\pi | + | \rho_{2k,j} - 8\pi |)
+ O(\varepsilon_k^{2\tau-1})(| \rho_{1k,j} - 8\pi |^2 + | \rho_{2k,j} - 8\pi |^2) + O(\varepsilon_k).
\]

Since the problem is considered locally, for simplicity of notations we omit the subscript \( j \) if there is no confusion. Similarly we use \( \rho_{1k}^1, \rho_{2k}^2 \) to denote \( \rho_{1k,j} \) and \( \rho_{2k,j} \).

Proof. We set

\[
\psi_{1k,1}^b(y) = (\partial_{\nu_1} + \partial_{\nu_2})V_{1k}(y), \quad \psi_{1k,2}^b(y) = i(\partial_{\nu_1} - \partial_{\nu_2})V_{1k}(y),
\psi_{1k,1}^c(y) = (\partial_{\nu_1} + \partial_{\nu_2})V_{1k}(y), \quad \psi_{1k,2}^c(y) = i(\partial_{\nu_1} - \partial_{\nu_2})V_{1k}(y),
\psi_{2k,1}^b(y) = (\partial_{\nu_1} + \partial_{\nu_2})V_{2k}(y), \quad \psi_{2k,2}^b(y) = i(\partial_{\nu_1} - \partial_{\nu_2})V_{2k}(y),
\psi_{2k,1}^c(y) = (\partial_{\nu_1} + \partial_{\nu_2})V_{2k}(y), \quad \psi_{2k,2}^c(y) = i(\partial_{\nu_1} - \partial_{\nu_2})V_{2k}(y).
\]

It is easy to check that, on \( \partial B_{\frac{1}{2}} \),

\[
\psi_{1k,1} = \frac{4y_1}{|y|^2} + O\left(\frac{1}{|y|^3}\right), \quad \psi_{1k,2} = -\frac{4y_2}{|y|^2} + O\left(\frac{1}{|y|^3}\right),
\psi_{2k,1} = -\frac{2y_1}{|y|^2} + O\left(\frac{1}{|y|^3}\right), \quad \psi_{2k,2} = \frac{2y_2}{|y|^2} + O\left(\frac{1}{|y|^3}\right),
\psi_{1k,1}^b = \frac{4y_1}{|y|^2} + O\left(\frac{1}{|y|^3}\right), \quad \psi_{1k,2}^b = -\frac{4y_2}{|y|^2} + O\left(\frac{1}{|y|^3}\right),
\psi_{2k,1}^b = -\frac{2y_1}{|y|^2} + O\left(\frac{1}{|y|^3}\right), \quad \psi_{2k,2}^b = \frac{2y_2}{|y|^2} + O\left(\frac{1}{|y|^3}\right),
\psi_{1k,1}^c = -\frac{4y_1}{|y|^2} + O\left(\frac{1}{|y|^3}\right), \quad \psi_{1k,2}^c = \frac{4y_2}{|y|^2} + O\left(\frac{1}{|y|^3}\right),
\psi_{2k,1}^c = \frac{2y_1}{|y|^2} + O\left(\frac{1}{|y|^3}\right), \quad \psi_{2k,2}^c = -\frac{2y_2}{|y|^2} + O\left(\frac{1}{|y|^3}\right),
\]

Integrating by parts, we have

\[
\int_{B_{\frac{1}{2}}} (-\Delta \bar{\eta}_{1k}) \psi_{1k,1}' + (-\Delta \bar{\eta}_{2k}) \psi_{2k,1}'
\]

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\[
= \int_{\frac{\partial B}{\tau_n}} (-\Delta \psi_{1,k,1}^c) \tilde{\eta}_{1k} + (-\Delta \psi_{2,k,1}^c) \tilde{\eta}_{2k} + \int_{\partial B \setminus \frac{\partial B}{\tau_n}} \left( \tilde{\eta}_{1k} \frac{\partial \psi_{1,k,1}^c}{\partial \nu} - \psi_{1,k,1}^c \frac{\partial \tilde{\eta}_{1k}}{\partial \nu} \right) \\
+ \int_{\partial B \setminus \frac{\partial B}{\tau_n}} \left( \tilde{\eta}_{2k} \frac{\partial \psi_{2,k,1}^c}{\partial \nu} - \psi_{2,k,1}^c \frac{\partial \tilde{\eta}_{2k}}{\partial \nu} \right).
\]  

(81)

By the previous estimate on \( \partial B \frac{\partial B}{\tau_n} \), (70) and (71), a straightforward calculation shows that

\[
\int_{\partial B \frac{\partial B}{\tau_n}} \left( \frac{\partial \psi_{1,k,1}^c}{\partial \nu} \tilde{\eta}_{1k} - \frac{\partial \tilde{\eta}_{1k}}{\partial \nu} \psi_{1,k,1}^c \right) \\
= O(\varepsilon_k^2 | \ln \varepsilon_k |)(|\rho_{1,k}^0 - 8\pi| + |\rho_{2,k}^0 - 8\pi|) + O(\varepsilon_k^2), \quad i = 1, 2.
\]

(82)

where cancelation occurs due to the radial symmetry of \( \ln |y| \). On the other hand, we note that, by (73) and (74),

\[
3D_{1k}(y) = (2\tilde{\eta}_{1k} - \tilde{\eta}_{2k}) + 3\nabla Q_{1k}(p_k) \varepsilon_k y + O(\tilde{\eta}_{1k}^2 + \tilde{\eta}_{2k}^2) + O(\varepsilon_k^2 |y|^2), \\
3D_{2k}(y) = (2\tilde{\eta}_{2k} - \tilde{\eta}_{1k}) + 3\nabla Q_{2k}(p_k) \varepsilon_k y + O(\tilde{\eta}_{1k}^2 + \tilde{\eta}_{2k}^2) + O(\varepsilon_k^2 |y|^2).
\]

Then

\[
\int_{B \frac{\partial B}{\tau_n}} 3\rho_{1k} h_{1k}(p_k) e^{V_k} D_{1k}(y) \psi_{1,k,1}^c + \int_{B \frac{\partial B}{\tau_n}} 3\rho_{2k} h_{2k}(p_k) e^{V_k} D_{2k}(y) \psi_{2,k,1}^c \\
= \int_{B \frac{\partial B}{\tau_n}} \rho_{1k} h_{1k}(p_k) e^{V_k} (2\tilde{\eta}_{1k} - \tilde{\eta}_{2k}) \psi_{1,k,1}^c + \int_{B \frac{\partial B}{\tau_n}} \rho_{2k} h_{2k}(p_k) e^{V_k} (2\tilde{\eta}_{2k} - \tilde{\eta}_{1k}) \psi_{2,k,1}^c \\
+ 3\nabla Q_{1k}(p_k) \varepsilon_k \int_{B \frac{\partial B}{\tau_n}} \rho_{1k} h_{1k}(p_k) e^{V_k} \varepsilon_k y \psi_{1,k,1}^c \\
+ 3\nabla Q_{2k}(p_k) \varepsilon_k \int_{B \frac{\partial B}{\tau_n}} \rho_{2k} h_{2k}(p_k) e^{V_k} \varepsilon_k y \psi_{2,k,1}^c \\
+ O(\varepsilon_k^2)(|\rho_{1,k}^0 - 8\pi| + |\rho_{2,k}^0 - 8\pi|) + O(\varepsilon_k^2),
\]

(83)

where Proposition 4.4 is used to estimate \( \tilde{\eta}_{1k}^2 + \tilde{\eta}_{2k}^2 \). The equations of \( \tilde{\eta}_{1k} \) and \( \tilde{\eta}_{2k} \), (81), (82) and (83) give us that

\[
3\nabla Q_{1k}(p_k) \varepsilon_k \int_{B \frac{\partial B}{\tau_n}} \rho_{1k} h_{1k}(p_k) e^{V_k} \varepsilon_k y \psi_{1,k,1}^c \\
+ 3\nabla Q_{2k}(p_k) \varepsilon_k \int_{B \frac{\partial B}{\tau_n}} \rho_{2k} h_{2k}(p_k) e^{V_k} \varepsilon_k y \psi_{2,k,1}^c \\
= O(\varepsilon_k^2 | \ln \varepsilon_k |)(|\rho_{1,k}^0 - 8\pi| + |\rho_{2,k}^0 - 8\pi|) \\
+ O(\varepsilon_k^2)(|\rho_{1,k}^0 - 8\pi| + |\rho_{2,k}^0 - 8\pi|) + O(\varepsilon_k^2).
\]

(84)
Similarly, the above procedure can also be applied to \( (\psi_{i,k}^c, 2 \psi_{2k,1}^c, 3 \psi_{2k,2}^c) \) and then other three equalities can be gotten, which have the same form as (84) just by replacing \((\psi_{1,k,1}^c, 2 \psi_{1,k,1}^c + \psi_{2k,1}^c)\) by \((\psi_{1,k,2}^c, 2 \psi_{1,k,2}^c + \psi_{2k,2}^c)\), etc.

