HAUSDORFF DIMENSION OF RUPTURES FOR SOLUTIONS OF A SEMILINEAR ELLIPTIC EQUATION WITH SINGULAR NONLINEARITY

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ABSTRACT. We consider the following semilinear elliptic equation with singular nonlinearity:

$$\Delta u - \frac{1}{u^\alpha} + h(x) = 0 \text{ in } \Omega$$

where $\alpha > 1$, $h(x) \in C^1(\Omega)$ and $\Omega$ is an open subset in $\mathbb{R}^n$, $n \geq 2$. Let $u$ be a non-negative finite energy stationary solution and $\Sigma = \left\{ x \in \Omega : \lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u| \right\}$ exists, and is equal to 0 be the rupture set of $u$. We show that the Hausdorff dimension of $\Sigma$ is less than or equal to $\frac{(n-2)\alpha + (n+2)}{\alpha + 1}$.

1. Introduction

Let $\Omega$ be an open subset in $\mathbb{R}^n$ ($n \geq 2$). In this paper we consider partial regularity for nonnegative solutions of the following equation

$$\Delta u - \frac{1}{u^\alpha} + h(x) = 0 \text{ in } \Omega$$

(1.1)

where $\alpha > 1$, $h \in C^1(\Omega)$ such that $||h||_{L^\infty(\Omega)} \leq a$ and $||\nabla h||_{L^\infty(\Omega)} \leq b$ for some constants $a, b > 0$. In particular, we are concerned with the Hausdorff dimension of the zero set:

$$\Sigma = \left\{ x \in \Omega : \lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u| \right\} \text{ exists, and is equal to } 0 \right\}.$$  

(1.2)

Problem (1.1) arises in the study of steady states of thin films. Equations of the type

$$u_t = -\nabla \cdot (f(u) \nabla u) - \nabla \cdot (g(u) \nabla u)$$

(1.3)

have been used to model the dynamics of thin films of viscous fluids, where $z = u(x,t)$ is the height of the air/liquid interface. The zero set $\Sigma$ defined in (1.2) is the liquid/solid interface and is sometimes called set of ruptures. The coefficient $f(u)$ reflects surface tension effects- a typical choice is $f(u) = u^3$. The coefficient of the

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second-order term can reflect additional forces such as gravity \( g(u) = u^3 \), van der Waals interactions \( g(u) = u^m, m < 0 \). For backgrounds on (1.3), we refer to [BP1, BP2, LP1, LP2, LP3, WB] and the references therein.

In general, let us assume that \( f(u) = u^p, g(u) = u^m \), where \( p, m \in \mathbb{R} \). Then a steady-state equation for (1.3) with Neumann boundary condition becomes

\[
(1.4) \quad \Delta u + \frac{u^p}{q} - C = 0 \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega,
\]

where \( q = m - p + 1 \) and \( C \) is some constant. (Here we have assumed that \( q \neq 0 \). If \( q = 0 \), we have to replace \( \frac{u^p}{q} \) by \( \log u \).) For thin films under van der Waals forces, we have \( f(u) = u^3, g(u) = u^m, q = m - 2 < -2 \). The one-dimensional steady-state problem of (1.3) has been studied thoroughly in [LP1, LP3] and the references therein. Numerical work on two-dimensional van der Waals driven rupture in (1.3) suggested that the rupture can occur in points [BBD, HLU] or rings [WB, YD, YH].

The main result of our paper is to give an estimate on the Hausdorff dimension of the rupture set \( \Sigma \). Roughly speaking, we prove that the Hausdorff dimension of \( \Sigma \) is less than or equal to \( ((n - 2)\alpha + (n + 2)) / (\alpha + 1) \).

We begin with some definitions. We call \( u \) a nonnegative finite energy solution of (1.1) in \( \Omega \) if \( u \geq 0 \) in \( \Omega \), \( u \) satisfies (1.1) pointwisely in \( \Omega \setminus \Sigma \), and the energy of \( u \)

\[
E(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx + \frac{1}{1 - \alpha} \int_\Omega u^1 - \alpha u \, dx - \int_\Omega h(x)u \, dx
\]

is finite.

We also say that such a finite energy solution \( u \) is stationary if, in addition, it satisfies

\[
(1.5) \quad \int_\Omega \left[ \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial \phi^j}{\partial x_i} - \frac{1}{2} |\nabla u|^2 \frac{\partial \phi^j}{\partial x_i} - \frac{1}{1 - \alpha} u^{1 - \alpha} \frac{\partial \phi^j}{\partial x_i} + u \frac{\partial h}{\partial x_i} \phi^j + u \frac{\partial \phi^j}{\partial x_i} \right] \, dx = 0
\]

for all regular vector field \( \phi \) with compact supports in \( \Omega \) (summation over \( i \) and \( j \) is understood).