Now we are in position to finish the proof of Proposition 5.1. We need to show that the corresponding coefficient matrix is non-degenerate, from which the proposition follows. Since \( \Delta(2\psi_{1,k,1}^c + \psi_{2k,1}^c) + 3\rho_{1,k}^c h_{1}(p_k) e^{V_1} y_1^1 \psi_{1,k,1}^c = 0 \), one has

\[
3 \int_{B_{\frac{1}{\varepsilon_k}}} \rho_{1,k}^c h_{1}(p_k) e^{V_1} y_1^1 \psi_{1,k,1}^c = - \int_{\partial B_{\frac{1}{\varepsilon_k}}} \Delta(2\psi_{1,k,1}^c + \psi_{2k,1}^c) y_1
\]

\[
= - \int_{\partial B_{\frac{1}{\varepsilon_k}}} \frac{\partial y_1}{\partial \nu} (2\psi_{1,k,1}^c + \psi_{2k,1}^c) - \frac{\partial (2\psi_{1,k,1}^c + \psi_{2k,1}^c)}{\partial \nu} y_1
\]

\[
= -12 \int_{\partial B_{\frac{1}{\varepsilon_k}}} \frac{y_1^2}{|y|^3} + O(\varepsilon_k) = -12\pi + O(\varepsilon_k),
\]

and

\[
3 \int_{B_{\frac{1}{\varepsilon_k}}} \rho_{1,k}^c h_{1}(p_k) e^{V_1} y_2^1 \psi_{1,k,1}^c = - \int_{B_{\frac{1}{\varepsilon_k}}} \Delta(2\psi_{1,k,1}^c + \psi_{2k,1}^c) y_2
\]

\[
= - \int_{\partial B_{\frac{1}{\varepsilon_k}}} \frac{\partial y_2}{\partial \nu} (2\psi_{1,k,1}^c + \psi_{2k,1}^c) - \frac{\partial (2\psi_{1,k,1}^c + \psi_{2k,1}^c)}{\partial \nu} y_2
\]

\[
= O(\varepsilon_k).
\]

Similarly we can prove that

\[
\int_{B_{\frac{1}{\varepsilon_k}}} \rho_{2,k}^c h_{2}(p_k) e^{V_2} y_1^1 \psi_{2k,1}^c = O(\varepsilon_k), \quad \int_{B_{\frac{1}{\varepsilon_k}}} \rho_{2,k}^c h_{2}(p_k) e^{V_2} y_2^1 \psi_{2k,1}^c = O(\varepsilon_k),
\]

\[
\int_{B_{\frac{1}{\varepsilon_k}}} \rho_{1,k}^c h_{1}(p_k) e^{V_1} y_1^1 \psi_{1,k,2}^c = O(\varepsilon_k),
\]

\[
\int_{B_{\frac{1}{\varepsilon_k}}} \rho_{1,k}^c h_{1}(p_k) e^{V_1} y_2^1 \psi_{1,k,2}^c = -12 \int_{\partial B_{\frac{1}{\varepsilon_k}}} \frac{y_1^2}{|y|^3} + O(\varepsilon_k) = -12\pi + O(\varepsilon_k),
\]

\[
\int_{B_{\frac{1}{\varepsilon_k}}} \rho_{2,k}^c h_{2}(p_k) e^{V_2} y_1^1 \psi_{2k,2}^c = O(\varepsilon_k), \quad \int_{B_{\frac{1}{\varepsilon_k}}} \rho_{2,k}^c h_{2}(p_k) e^{V_2} y_2^1 \psi_{2k,2}^c = O(\varepsilon_k),
\]

\[
\int_{B_{\frac{1}{\varepsilon_k}}} \rho_{1,k}^c h_{1}(p_k) e^{V_1} y_1^1 \psi_{1,k,1}^b = O(\varepsilon_k), \quad \int_{B_{\frac{1}{\varepsilon_k}}} \rho_{1,k}^c h_{1}(p_k) e^{V_1} y_2^1 \psi_{1,k,1}^b = O(\varepsilon_k),
\]

\[
\int_{B_{\frac{1}{\varepsilon_k}}} \rho_{2,k}^c h_{2}(p_k) e^{V_2} y_1^1 \psi_{2k,1}^b = -24 \int_{\partial B_{\frac{1}{\varepsilon_k}}} \frac{y_1^2}{|y|^3} + O(\varepsilon_k) = -24\pi + O(\varepsilon_k),
\]

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\[
\int_{B_{\frac{1}{\varepsilon_k}}} \rho_{2k} h_{2k}(p_k) e^{V_k y_2 \psi_{2k,1}^b} = O(\varepsilon_k),
\]
\[
\int_{B_{\frac{1}{\varepsilon_k}}} \rho_{1k} h_{1k}(p_k) e^{V_k y_1 \psi_{1k,2}^b} = O(\varepsilon_k), \quad \int_{B_{\frac{1}{\varepsilon_k}}} \rho_{1k} h_{1k}(p_k) e^{V_k y_2 \psi_{1k,2}^b} = O(\varepsilon_k),
\]
\[
\int_{B_{\frac{1}{\varepsilon_k}}} \rho_{2k} h_{2k}(p_k) e^{V_k y_1 \psi_{2k,2}^b} = O(\varepsilon_k),
\]
\[
\int_{B_{\frac{1}{\varepsilon_k}}} \rho_{2k} h_{2k}(p_k) e^{V_k y_2 \psi_{2k,2}^b} = -24 \int_{\partial B_{\frac{1}{\varepsilon_k}}} \frac{y_2^2}{|y|^3} + O(\varepsilon_k) = -24\pi + O(\varepsilon_k).
\]

The above computation obviously implies the non-degeneracy of the coefficient matrix. The proof is thus completed. \(\square\)

6 Estimates of \(\rho_{i,k,j} - 8\pi\)

In this section we estimate the convergence rate of \(\rho_{1,k,j} \to 8\pi\) and \(\rho_{2,k,j} \to 8\pi\) in terms of the blow-up values.

**Lemma 6.1.** There holds
\[
\rho_{1,k,j} - 8\pi = -\frac{1}{3} \int_{\partial B_{\delta}(p_{k,j})} \frac{\partial \eta_{1k,j}}{\partial \nu} + O(\varepsilon_k^2),
\]
(85)
\[
\rho_{2,k,j} - 8\pi = -\frac{1}{3} \int_{\partial B_{\delta}(p_{k,j})} \frac{\partial \eta_{2k,j}}{\partial \nu} + O(\varepsilon_k^2).
\]
(86)

**Proof.** By the definition of \(\rho_{1,k,j}\) and \(\eta_{k,j}\), we have
\[
3\rho_{1,k,j} = \int_{B_{\delta}(p_{k,j})} 3\rho_{1k} h_{1k} e^{u_{1k}} = -\int_{B_{\delta}(p_k)} \Delta(2\bar{u}_{1k} + \bar{u}_{2k})
\]
\[
= -\int_{B_{\delta}(p_{k,j})} \Delta \eta_{1k,j} - \int_{B_{\delta}(p_{k,j})} \Delta (2\bar{U}_{1k,j} + U_{2k,j})
\]
\[
= -\int_{\partial B_{\delta}(p_{k,j})} \frac{\partial \eta_{1k,j}}{\partial \nu} + \int_{B_{\frac{1}{\varepsilon_k}}} 3\rho_{1k} h_{1k}(p_k) e^{V_{1k,j}}
\]
\[
= 24\pi - \int_{\partial B_{\delta}(p_{k,j})} \frac{\partial \eta_{1k,j}}{\partial \nu} + O(\varepsilon_k^2).
\]
(86) can be proved similarly. \(\square\)

**Proposition 6.2.** We have
\[
\rho_{1,k,j} - 8\pi = C_{1,k,j} \Delta x Q_{1,k,j}(p_{k,j}) \varepsilon_{k,j}^2 \ln \varepsilon_{k,j} + O(\varepsilon_k^2),
\]
\[
\rho_{2,k,j} - 8\pi = C_{2,k,j} \Delta x Q_{2,k,j}(p_{k,j}) \varepsilon_{k,j}^2 \ln \varepsilon_{k,j} + O(\varepsilon_k^2),
\]
where \(C_{1,k,j}\) and \(C_{2,k,j}\) are positive constants uniformly bounded below from 0 and above from \(\infty\).
Proof. We omit the subscript $j$ as in the previous section. Define
\[
\psi_1^{a_1}(y) = \partial_{a_{1k}} V_{1k}(y), \quad \psi_1^{a_2}(y) = \partial_{a_{2k}} V_{2k}(y), \quad \psi_2^{a_1}(y) = \partial_{a_{1k}} V_{2k}(y), \quad \psi_2^{a_2}(y) = \partial_{a_{2k}} V_{2k}(y).
\]
For $|y| \to \infty$,
\[
\psi_1^{a_1} = O(\frac{1}{|y|^2}), \quad \psi_2^{a_1} = O(\frac{1}{|y|^3}),
\]
\[
\psi_1^{a_2} = \frac{2}{a_{1k}} + O(\frac{1}{|y|^2}), \quad \psi_2^{a_2} = \frac{2}{a_{2k}} + O(\frac{1}{|y|^2}),
\]
\[
\partial_{\psi_1^{a_1}} = O(\frac{1}{|y|}), \quad \partial_{\psi_2^{a_1}} = O(\frac{1}{|y|^2}),
\]
\[
\partial_{\psi_1^{a_2}} = \frac{1}{|y|^2}, \quad \partial_{\psi_2^{a_2}} = \frac{1}{|y|^2}.
\]
It is then easy to check that
\[
\int_{\partial B_{\frac{1}{T_k}}} \left( \frac{\partial \psi_1^{a_1}}{\partial \nu} \tilde{\eta}_{1k} - \frac{\partial \tilde{\eta}_{1k}}{\partial \nu} \psi_1^{a_1} \right) = O(\varepsilon_k^2 \ln \varepsilon_k) (|\rho_1^{a_1} - 8\pi| + |\rho_2^{a_1} - 8\pi|) + O(\varepsilon_k^2),
\]
and
\[
\int_{\partial B_{\frac{1}{T_k}}} \left( \frac{\partial \psi_2^{a_2}}{\partial \nu} \tilde{\eta}_{2k} - \frac{\partial \tilde{\eta}_{2k}}{\partial \nu} \psi_2^{a_2} \right) = -\frac{2}{a_{1k}} \int_{\partial B_{\frac{1}{T_k}}} \frac{\partial \tilde{\eta}_{2k}}{\partial \nu} + O(\varepsilon_k^2 \ln \varepsilon_k) (|\rho_1^{a_2} - 8\pi| + |\rho_2^{a_2} - 8\pi|) + O(\varepsilon_k^2).
\]
So we have
\[
\int_{B_{\frac{1}{T_k}}} (-\Delta \tilde{\eta}_{1k}) \psi_1^{a_1} + (-\Delta \tilde{\eta}_{2k}) \psi_2^{a_1}
\]
\[
= \int_{B_{\frac{1}{T_k}}} (-\Delta \psi_1^{a_1}) \tilde{\eta}_{1k} + (-\Delta \psi_2^{a_1}) \tilde{\eta}_{2k}
\]
\[
+ \int_{\partial B_{\frac{1}{T_k}}} \left( \frac{\partial \psi_1^{a_1}}{\partial \nu} \tilde{\eta}_{1k} - \frac{\partial \tilde{\eta}_{1k}}{\partial \nu} \psi_1^{a_1} \right) + \int_{\partial B_{\frac{1}{T_k}}} \left( \frac{\partial \psi_2^{a_2}}{\partial \nu} \tilde{\eta}_{2k} - \frac{\partial \tilde{\eta}_{2k}}{\partial \nu} \psi_2^{a_2} \right)
\]
\[
= \int_{B_{\frac{1}{T_k}}} (-\Delta \psi_1^{a_1}) \tilde{\eta}_{1k} + (-\Delta \psi_2^{a_2}) \tilde{\eta}_{2k} - \frac{2}{a_{1k}} \int_{\partial B_{\frac{1}{T_k}}} \frac{\partial \tilde{\eta}_{2k}}{\partial \nu} + O(\varepsilon_k^2)
\]
\[
+ O(\varepsilon_k^2 \ln \varepsilon_k) (|\rho_1^{a_1} - 8\pi| + |\rho_2^{a_1} - 8\pi|).
\]
On the other hand, using Proposition 5.1, we obtain that

\[ 3D_{1k}(y) = (2\bar{\eta}_{1k} - \bar{\eta}_{2k}) + \frac{3}{2} \nabla^2 Q_{1k}(p_k) \varepsilon_k^2 y^2 + \bar{g}_1(y), \]  

\[ 3D_{2k}(y) = (2\bar{\eta}_{2k} - \bar{\eta}_{1k}) + \frac{3}{2} \nabla^2 Q_{2k}(p_k) \varepsilon_k^2 y^2 + \bar{g}_2(y), \]

where

\[
\bar{g}_i(y) = O \left( \left| \nabla Q_{ik}(p_{k,j}) \varepsilon_k^3 |y| + \bar{\eta}_{1k}^2 + \bar{\eta}_{2k}^2 + \varepsilon_k^{1+\beta} |y|^{2+\beta} \right) \right.
\]
\[ = O(\varepsilon_k^{1+\gamma} |y|(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|) + O(\varepsilon_k^{2+\gamma} |y|(|\rho_{1k}^0 - 8\pi|^2 + |\rho_{2k}^0 - 8\pi|^2))
\]
\[ + O \left( \varepsilon_k^2 |y| + \bar{\eta}_{1k}^2 + \bar{\eta}_{2k}^2 + \varepsilon_k^{2+\beta} |y|^{2+\beta} \right) \]
\[ = O(\varepsilon_k^2 (1 + |y|) + O(\varepsilon_k^{2+\beta} |y|^{2+\beta}) + O(\varepsilon_k^{2+\gamma}) (1 + |y|)^{2+\gamma} (|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|). \]

It is easy to see that

\[ \int_{B_{\frac{1}{\varepsilon_k}}} e^{V_{1k} \psi_{1k}^a \bar{g}_i} = O(\varepsilon_k^{2+\gamma}) (|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|) + O(\varepsilon_k^2). \]

By the equations of \( \bar{\eta}_{1k} \) and \( \bar{\eta}_{2k} \), it holds that

\[ \int_{B_{\frac{1}{\varepsilon_k}}} (-\Delta \bar{\eta}_{1k}) \psi_{1k}^a + (-\Delta \bar{\eta}_{2k}) \psi_{2k}^a \]
\[ = \int_{B_{\frac{1}{\varepsilon_k}}} 3\rho_{1k} h_{1k}(p_k)e^{V_{1k} D_{1k} \psi_{1k}^a} + 3\rho_{2k} h_{2k}(p_k)e^{V_{2k} D_{2k} \psi_{2k}^a}. \]  

From (90) we have that, since \( \psi_{1k}^a = O(\frac{1}{|y|}) \) as \( |y| \to \infty \),

\[ \int_{B_{\frac{1}{\varepsilon_k}}} 3\rho_{1k} h_{1k}(p_k)e^{V_{1k} D_{1k} \psi_{1k}^a} = \int_{B_{\frac{1}{\varepsilon_k}}} \rho_{1k} h_{1k}(p_k)e^{V_{1k} \psi_{1k}^a (2\bar{\eta}_{1k} - \bar{\eta}_{2k}) + O(\varepsilon_k^2)} + O(\varepsilon_k^{2+\gamma}) (|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|). \]

From (91) we know that

\[ \int_{B_{\frac{1}{\varepsilon_k}}} 3\rho_{2k} h_{2k}(p_k)e^{V_{2k} D_{2k} \psi_{2k}^a} \]
\[ = \int_{B_{\frac{1}{\varepsilon_k}}} \rho_{2k} h_{2k}(p_k)e^{V_{2k} \psi_{2k}^a (2\bar{\eta}_{2k} - \bar{\eta}_{1k}) + O(\varepsilon_k^{2+\gamma}) (|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|)} + \frac{\varepsilon_k^2}{2} \int_{B_{\frac{1}{\varepsilon_k}}} 3\rho_{2k} h_{2k}(p_k)e^{V_{2k} \psi_{2k}^a \nabla^2 Q_{2k}(p_k)y^2 + O(\varepsilon_k^2)}. \]  

We next claim that
\[
\int_{B_{\frac{R}{\varepsilon}}} 3\rho_{2k} h_{2k}(p_k)e^{V_{2k}\psi_{2k}^0} Q_{2k}(p_k) y^2 = \frac{\Delta x Q_{2k}(p_k)}{2} \int_{B_{\frac{R}{\varepsilon}}} 3\rho_{2k} h_{2k}(p_k)e^{V_{2k}\psi_{2k}^0} |y|^2 + O(1). \tag{95}
\]

Indeed, direct calculations show that
\[
\int_{B_{\frac{R}{\varepsilon}}} 3\rho_{2k} h_{2k}(p_k)e^{V_{2k}\psi_{2k}^0} (y_1^2 - y_2^2) = -\int_{B_{\frac{R}{\varepsilon}}} \Delta \psi_{2k}^0(y_1^2 - y_2^2) = O(1) \tag{96}
\]
and
\[
\int_{B_{\frac{R}{\varepsilon}}} 3\rho_{2k} h_{2k}(p_k)e^{V_{2k}\psi_{2k}^0} y_1 y_2 = -\int_{B_{\frac{R}{\varepsilon}}} \Delta \psi_{2k}^0 y_1 y_2 = O(1). \tag{97}
\]

The claim (95) follows from (96) and (97). Therefore (94) implies that
\[
\int_{B_{\frac{R}{\varepsilon}}} \rho_{2k} h_{2k}(p_k)e^{V_{2k}} D_{2k} \psi_{2k}^0 = \\
\int_{B_{\frac{R}{\varepsilon}}} \rho_{2k} h_{2k}(p_k)e^{V_{2k}} \psi_{2k}^0(2\eta_{2k} - \eta_{1k}) + O(\varepsilon_k^2)(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|) + \\
\int_{B_{\frac{R}{\varepsilon}}} \frac{\Delta x Q_{2k}(p_k)}{4} \int_{B_{\frac{R}{\varepsilon}}} 3\rho_{2k} h_{2k}(p_k)e^{V_{2k}\psi_{2k}^0} |y|^2 + O(\varepsilon_k^2). \tag{98}
\]

Finally combining (89), (92), (93) and (98), we obtain that
\[
-\frac{2}{\alpha_k} \int_{\partial B_{\frac{R}{\varepsilon}}} \frac{\partial \eta_{2k}}{\partial \nu} = \frac{\varepsilon_k^2 \Delta x Q_{2k}(p_k)}{4} \int_{B_{\frac{R}{\varepsilon}}} 3\rho_{2k} h_{2k}(p_k)e^{V_{2k}\psi_{2k}^0} |y|^2 + O(\varepsilon_k^2)(|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|) + O(\varepsilon_k^2). \tag{99}
\]

Furthermore we perform a similar procedure by replacing \((\psi_{1k}^0, \psi_{2k}^0)\) by \((\psi_{1k}^0, \psi_{2k}^0)\). It is easy to see that

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\[
\int_{B_{\frac{1}{2k}}} (-\Delta \tilde{\eta}_{1k}) \psi_{1k}^{a_1} + (-\Delta \tilde{\eta}_{2k}) \psi_{2k}^{a_2} = \int_{B_{\frac{1}{2k}}} (-\Delta \psi_{1k}^{a_1}) \tilde{\eta}_{1k} + (-\Delta \psi_{2k}^{a_2}) \tilde{\eta}_{2k} + \frac{2}{a_{2k}} \int_{\partial B_{\frac{1}{2k}}} \frac{\partial \tilde{\eta}_{1k}}{\partial \nu} + \frac{2}{a_{2k}} \int_{\partial B_{\frac{1}{2k}}} \frac{\partial \tilde{\eta}_{2k}}{\partial \nu} + O(\varepsilon_k^2),
\]

\[
\int_{B_{\frac{1}{2k}}} 3\rho_{1k} h_{1k}(p_k) e^{V_{1k}} D_{1k} \psi_{1k}^{a_1}
\]

\[
= \int_{B_{\frac{1}{2k}}} \rho_{1k} h_{1k}(p_k) e^{V_{1k}} \psi_{1k}^{a_1} (2\tilde{\eta}_{1k} - \tilde{\eta}_{2k}) + O(\varepsilon_k^2) (|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|)
\]

\[
+ \frac{\varepsilon_k^2}{4} \int_{B_{\frac{1}{2k}}} 3\rho_{1k} h_{1k}(p_k) e^{V_{1k}} \psi_{1k}^{a_1} |y|^2 + O(\varepsilon_k^2)
\]
and

\[
\int_{B_{\frac{1}{2k}}} 3\rho_{2k} h_{2k}(p_k) e^{V_{2k}} D_{2k} \psi_{2k}^{a_2}
\]

\[
= \int_{B_{\frac{1}{2k}}} \rho_{2k} h_{2k}(p_k) e^{V_{2k}} \psi_{2k}^{a_2} (2\tilde{\eta}_{2k} - \tilde{\eta}_{1k}) + O(\varepsilon_k^2) (|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|)
\]

\[
+ \frac{\varepsilon_k^2}{4} \int_{B_{\frac{1}{2k}}} 3\rho_{2k} h_{2k}(p_k) e^{V_{2k}} \psi_{2k}^{a_2} |y|^2 + O(\varepsilon_k^2).
\]