For finite energy solutions \( u \in H^1(\Omega) \) and \( \int_\Omega u^{1 - \alpha}(x) \, dx < \infty \) the identity (1.5) is obtained by assuming that the functional \( E(u) \) is stationary with respect to domain variations, that is,

\[
\frac{d}{dt} E(u_t) \bigg|_{t=0} = 0
\]

where \( u_t(x) = u(x + t\phi(x)), \phi(x) = (\phi^1(x), \phi^2(x), \ldots, \phi^n(x)) \). (The identity (1.5) can also be obtained by multiplying (1.1) by \( \phi \cdot \nabla u \) and integrating it by parts in \( \Omega \) (if it can be integrated by parts)). Examples of stationary solution include minimizers
of the energy functional $E(u)$ (if they exist). The concept of stationary solutions was introduced in [Ev].

Let us define

$$\mathcal{E}_\alpha = \{ u \in H^1(\Omega) : u \geq 0 \text{ in } \Omega, \int_\Omega u^{1-\alpha}(x)dx < \infty \}.$$ 

Let $u \in \mathcal{E}_\alpha$ be a finite energy solution of (1.1). We easily see that away from $\Sigma$, the classical regularity theory ensures that $u$ is regular. Therefore $\Sigma$ is the set of singularities of $u^{-1}$. Moreover, by the definition, $\Sigma$ is a relatively closed subset of $\Omega$.

Our partial regularity result is the following theorem.

**Theorem 1.1.** Let $\alpha > 1$ be given. If $u \in \mathcal{E}_\alpha$ is a finite energy solution of (1.1), which is stationary, then $u$ is smooth outside a closed rupture set of $u$ with locally finite Hausdorff $\mu$-dimensional measure, where $\mu = \frac{(n-2)\alpha + (n+2)}{(\alpha+1)}$. In other words, the Hausdorff dimension of $\Sigma$ is less than or equal to $\mu$.

In a recent paper [JL], Jiang and Lin studied the weak solution of (1.1) in the sense that $u \in H^1_{loc}(\Omega)$, $u \in L^1(\Omega)$ and $u^{-\alpha} \in L^1(\Omega)$. Using an important Poincaré type inequality, they found that $H^s(\Sigma) = 0$, where $s = n - 2 + \frac{4}{\alpha+2}$. On the other hand, here we assume that $u^{1-\alpha} \in L^1(\Omega)$ which is weaker than $u^{-\alpha} \in L^1(\Omega)$. But the Hausdorff dimension of $\Sigma$ obtained in this paper is larger than that obtained in [JL].

We will first establish a monotonicity inequality for the nonnegative finite energy stationary solutions $u \in \mathcal{E}_\alpha$ of (1.1). Then, using such monotonicity of the energy of $u$, we obtain the measure estimate of the singular set $\Sigma$ of $u^{-1}$. This estimate on $\Sigma$ may have potential applications on the estimates for ruptures of thin films. For example, if $n = 2, \alpha > 3$, Theorem 1.1 implies that there are no finite energy stationary solutions with ring ruptures.

We don’t know if $\mu = ((n-2)\alpha + (n+2))/(\alpha + 1)$ is the optimal.

About the applicability of Theorem 1.1, we see that under the flow (1.3) and the fact that the pressure is constant (i.e., $\frac{1}{|\Omega|} \int_\Omega u(x, t) \equiv constant$), the energy of $u(x, t)$ is decreasing with respect to $t$. Thus, if we start with a finite energy initial data, then the limit of $u(x, t)$ as $t \to +\infty$ (if exists) is also of finite energy. We also believe only local minimizers of $E(u)$ are stable with respect to the flow (1.3). Our theorem gives estimates on Hausdorff dimension of ruptures of stable attractors.

A different kind of problem

$$\Delta u + k(x)\frac{1}{u^\alpha} = 0 \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial \Omega \quad (1.6)$$
was studied in [CR, De, GHW, Go, GL] and the references therein, where \( k(x) > 0 \). The regularity of \( \nabla u \) is obtained. Problem (1.6) is fundamentally different from (1.1): the sign of nonlinearity makes the Maximum Principle applicable to (1.6) which allow the use of e.g. a super-sub solutions scheme. In fact the following problem

\[
\Delta u + \frac{1}{u^\alpha} - h(x) = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial \Omega,
\]

possesses a (unique) positive solution in case that \( h \) is, for example, positive.

An interesting problem is to construct solutions with ruptures to (1.4). In this regard, we remark that when \( \Omega \) is the unit ball \( B \) of \( \mathbb{R}^n \), the problem

\[
(1.7) \quad \Delta u - \frac{1}{u^\alpha} + h(x) = 0 \quad \text{in } B, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial B
\]

has been studied for \( h(x) = h(|x|) \) in [DH]. They showed that (1.7) has a nonnegative radial solution \( u \in C^0(B) \) satisfying

\[
c_1 r^{2(\alpha + 1)} \leq u(r) \leq c_2, \quad c_1, c_2 > 0.
\]

It is unknown if the solution constructed in [DH] has ruptures. It appears that it is a quite difficult problem in constructing rupture solutions in higher dimension. Partial progress has been done in [GW].