So we get that

\[
- \frac{2}{a_{2k}} \int_{\partial B_{\frac{1}{2k}}} \frac{\partial \tilde{\eta}_{1k}}{\partial \nu} + \frac{2}{a_{2k}} \int_{\partial B_{\frac{1}{2k}}} \frac{\partial \tilde{\eta}_{2k}}{\partial \nu}
\]

\[
= \frac{\varepsilon_k^2}{4} \int_{B_{\frac{1}{2k}}} 3\rho_{1k} h_{1k}(p_k) e^{V_{1k}} \psi_{1k}^{a_1} |y|^2
\]

\[
+ \frac{\varepsilon_k^2}{4} \int_{B_{\frac{1}{2k}}} 3\rho_{2k} h_{2k}(p_k) e^{V_{2k}} \psi_{2k}^{a_2} |y|^2
\]

\[
+ O(\varepsilon_k^2) (|\rho_{1k}^0 - 8\pi| + |\rho_{2k}^0 - 8\pi|) + O(\varepsilon_k^2) \quad \text{(100)}.
\]

It follows from (99) and (100) that

\[
- \frac{2}{a_{2k}} \int_{\partial B_{\frac{1}{2k}}} \frac{\partial \tilde{\eta}_{1k}}{\partial \nu} = \frac{\varepsilon_k^2}{4} \int_{B_{\frac{1}{2k}}} 3\rho_{1k} h_{1k}(p_k) e^{V_{1k}} \psi_{1k}^{a_1} |y|^2
\]

\[
+ \frac{\varepsilon_k^2}{4} \int_{B_{\frac{1}{2k}}} 3\rho_{2k} h_{2k}(p_k) e^{V_{2k}} \psi_{2k}^{a_2} \left( \frac{a_{1k}}{a_{2k}} \phi_{1k}^{a_1} + \psi_{2k}^{a_2} \right) |y|^2
\]

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\[ + O(\varepsilon_k^2)\epsilon_k^2 \left| \frac{\rho_1^0 - 8\pi}{\rho_1^0 - 8\pi} \right| + O(\varepsilon_k^2) \]

\[ = \frac{\epsilon_k^2}{4} \Delta_k [Q_{1k} (p_k)] + O(\varepsilon_k^2) \]

\[ + O(\varepsilon_k^2)\left| \frac{\rho_1^0 - 8\pi}{\rho_1^0 - 8\pi} \right| + O(\varepsilon_k^2), \quad (101) \]

where \( \Delta_k = \frac{\rho_1^0 - 8\pi}{\rho_1^0 - 8\pi} \) is used in the last equality.

Since \( e^{V_{1k}} \sim |y|^{-4} \) for \( |y| \) large, Lemma 6.1, (99) and (101) give that

\[ \rho_1^0 - 8\pi = C_1 \Delta_k [Q_{1k} (p_k)] \epsilon_k^2 \left| \ln \epsilon_k \right| + O(\varepsilon_k^2)\]

\[ \rho_2^0 - 8\pi = C_2 \Delta_k [Q_{2k} (p_k)] \epsilon_k^2 \left| \ln \epsilon_k \right| + O(\varepsilon_k^2), \]

where \( C_{ik} \) is a constant such that \( 0 < C_1 < C_{ik} < C_2 < \infty \) \((i = 1, 2)\). Obviously we also have that

\[ \rho_1^0 - 8\pi = C_1 \Delta_k [Q_{1k} (p_k)] \epsilon_k^2 \left| \ln \epsilon_k \right| + O(\varepsilon_k^2), \]

\[ \rho_2^0 - 8\pi = C_2 \Delta_k [Q_{2k} (p_k)] \epsilon_k^2 \left| \ln \epsilon_k \right| + O(\varepsilon_k^2). \]

The proof is complete. \( \square \)

By Proposition 6.2, we now have a sharper estimate for \( \nabla Q_{1k,j} (p_k,j) \):

\[ |\nabla Q_{1k,j} (p_k,j)| + |\nabla Q_{2k,j} (p_k,j)| = O(\varepsilon_k). \]

Hence Lemma 4.1, Lemma 4.3 and Proposition 4.4 hold for any \( \tau \in (0, 1) \).

## 7 Estimates for \( \nabla^2 Q_{1k,j} \) and \( \nabla^2 Q_{2k,j} \)

In this section we make use of the remaining two kernel elements to obtain the estimates on the second derivatives of the \( Q_{ik,j} \)'s. For convenience we still omit the subscript \( j \).

**Proposition 7.1.** It holds that

\[ 6\pi (\partial_{11} - \partial_{22}) [Q_{2k} (p_k) - Q_{1k} (p_k)] + \frac{T_{1k,1}}{4} \Delta_k [Q_{1k} (p_k)] + \frac{T_{2k,1}}{4} \Delta_k [Q_{2k} (p_k)] = O(\varepsilon_k), \]

\[ 12\pi \partial_{ij} [Q_{2k} (p_k) - Q_{1k} (p_k)] + \frac{T_{1k,2}}{4} \Delta_k [Q_{1k} (p_k)] + \frac{T_{2k,2}}{4} \Delta_k [Q_{2k} (p_k)] = O(\varepsilon_k), \]

where \( T_{1k,1}, T_{2k,1}, T_{1k,2} \) and \( T_{2k,2} \) are four constants defined by

\[ T_{1k,1} = \int_{\mathbb{R}^2} 3p_{1k} h_{1k} (p_k) e^{V_{1k} \psi_{1k,1}^d} |y|^2, \]

\[ T_{2k,1} = \int_{\mathbb{R}^2} 3p_{2k} h_{2k} (p_k) e^{V_{2k} \psi_{2k,1}^d} |y|^2, \]

\[ T_{1k,2} = \int_{\mathbb{R}^2} 3p_{1k} h_{1k} (p_k) e^{V_{1k} \psi_{1k,2}^d} |y|^2, \]

\[ T_{2k,2} = \int_{\mathbb{R}^2} 3p_{2k} h_{2k} (p_k) e^{V_{2k} \psi_{2k,2}^d} |y|^2. \]
\[ T_{2k,2} = \int_{\mathbb{R}^3} 3\rho_{2k} h_{2k}(p_k) e^{V_{2k}\psi_{2k,2}^d} |y|^2. \]

and

\[ \psi_{1k,1}^d(y) = (\partial_{\alpha_k} + \partial_{\beta_k}) V_{1k}(y), \quad \psi_{2k,1}^d(y) = (\partial_{\alpha_k} - \partial_{\beta_k}) V_{1k}(y), \]

\[ \psi_{1k,2}^d(y) = i(\partial_{\alpha_k} - \partial_{\beta_k}) V_{1k}(y), \quad \psi_{2k,2}^d(y) = i(\partial_{\alpha_k} + \partial_{\beta_k}) V_{2k}(y). \]

**Remark 7.2.** These constants \( T_{ik,j}, i, j = 1, 2 \) may be nonzero. See the remark at the end of this section.

**Proof.** As \( |y| \to \infty \), a Taylor expansion gives that

\[ \psi_{1k,1}^d = -\frac{6(y_1^2 - y_2^2)}{|y|^4} + O\left(\frac{1}{|y|^3}\right), \quad \frac{\partial \psi_{1k,1}^d}{\partial \nu} = \frac{12(y_1^2 - y_2^2)}{|y|^6} + O\left(\frac{1}{|y|^4}\right), \]

\[ \psi_{2k,1}^d = \frac{6(y_1^2 - y_2^2)}{|y|^4} + O\left(\frac{1}{|y|^3}\right), \quad \frac{\partial \psi_{2k,1}^d}{\partial \nu} = -\frac{12(y_1^2 - y_2^2)}{|y|^6} + O\left(\frac{1}{|y|^4}\right), \]

\[ \psi_{1k,2}^d = -\frac{12y_1y_2}{|y|^4} + O\left(\frac{1}{|y|^3}\right), \quad \frac{\partial \psi_{1k,2}^d}{\partial \nu} = \frac{24y_1y_2}{|y|^6} + O\left(\frac{1}{|y|^4}\right), \]

\[ \psi_{2k,2}^d = \frac{12y_1y_2}{|y|^4} + O\left(\frac{1}{|y|^3}\right), \quad \frac{\partial \psi_{2k,2}^d}{\partial \nu} = -\frac{24y_1y_2}{|y|^6} + O\left(\frac{1}{|y|^4}\right). \]

Then using the estimate (70) and (71), we have

\[ \int_{B \setminus \bar{h}} (-\Delta \bar{\eta}_{1k}) \psi_{1k,1}^d + (-\Delta \bar{\eta}_{2k}) \psi_{2k,1}^d \]

\[ = \int_{B \setminus \bar{h}} (-\Delta \psi_{1k,1}^d) \bar{\eta}_{1k} + (-\Delta \psi_{2k,1}^d) \bar{\eta}_{2k} + O(\varepsilon_k^2). \quad (102) \]

Since \( h_{1k} \) and \( h_{2k} \) are of \( C^{2,\beta}(\Omega) \), it holds that

\[ 3D_{1k} = (2\bar{\eta}_{1k} - \bar{\eta}_{2k}) + 3\nabla Q_{1k}(p_k)\varepsilon_k y + \frac{3\nabla^2 Q_{1k}(p_k)}{2} \varepsilon_k^2 y^2 + O(\varepsilon_k^{2+\beta})(1 + |y|)^{2+\beta}, \]

\[ 3D_{2k} = (2\bar{\eta}_{2k} - \bar{\eta}_{1k}) + 3\nabla Q_{2k}(p_k)\varepsilon_k y + \frac{3\nabla^2 Q_{2k}(p_k)}{2} \varepsilon_k^2 y^2 + O(\varepsilon_k^{2+\beta})(1 + |y|)^{2+\beta}. \]

Since

\[ \int_{B \setminus \bar{h}} 3\rho_{1k} h_{1k}(p_k) e^{V_{1k}\psi_{1k,1}^d} y = -\int_{B \setminus \bar{h}} \Delta (2\psi_{1k,1}^d + \psi_{2k,1}^d) y, \]

\[ = \int_{B \setminus \bar{h}} \partial_y (2\psi_{1k,1}^d + \psi_{2k,1}^d) - \frac{\partial (2\psi_{1k,1}^d + \psi_{2k,1}^d)}{\partial \nu} y = O(\varepsilon_k), \]

\[ 30 \]
and clearly
\[ \int_{B_{\frac{\epsilon_k}{h}}} 3\rho_{2k} h_{2k}(p_k) e^{V_{1k} \psi_{2k,1}^d} y = O(\epsilon_k), \]
a direct computation shows that
\[ \int_{B_{\frac{\epsilon_k}{h}}} 3\rho_{1k} h_{1k}(p_k) e^{V_{1k} D_{1k}(y) \psi_{1k,1}^d} \]
\[ = \int_{B_{\frac{\epsilon_k}{h}}} \rho_{1k} h_{1k}(p_k) e^{V_{1k} (2\eta_{1k} - \eta_{2k}) \psi_{1k,1}^d} + \frac{\nabla^2 Q_{1k}(p_k)}{2} \epsilon_k^2 \int_{B_{\frac{\epsilon_k}{h}}} 3\rho_{1k} h_{1k}(p_k) e^{V_{1k} y^2 \psi_{1k,1}^d} + O(\epsilon_k^{2+\beta}), \]
and
\[ \int_{B_{\frac{\epsilon_k}{h}}} 3\rho_{2k} h_{2k}(p_k) e^{V_{2k} D_{2k}(y) \psi_{2k,1}^d} \]
\[ = \int_{B_{\frac{\epsilon_k}{h}}} \rho_{2k} h_{2k}(p_k) e^{V_{2k} (2\eta_{2k} - \eta_{1k}) \psi_{2k,1}^d} + \frac{\nabla^2 Q_{2k}(p_k)}{2} \epsilon_k^2 \int_{B_{\frac{\epsilon_k}{h}}} 3\rho_{2k} h_{2k}(p_k) e^{V_{2k} y^2 \psi_{2k,1}^d} + O(\epsilon_k^{2+\beta}). \]

It is obviously that
\[ \nabla^2 Q_{1k}(p_k) y^2 = \frac{1}{2} (\partial_{11} - \partial_{22}) Q_{1k}(p_k)(y_1^2 - y_2^2) + \frac{1}{2} \Delta Q_{1k}(p_k)|y|^2 + 2\partial_{12} Q_{1k}(p_k)y_1 y_2, \]
\[ \nabla^2 Q_{2k}(p_k) y^2 = \frac{1}{2} (\partial_{11} - \partial_{22}) Q_{2k}(p_k)(y_1^2 - y_2^2) + \frac{1}{2} \Delta Q_{2k}(p_k)|y|^2 + 2\partial_{12} Q_{2k}(p_k)y_1 y_2. \]

We can further check that
\[ \int_{B_{\frac{\epsilon_k}{h}}} 3\rho_{1k} h_{1k}(p_k) e^{V_{1k} \psi_{1k,1}^d (y_1^2 - y_2^2)} = - \int_{B_{\frac{\epsilon_k}{h}}} \Delta (2\psi_{1k,1}^d + \psi_{2k,1}^d)(y_1^2 - y_2^2) \]
\[ = \int_{\partial B_{\frac{\epsilon_k}{h}}} \frac{\partial (y_1^2 - y_2^2)}{\partial \nu}(2\psi_{1k,1}^d + \psi_{2k,1}^d) - \frac{\partial (2\psi_{1k,1}^d + \psi_{2k,1}^d)}{\partial \nu}(y_1^2 - y_2^2) \]
\[ = -24 \int_{\partial B_{\frac{\epsilon_k}{h}}} \frac{(y_1^2 - y_2^2)^2}{|y|^6} + O(\epsilon_k) = -24\pi + O(\epsilon_k) \]
\[ (103) \]
and similarly
\[ \int_{B_{\frac{\epsilon_k}{h}}} 3\rho_{1k} h_{1k}(p_k) e^{V_{1k} \psi_{1k,1}^d y_1 y_2} = - \int_{B_{\frac{\epsilon_k}{h}}} \Delta (2\psi_{1k,1}^d + \psi_{2k,1}^d) y_1 y_2 \]
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\[ \int_{\Omega} \frac{2y_1y_2}{|y|} (2\psi_{1,k,1}^d + \psi_{2,k,1}^d) - \frac{\partial(2\psi_{1,k,1}^d + \psi_{2,k,1}^d)}{\partial y} y_1y_2 = O(\varepsilon_k). \]  

Furthermore, \( T_{1,k,1} \) is such that

\[ \int_{B_{\frac{\varepsilon_k^2}{\kappa}}} 3\rho_{1,k}h_{1,k}(p_k)e^{V_{1,k}\psi_{1,k,1}^d}|y|^2 = T_{1,k,1} + O(\varepsilon_k^2). \]  

Thus, (103), (105) and (104) imply that, by symmetry,

\[ \frac{\nabla^2 Q_{1,k}(p_k)}{2} \varepsilon_k^2 \int_{B_{\frac{\varepsilon_k^2}{\kappa}}} 3\rho_{1,k}h_{1,k}(p_k)e^{V_{1,k}y^2\psi_{1,k,1}^d} = -6\pi(\partial_{11} - \partial_{22})Q_{1,k}(p_k)\varepsilon_k^2 + \frac{1}{4}\Delta Q_{1,k}(p_k)T_{1,k,1}\varepsilon_k^2 + O(\varepsilon_k^3). \]

Analogously, we have

\[ \frac{\nabla^2 Q_{2,k}(p_k)}{2} \varepsilon_k^2 \int_{B_{\frac{\varepsilon_k^2}{\kappa}}} 3\rho_{2,k}h_{2,k}(p_k)e^{V_{2,k}y^2\psi_{2,k,1}^d} = 6\pi(\partial_{11} - \partial_{22})Q_{2,k}(p_k)\varepsilon_k^2 + \frac{1}{4}\Delta Q_{2,k}(p_k)T_{2,k,1}\varepsilon_k^2 + O(\varepsilon_k^3), \]

where the constant \( T_{2,k,1} \) satisfies that

\[ \int_{B_{\frac{\varepsilon_k^2}{\kappa}}} 3\rho_{2,k}h_{2,k}(p_k)e^{V_{2,k}\psi_{2,k,1}^d}|y|^2 = T_{2,k,1} + O(\varepsilon_k^2). \]

We conclude that

\[ \int_{B_{\frac{\varepsilon_k^2}{\kappa}}} 3\rho_{1,k}h_{1,k}(p_k)e^{V_{1,k}D_{1,k}(y)\psi_{1,k,1}^d} + \int_{B_{\frac{\varepsilon_k^2}{\kappa}}} 3\rho_{2,k}h_{2,k}(p_k)e^{V_{2,k}D_{2,k}(y)\psi_{2,k,1}^d} \]

\[ = \int_{B_{\frac{\varepsilon_k^2}{\kappa}}} \rho_{1,k}h_{1,k}(p_k)e^{V_{1,k}(2\tilde{\eta}_{1,k} - \tilde{\eta}_{2,k})\psi_{1,k,1}^d} + \int_{B_{\frac{\varepsilon_k^2}{\kappa}}} \rho_{2,k}h_{2,k}(p_k)e^{V_{2,k}(2\tilde{\eta}_{2,k} - \tilde{\eta}_{1,k})\psi_{2,k,1}^d} \]

\[ + 6\pi(\partial_{11} - \partial_{22})[Q_{2,k}(p_k) - Q_{1,k}(p_k)]\varepsilon_k^2 + \left[ \frac{T_{1,k}}{4}\Delta Q_{1,k}(p_k) + \frac{T_{2,k}}{4}\Delta Q_{2,k}(p_k) \right] \varepsilon_k^2 + O(\varepsilon_k^{2+\beta}). \]  

Using (102) and (106), we get that

\[ 6\pi(\partial_{11} - \partial_{22})[Q_{2,k}(p_k) - Q_{1,k}(p_k)] + \frac{T_{1,k,1}}{4}\Delta Q_{1,k}(p_k) + \frac{T_{2,k,1}}{4}\Delta Q_{2,k}(p_k) = O(\varepsilon_k^\beta). \]
Repeating the above procedure and using
\[
\int_{B_{\frac{1}{\epsilon}}^{+}} 3\rho_{1k} h_{1k}(p_{k}) e^{V_{2k}(y)} \psi^{d}_{1k,2} (y_{1}^{2} - y_{2}^{2}) = O(\epsilon_{k}),
\]
\[
\int_{B_{\frac{1}{\epsilon}}^{+}} 3\rho_{1k} h_{1k}(p_{k}) e^{V_{2k}(y)} \psi^{d}_{1k,2} |y|^{2} = T_{1k,2} + O(\epsilon_{k}^{2}),
\]
\[
\int_{B_{\frac{1}{\epsilon}}^{+}} 3\rho_{1k} h_{1k}(p_{k}) e^{V_{2k}(y)} \psi^{d}_{1k,2} y_{1} y_{2} = -12\pi + O(\epsilon_{k}),
\]
\[
\int_{B_{\frac{1}{\epsilon}}^{+}} 3\rho_{2k} h_{2k}(p_{k}) e^{V_{2k}(y)} \psi^{d}_{2k,2} (y_{1}^{2} - y_{2}^{2}) = O(\epsilon_{k}),
\]
\[
\int_{B_{\frac{1}{\epsilon}}^{+}} 3\rho_{2k} h_{2k}(p_{k}) e^{V_{2k}(y)} \psi^{d}_{2k,2} |y|^{2} = T_{2k,2} + O(\epsilon_{k}^{2}),
\]
\[
\int_{B_{\frac{1}{\epsilon}}^{+}} 3\rho_{2k} h_{2k}(p_{k}) e^{V_{2k}(y)} \psi^{d}_{2k,2} y_{1} y_{2} = 12\pi + O(\epsilon_{k}),
\]
we obtain that
\[
\int_{B_{\frac{1}{\epsilon}}^{+}} (-\Delta \tilde{h}_{k}) \psi^{d}_{1k,2} + (-\Delta \tilde{h}_{2k}) \psi^{d}_{2k,2}
\]
\[
= \int_{B_{\frac{1}{\epsilon}}^{+}} (-\Delta \psi^{d}_{1k,2}) \tilde{h}_{1k} + (-\Delta \psi^{d}_{2k,2}) \tilde{h}_{2k} + O(\epsilon_{k}^{3}),
\]
and
\[
\int_{B_{\frac{1}{\epsilon}}^{+}} 3\rho_{1k} h_{1k}(p_{k}) e^{V_{2k}(y)} \psi^{d}_{1k,2} + \int_{B_{\frac{1}{\epsilon}}^{+}} 3\rho_{2k} h_{2k}(p_{k}) e^{V_{2k}(y)} \psi^{d}_{2k,2}
\]
\[
= \int_{B_{\frac{1}{\epsilon}}^{+}} \rho_{1k} h_{1k}(p_{k}) e^{V_{2k}(2\tilde{h}_{k} - \tilde{h}_{2k})} \psi^{d}_{1k,2} + \int_{B_{\frac{1}{\epsilon}}^{+}} \rho_{2k} h_{2k}(p_{k}) e^{V_{2k}(2\tilde{h}_{2k} - \tilde{h}_{1k})} \psi^{d}_{2k,2}
\]
\[
+ 12\pi \delta_{12} [Q_{2k}(p_{k}) - Q_{1k}(p_{k})] \epsilon_{k}^{2} + \left[ \frac{T_{1k,2}}{4} \Delta Q_{1k}(p_{k}) + \frac{T_{2k,2}}{4} \Delta Q_{2k}(p_{k}) \right] \epsilon_{k}^{2}
\]
\[
+ O(\epsilon_{k}^{2+\beta}).
\]
Therefore we have
\[
12\pi \delta_{12} [Q_{2k}(p_{k}) - Q_{1k}(p_{k})] + \frac{T_{1k,2}}{4} \Delta Q_{1k}(p_{k}) + \frac{T_{2k,2}}{4} \Delta Q_{2k}(p_{k}) = O(\epsilon_{k}^{\beta}).
\]
The proof is concluded. \(\square\)
Remark 7.3. The coefficients $T_{1k,j}$ may be nonzero. Assuming $b = c = 0$ and $d \in \mathbb{R}$, we compute $T_{1,1}$ for $|d|$ small. We drop the dependence on $k$ and $d$. Then

$$2\psi_{1,1} + \psi_{2,1} = \frac{y_1^2 - y_2^2 + d}{a_1^2 + a_2^2|y|^2 + |y + d|^2}.$$ 

Then it is easy to see that

$$T_{1,1} \sim \int_{R^2} e^{Vi} \psi_{1,1}|y|^2 = \lim_{R \to +\infty} \int_{B_R(0)} (-\Delta (2\psi_{1,1} + \psi_{2,1})) |y|^2$$

$$= -4 \lim_{R \to +\infty} \int_{B_R(0)} (2\psi_{1,1} + \psi_{2,1}),$$

where

$$\int_{B_R(0)} (2\psi_{1,1} + \psi_{2,1}) = \int_{B_R(0)} \frac{y_1^2 - y_2^2}{a_1^2 + a_2^2|y|^2 + |y + d|^2} + d \int_{B_R(0)} \frac{1}{a_1^2 + a_2^2|y|^2 + |y|^4} + O(d^2)

= \int_{B_R(0)} \frac{y_1^2 - y_2^2}{a_1^2 + a_2^2|y|^2 + |y|^4 |y + d|^2} - d \int_{B_R(0)} \frac{1}{a_1^2 + a_2^2|y|^2 + |y|^4} + O(d^2)

= \int_{B_R(0)} \frac{-2d(y_1^2 - y_2^2)^2}{(a_1^2 + a_2^2|y|^2 + |y|^4)^2} + d \int_{B_R(0)} \frac{1}{a_1^2 + a_2^2|y|^2 + |y|^4} + O(d^2)

= d \int_{B_R(0)} \frac{a_1^2 + a_2^2|y|^2}{(a_1^2 + a_2^2|y|^2 + |y|^2|^2)^2} + O(d^2).$$

Thus $\frac{T_{1,1}}{d}$ approaches a nonzero constant as $d \to 0$.

8 Proof of Theorem 1.2

Applying Proposition 6.2 to Proposition 5.1 leads to

$$\nabla Q_{1k,j}(p_{k,j}) = O(\varepsilon_k), \quad \nabla Q_{2k,j}(p_{k,j}) = O(\varepsilon_k),$$

which implies that

$$8\pi \nabla_x H(p_{k,j},p_{k,j}) + 8\pi \sum_{j=1}^{m} \nabla_x G(p_{k,j},p_{k,t}) + \nabla \ln h_{1k}(p_{k,j}) = O(\varepsilon_k),$$

$$8\pi \nabla_x H(p_{k,j},p_{k,j}) + 8\pi \sum_{j=1}^{m} \nabla_x G(p_{k,j},p_{k,t}) + \nabla \ln h_{2k}(p_{k,j}) = O(\varepsilon_k).$$

This proves (24)-(25).

Similarly from Proposition 7.1 we have

$$6\pi (\partial_{11} - \partial_{22}) [\ln h_{2k}(p_{k,j}) - \ln h_{1k}(p_{k,j})]$$

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\[ T_{1k,1}^j \Delta \ln h_{1k}(p_{k,j}) + T_{2k,1}^j \Delta \ln h_{2k}(p_{k,j}) = O(\varepsilon_k^\beta), \]

\[ 12 \pi \partial_{12} [\ln h_{2k}(p_{k,j}) - \ln h_{1k}(p_{k,j})] \]

\[ + T_{1k,2}^j \Delta \ln h_{1k}(p_{k,j}) + T_{2k,2}^j \Delta \ln h_{2k}(p_{k,j}) = O(\varepsilon_k^\beta), \]

which proves (26)-(27).

It remains to estimate \( \rho_{1k} - 8m \pi \) and \( \rho_{2k} - 8m \pi \). Recall that

\[ \rho_{ik} = \sum_{j=1}^m \rho_{ik,j} + O(\varepsilon_k^2) \quad \text{for } i = 1, 2. \]

Noting that \( \Delta_x Q_{ik,j}(p_{k,j}) = \Delta \ln h_{ik}(p_{k,j}) \) \( (i = 1, 2) \) and using Proposition 6.2, we easily have that

\[ \rho_{1k} - 8m \pi = \sum_{j=1}^m C_{1k,j} \Delta \ln h_{1k}(p_{k,j}) \varepsilon_{k,j}^2 | \ln \varepsilon_{k,j} | + O(\varepsilon_k^2), \]

\[ \rho_{2k} - 8m \pi = \sum_{j=1}^m C_{2k,j} \Delta \ln h_{2k}(p_{k,j}) \varepsilon_{k,j}^2 | \ln \varepsilon_{k,j} | + O(\varepsilon_k^2). \]

Hence (22)-(23) are established.

This completes the proof of Theorem 1.1.

9 The Case on a Surface: Proof of Theorem 1.1

In this section, we consider the following Toda system of SU(3)

\[ \begin{cases} 
-\Delta_g u_{1k} = 2\rho_{1k} \left( \frac{h_{1k} e^{u_{1k}}}{f_M h_{1k} e^{u_{1k}} \ln h_{1k} e^{u_{1k}}} - 1 \right) - \rho_{2k} \left( \frac{h_{2k} e^{u_{2k}}}{f_M h_{2k} e^{u_{2k}} \ln h_{2k} e^{u_{2k}}} - 1 \right) & \text{on } M, \\
-\Delta_g u_{2k} = 2\rho_{2k} \left( \frac{h_{2k} e^{u_{2k}}}{f_M h_{2k} e^{u_{2k}} \ln h_{2k} e^{u_{2k}}} - 1 \right) - \rho_{1k} \left( \frac{h_{1k} e^{u_{1k}}}{f_M h_{1k} e^{u_{1k}} \ln h_{1k} e^{u_{1k}}} - 1 \right) & \text{on } M, \end{cases} \tag{107} \]

where \((M, g)\) is a closed Riemann surface, \( \Delta_g \) is the Laplace-Beltrami operator. Here we normalize the volume as \( |M| = 1 \). In this system, \( \rho_{1k} \) and \( \rho_{2k} \) are two constants, \( h_{1k}(x) \) and \( h_{2k}(x) \) are two positive functions converging to \( h_1(x) \) and \( h_2(x) \) respectively in \( C^2(M) \) as \( k \to \infty \).

Let \( \bar{u}_{1k}, \bar{u}_{2k}, \varepsilon_{k,j} \) and other symbols be defined as before. Set

\[ w_{1k} = 2u_{1k} + u_{2k} - 3 \sum_{j=1}^m \rho_{1k,j} G(x, p_{k,j}) - 2 \bar{u}_{1k} - \bar{u}_{2k}, \]

\[ w_{2k} = u_{1k} + 2u_{2k} - 3 \sum_{j=1}^m \rho_{2k,j} G(x, p_{k,j}) - \bar{u}_{1k} - 2 \bar{u}_{2k}, \]
where $\bar{u}_{1k}, \bar{u}_{2k}$ are the averages of $u_{1k}, u_{2k}$, and $G(x, p)$ is the Green function of $\Delta_g$ on $M$ with singularity at $p$. Then by the same method, we have the similar estimates as in Lemma 3.4.

Lemma 9.1. It holds that, for $i = 1, 2$,

$$|w_{ik}| + |\nabla w_{ik}| = O(\varepsilon_k) \quad \text{for } x \in M \setminus \bigcup_{j=1}^{n} B_{\delta_j}(p_j).$$

Recall that

$$\begin{cases}
\Delta_g \bar{u}_{1k} + 2\rho_{1k} (h_{1k} e^{\bar{u}_{1k}} - 1) - \rho_{2k} (h_{2k} e^{\bar{u}_{2k}} - 1) = 0 & \text{on } M, \\
\Delta_g \bar{u}_{2k} - \rho_{1k} (h_{1k} e^{\bar{u}_{1k}} - 1) + 2\rho_{2k} (h_{2k} e^{\bar{u}_{2k}} - 1) = 0 & \text{on } M.
\end{cases} \quad (108)$$

Since the computation from Section 4 to Section 8 is local. We introduce some notation for local computation. Note that isothermal coordinates always exist on Riemann surfaces. When (108) is considered locally, it is convenient to introduce a local coordinate $x$ (still denoted by $x$) such that $p_{k,j}$ has the coordinate 0 and the metric $g_{ij} = e^{\phi} \delta_{ij}$ with $\phi(0) = 0$ and $\nabla \phi(0) = 0$. In this case, (108) is reduced to

$$\begin{cases}
\Delta \bar{u}_{1k} + 2\rho_{1k} e^{\phi} (h_{1k} e^{\bar{u}_{1k}} - 1) - \rho_{2k} e^{\phi} (h_{2k} e^{\bar{u}_{2k}} - 1) = 0 & \text{in } B_{\delta}(0), \\
\Delta \bar{u}_{2k} - \rho_{1k} e^{\phi} (h_{1k} e^{\bar{u}_{1k}} - 1) + 2\rho_{2k} e^{\phi} (h_{2k} e^{\bar{u}_{2k}} - 1) = 0 & \text{in } B_{\delta}(0),
\end{cases}$$

where $\Delta$ stands for the Laplacian in $\mathbb{R}^2$.

Furthermore, we set

$$\bar{u}_{1k}(x) = \bar{u}_{1k} - (2\rho_{1k} - \rho_{2k}) f_k(x) \quad \text{and} \quad \bar{u}_{2k}(x) = \bar{u}_{2k} - (2\rho_{2k} - \rho_{1k}) f_k(x),$$

where the function $f_k$ is defined by

$$\begin{cases}
\Delta f_k = e^{\phi} & \text{for } |x| \leq \delta_0, \\
f_k(0) = 0, \quad \nabla f_k(0) = 0.
\end{cases}$$

Clearly $(\bar{u}_{1k}, \bar{u}_{2k})$ satisfies that

$$\begin{cases}
\Delta \bar{u}_{1k} + 2\rho_{1k} \bar{h}_{1k} e^{\bar{u}_{1k}} - \rho_{2k} \bar{h}_{2k} e^{\bar{u}_{2k}} = 0 & \text{for } |x| \leq \delta_0, \\
\Delta \bar{u}_{2k} - \rho_{1k} \bar{h}_{1k} e^{\bar{u}_{1k}} + 2\rho_{2k} \bar{h}_{2k} e^{\bar{u}_{2k}} = 0 & \text{for } |x| \leq \delta_0, \quad (109)
\end{cases}$$

where

$$\bar{h}_{1k}(x) = e^{\phi} h_{1k} e^{(2\rho_{1k} - \rho_{2k}) f_k}, \quad \bar{h}_{2k}(x) = e^{\phi} h_{2k} e^{(2\rho_{2k} - \rho_{1k}) f_k}.$$ 

Thus the similar proceeding from Section 4 to Section 8 can be carried out to (109). Note that now

$$Q_{1k,j}(x) = 2\bar{G}_{1k,j}(x) - \bar{G}_{2k,j}(x) + \ln \bar{h}_{1k}$$

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\[ Q_{2k,j}(x) = 2\tilde{G}_{2k,j}(x) - \tilde{G}_{1k,j}(x) + \ln h_{2k} + \phi + (2\rho_{2k} - \rho_{1k}) f_k. \]

Using
\[ \nabla \phi(p_{k,j}) = \nabla f_k(p_{k,j}) = 0, \]
\[ \Delta \phi(p_{k,j}) = -2K(p_{k,j}) \quad \text{where } K \text{ is the Gauss curvature}, \]
\[ (2\rho_{1k} - \rho_{2k}) \Delta f_k(p_{k,j}) = 8m\pi + 2(\rho_{1k} - 8m\pi) - (\rho_{2k} - 8m\pi), \]
\[ (2\rho_{2k} - \rho_{1k}) \Delta f_k(p_{k,j}) = 8m\pi + 2(\rho_{2k} - 8m\pi) - (\rho_{1k} - 8m\pi), \]
\[ (\partial_{11} - \partial_{22}) [(2\rho_{1k} - \rho_{2k}) f_k - (2\rho_{2k} - \rho_{1k}) f_k] = O(|\rho_{1k} - 8m\pi| + |\rho_{2k} - 8m\pi|), \]
\[ \partial_{12} [(2\rho_{1k} - \rho_{2k}) f_k - (2\rho_{2k} - \rho_{1k}) f_k] = O(|\rho_{1k} - 8m\pi| + |\rho_{2k} - 8m\pi|), \]
we therefore obtain Theorem 1.1.

Appendix: Proof of Lemma 4.1

Here we give the proof of Lemma 4.1. We shall follow the proof in [4]. Several key ingredients needed already exist: first of all, we have the non-degeneracy of entire solution; secondly, each \( \tilde{\eta}_{k,j} \) satisfies a linear equation with potential decaying like \(|z|^{-4}\):
\[ e^{V_{k,j}(z)} = \frac{c_i}{|z|^4} + O\left(\frac{1}{|z|^8}\right) \quad i = 1, 2. \] (110)

Lastly, we have two bounded (non decaying) kernels \( \psi_{1k}^0, \psi_{2k}^0 \) (as defined at (87)-(88)).

For \( 0 < \tau \leq \tau_0 \), let
\[ R = \frac{\delta}{\varepsilon_{k,j}}, \quad \alpha = \varepsilon_{k,j}^\tau + \varepsilon_{k,j} - \sup_{\mathcal{B} \leq |z| \leq R} (|\tilde{\eta}_{1k,j}| + |\tilde{\eta}_{2k,j}|) \]
and
\[ N_k^i = \sup_{|z| \leq R} \frac{|\tilde{\eta}_{k,j}|}{\alpha(1 + |z|)^\tau}, \quad i = 1, 2. \]

We claim that
\[ N_k^i \leq C \] (111)

for some constant \( C \). To prove (111), we follow the proof in [4] and divide it into several steps. We prove it by contradiction. Without loss of generality, we may assume that \( N_k := N_k^1 \geq N_k^2 \). Assume that
\[ N_k \to +\infty \quad \text{as } k \to +\infty. \] (112)
Step 1. Let $|\tilde{\eta}_{1k,j}(y)| + |\tilde{\eta}_{2k,j}(y)| = o(\alpha N_k)$ in any compact set.

$$\tilde{\eta}_{1k,j}(y) = \frac{\tilde{\eta}_{1k,j}(y)}{||(1 + |y|)^{-\tau} (|\tilde{\eta}_{1k,j}| + |\tilde{\eta}_{2k,j}|)) ||_{L^\infty(B_{\frac{1}{N_k}})}},$$

$$\tilde{\eta}_{2k,j}(y) = \frac{\tilde{\eta}_{2k,j}(y)}{||(1 + |y|)^{-\tau} (|\tilde{\eta}_{1k,j}| + |\tilde{\eta}_{2k,j}|)) ||_{L^\infty(B_{\frac{1}{N_k}})}},$$

Clearly $\tilde{\eta}_{1k,j}$ and $\tilde{\eta}_{2k,j}$ are locally bounded. Recall that $a_{1k,j}$, $a_{2k,j}$, $b_{k,j}$, $c_{k,j}$ and $d_{k,j}$ are determined by (60)-(64) such that $0 < C_1 < a_{1k,j}, a_{2k,j} < C_2 < \infty$ and the other coefficients are of order $O(1)$ in $k$. Recall the system (77) for $\tilde{\eta}_{ik,j}$ and we find that the inhomogeneous terms in the equations of $\tilde{\eta}_{ik,j}$ are $O\left(\frac{e^{\frac{\tau}{2}N_k}}{\alpha N_k(1 + |y|)^{\tau}}\right)$. Obviously, $\frac{e^{\frac{\tau}{2}N_k}}{\alpha N_k} \to 0$ as $k$ goes to infinity, so the inhomogeneous terms tend to zero. Standard elliptic regularity then implies that there exist $\bar{\eta}_{\infty}$ such that $\tilde{\eta}_{ik,j} \to \bar{\eta}_{\infty}$ in $C^2_{\infty}(\mathbb{R}^2)$ and

$$-\Delta \bar{\eta}_{1\infty} = e^{v_1} (2\bar{\eta}_{1\infty} - \bar{\eta}_{2\infty}), \quad -\Delta \bar{\eta}_{2\infty} = e^{v_2} (2\bar{\eta}_{2\infty} - \bar{\eta}_{1\infty}),$$

where $(v_1, v_2)$ is the entire solution with the parameters determined by the convergence. Since $\bar{\eta}_{1\infty} = O(1 + |y|^{\tau})$, $\bar{\eta}_{2\infty} = O(1 + |y|^{\tau})$, by Theorem 2.1 we deduce that $\tilde{\eta}_{\infty} = \sum_{\ell=1}^{8} \gamma_{\ell} Z_{\ell}$ where $Z_1 = (\frac{\partial a_{1k,j}}{\partial z_{2j}}, w_1), Z_2 = (\frac{\partial a_{2k,j}}{\partial z_{2j}}, w_2), \ldots$. Since the choices of $a_{1k,j}$, $a_{2k,j}$, $b_{k,j}$, $c_{k,j}$ and $d_{k,j}$ imply that

$$\nabla^\ell \tilde{\eta}_{\infty}(0) = \nabla^\ell \bar{\eta}_{\infty}(0) = 0 \quad \text{for} \quad \ell = 0, 1,$$

$$\partial_1 \tilde{\eta}_{\infty}(0) = \partial_2 \bar{\eta}_{\infty}(0), \quad \partial_1 \bar{\eta}_{\infty}(0) = 0,$$

we deduce that $\gamma_{\ell} = 0$ and hence $\tilde{\eta}_{1\infty} = \tilde{\eta}_{2\infty} \equiv 0$. So $\tilde{\eta}_{ik,j} \to 0$ as $k \to \infty$ for $i = 1, 2$. This concludes Step 1.

Step 2. There exists $C_1$ such that

$$|y| |\nabla \tilde{\eta}_{ik,j}(y)| \leq C_1 \left( \sup_{\frac{1}{N_k} \leq |z| \leq 2|y|} (|\tilde{\eta}_{1k,j}(z)| + |\tilde{\eta}_{2k,j}(z)|) + \varepsilon_k^{\frac{2}{\alpha}} \right).$$

This is the standard gradient estimates. We omit the proof (see [4]).

Step 3. It holds that

$$N_{ik}^* := \sup_{|z| \leq R} \sup_{|z'\leq |z|} \frac{|\tilde{\eta}_{ik}(z) - \tilde{\eta}_{ik}(z')|}{\alpha(1 + |z|)^{\tau}} = o(N_k), \quad i = 1, 2. \quad (114)$$

We prove it by contradiction. Assume that $N_{ik}^* \geq c_0 N_k$ for some $c_0 > 0$. Without loss of generality, we might assume $N_{1k}^* = \max(N_{1k}^*, N_{2k}^*)$. Let $z_k'$ and $z_k''$ be such that $|z_k'| = |z_k''|$ and

$$N_{1k}^* = \frac{|\tilde{\eta}_{1k,j}(z_k') - \tilde{\eta}_{1k,j}(z_k'')|}{\alpha(1 + |z_k'|)^{\tau}}.$$
As in [4], we can prove $|z_k^*| = |z_k''| < \frac{\nu}{2}$ and the angle between $\overrightarrow{O z_k^*}$ and $\overrightarrow{O z_k''}$ \( \theta_k > \theta_0 > 0 \). This follows from the gradient estimate of Step 2 and \( \tau \leq \tau_0 \).

Without loss of generality, we may assume $z_k^*$ and $z_k''$ are symmetric with respect to $z_1$-axis and $\tilde{\eta}_{k, j}(z_k') > \tilde{\eta}_{k, j}(z_k'')$.

Set \[ \omega_k^*(z) = \tilde{\eta}_{k, j}(z) - \tilde{\eta}_{k, j}(z^-) \quad \text{for } z > 0, \]

where $z = (z_1, z_2)$ and $z^- = (z_1, -z_2)$, and set \[ \omega_k(z) = \frac{\omega_k^*(z)}{(1 + z_2)^{1/2}}. \]

Let $z_k^*$ be the maximum of $|\omega_k(z)|$ in $B_R^+ = \{ |z| \leq R, z_2 > 0 \}$ and denote the maximum of $|\omega_k(z)|$ by $N_k^{**}$. Then by the assumption, $N_k^{**} \geq c_1 N_1^{**} \geq c_2 N_k$ for some positive constants $c_1$ and $c_2$, due to $\theta_k > \theta_0 > 0$. In particular, $N_k^{**} \to +\infty$ as $k \to +\infty$. By (70), (71) and Step 1, $1 << |z_k^*| \leq \frac{\nu}{2}$.

Straightforward computations show that $\omega_k(z)$ satisfies, for $|z| > 1$,

\[ \Delta \omega_k(z) + 2\tau \nabla \log(1 + z_2) \nabla \omega_k \leq \frac{\tau(1 - \tau)}{(1 + z_2)^2} \omega_k + O \left( \frac{z_k^*}{|z|^2(1 + z_2)^2} \right). \]

At $z = z_k^*$, we have $\nabla \omega_k = 0$, $\Delta \omega_k \leq 0$ and thus we obtain

\[ c_2 N_k \leq N_k^{**} = \omega_k(z_k^*) \leq C \frac{(1 + z_2)^2}{|z|^4(1 + z_2)^2} (|\tilde{\eta}_{k, j}| + |\tilde{\eta}_{2k, j}|) + C \]

\[ \leq C N_k |z_k^*|^{-2} + C, \]

which clearly gives a contradiction. This proves Step 3.

**Step 4** Set the radial average of $\tilde{\eta}_{k, j}$ as

\[ \varphi_{k, j}(r) = \frac{1}{2\pi} \int_0^{2\pi} \tilde{\eta}_{k, j}(re^{i\theta}) \, d\theta, \quad i = 1, 2. \]

By Step 3, $\max_{i=1,2} \sup_{|z| \leq R} \frac{|\tilde{\eta}_{k, j}|}{\alpha_1(1 + |z|)} \geq c_3 N_k$ for some positive constant $c_3$. Let us assume that $\hat{\varphi}_{k, j}(s_k) = \sup_{|z| \leq R} \frac{|\tilde{\eta}_{k, j}|}{\alpha_1(1 + |z|)} \geq C \alpha N_k$. By Step 3, we have $1 << s_k < \frac{R}{2}$. Then multiplying the equations for $\tilde{\eta}_{k, j}$ by the kernel functions $(\psi_{1k}^{a_1}, \psi_{2k}^{a_1})$ (as defined at (87)-(88)) respectively, we obtain, as in the proof of Estimate C of [4] and also as in the proof of (99), that

\[ \int_{|z| = r} \left[ \frac{\partial}{\partial \nu}(\tilde{\eta}_{k, j}(z)) \psi_{1k}^{a_1} - \frac{\partial}{\partial \nu}(\psi_{1k}^{a_1}) \tilde{\eta}_{k, j}(z) \right] \]

\[ + \int_{|z| = r} \left[ \frac{\partial}{\partial \nu}(\tilde{\eta}_{2k, j}(z)) \psi_{2k}^{a_1} - \frac{\partial}{\partial \nu}(\psi_{2k}^{a_1}) \tilde{\eta}_{2k, j}(z) \right] \]

\[ = 39 \]
\[
= \int \left[ \rho_{1k} h_{1k}(p_{k,j}) e^{V_{1k,j} \psi_{1k}^{a_1}} O(\vert \tilde{\eta}_{1,k,j} \vert^2 + \vert \tilde{\eta}_{2,k,j} \vert^2) \right] \\
+ \int \left[ \rho_{2k} h_{2k}(p_{k,j}) e^{V_{2k,j} \psi_{2k}^{a_1}} O(\vert \tilde{\eta}_{1,k,j} \vert^2 + \vert \tilde{\eta}_{2,k,j} \vert^2) \right] + O(\varepsilon_k^2).
\]

Using the asymptotic expansion of \((\psi_{1k}^{a_1}, \psi_{2k}^{a_1})\) (see (88))
\[
\psi_{1k}^{a_1} = O\left( \frac{1}{|y|^2} \right), \quad \frac{\partial \psi_{1k}^{a_1}}{\partial \nu} = O\left( \frac{1}{|y|^3} \right),
\]
\[
\psi_{2k}^{a_1} = \frac{2}{a_{1k}} + O\left( \frac{1}{|y|^2} \right), \quad \frac{\partial \psi_{2k}^{a_1}}{\partial \nu} = O\left( \frac{1}{|y|^3} \right),
\]
we obtain, similar to [4], that
\[
\vert \varphi_{2k}(r) \vert \leq c \left( \frac{\alpha N_k}{r^{2-\gamma}} + \varepsilon_k^2 \ln(r + 2) \right) + (\alpha N_k)^2 r^{-1}.
\]

Similarly, using the kernel \((\psi_{1k}^{a_2}, \psi_{2k}^{a_2})\), we may further obtain
\[
\vert \varphi_{1k}(r) \vert \leq c \left( \frac{\alpha N_k}{r^{2-\gamma}} + \varepsilon_k^2 \ln(r + 2) \right) + (\alpha N_k)^2 r^{-1}.
\]

Note that
\[
\alpha N_k = \sup_{|z| \leq R} \frac{|\tilde{\eta}_{1,k,j}(z)|}{(1 + |z|)^2} \to 0 \quad \text{as } k \to \infty,
\]

because \(\tilde{\eta}_{1,k,j}\) is uniformly bounded and \(\tilde{\eta}_{1,k,j}(z) \to 0\) uniformly in any compact set. Hence
\[
\alpha N_k (1 + s_k)^{\gamma} \leq |\varphi_{1k}(s_k)| \leq C \int_{r_0}^{s_k} |\varphi_{1k}'(r)| dr \leq C (\ln s_k)^2 (\varepsilon_k^2 + \alpha N_k) + r_0^{-1} \alpha N_k,
\]

where \(r_0\) is a fixed but large positive number. Since \(s_k \to \infty\), we get that
\[
N_k \leq \frac{C (\ln s_k)^2 \varepsilon_k^2}{(1 + s_k)^{\gamma}} = o(1) \quad \text{as } k \to \infty,
\]

which yields a contradiction. This completes the proof of Lemma 4.1.

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References