Our results here are in the same spirit of those in [Ev, Scn, Pa] where the Hausdorff dimensions of the blow up set of harmonic maps or some nonlinear elliptic problems are studied. The proof of Theorem 1.1 is divided into three steps:

**Step 1.** We show that if \( u \in \mathcal{E}_\alpha \), then \( u \in L^\infty_{A, \alpha}(\Omega) \).

**Step 2.** Fix \( x_0 \in \Omega \) such that \( B(x_0, 2r_0) \subset \Omega \). We show that there exists a constant \( C = C(a, b, \|u\|_{L^\infty(B(x_0, 2r_0), n)} ) \) such that the following functional

\[
E_u(x_0, r) = -\frac{\alpha + 1}{2(\alpha - 1)} r^{-\mu} \int_{B(x_0, r)} u^{1-\alpha} \, dx + \frac{1}{4} \frac{d}{dr} \left[ r^{-\mu} \int_{\partial B(x_0, r)} u^2 \, ds \right] - \frac{1}{4} r^{-\mu - 1} \int_{\partial B(x_0, r)} u^2 \, ds + C \int_0^r \xi^{n-\mu - 1} \, d\xi
\]

is an increasing function of \( r \in (0, r_0) \).

These are done in Section 2.

**Step 3.** Using the monotonicity formula, we show that there exists \( \epsilon^* > 0 \) such that for \( x_0 \in \Sigma \),

\[
\lim_{r \to 0^+} r^{-\mu} \int_{B(x_0, r)} \left[ |\nabla u|^2 + u^{1-\alpha} \right] \, dx \geq \epsilon^*
\]

which concludes the proof of Theorem 1.1.

This is done in Section 3.
Finally, we remark the negative power $u^{-\alpha}$ can be considered as **negative supercritical** in $\mathbb{R}^n, n \geq 2$. In fact, it is known ([ACW]) that $u^{-\alpha}$ is subcritical if $\alpha < 3$ and supercritical if $\alpha > 3$. (A naive reason is as follows: the critical Sobolev exponent is $\frac{n+2}{n-2}$ which equals $-3$ if $n = 1$.) Thus formally $u^{-\alpha}$ is supercritical for $n \geq 2$. Our results give estimates on the singular set for negative supercritical problem, which is new.

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2. **A Monotonicity Inequality**

In this section we shall obtain a key monotonicity inequality for **finite energy stationary** solutions $u \in \mathcal{E}_\alpha$ of (1.1). To do this, we first recall the following result.

**Lemma 2.1.** Let $f \geq 0$ in $\Omega$ and $g \in L^q(\Omega)$ for some $q > n/2$. Let $u$ be a nonnegative solution of the equation

$$
\Delta u = f + g \quad \text{in } \Omega.
$$

Then for any $B_2 \subset \Omega$, and $\|u\|_{H^1(B_{2R})} < \infty$, we have

$$
\sup_{B_R} u \leq c(n,q)(R^{-\frac{n}{2}}\|u\|_{L^q(B_{2R})} + R^{2-\frac{n}{q}}\|g\|_{L^n(B_{2R})}).
$$

**Proof.** Similar to the proof of Lemma 4.1 of [JL].

Lemma 2.1 (with $f = u^{-\alpha}$, $g = h$) implies that if $u \in \mathcal{E}_\alpha$ is a nonnegative solution of (1.1), then $u \in L_\infty^\infty(\Omega)$.

Now we establish the key monotonicity inequality for **finite energy stationary** solutions $u \in \mathcal{E}_\alpha$ of (1.1). We follow the notation in [Ev, Pa].

Fix $x_0 \in \Omega$ such that $B(x_0, 2r_0) \subset \Omega$, where $0 < r_0 \leq R$ and $R$ is given in Lemma 2.1. Let $r, m > 0$ be such that $r + m < r_0$. Set $\phi(x) = \xi(|x - x_0|)(x - x_0)$, where

$$
\xi(|x - x_0|) \equiv \begin{cases} 
1 & \text{for } |x - x_0| \leq r, \\
1 + \frac{r - |x - x_0|}{m} & \text{for } r \leq |x - x_0| \leq r + m, \\
0 & \text{for } |x - x_0| \geq r + m.
\end{cases}
$$
We derive from (1.5), letting \( m \to 0^+ \), that the following identity holds
\[
\frac{n}{\alpha - 1} \int_{B(x_0, r)} u^{1-\alpha} dx - \frac{n - 2}{2} \int_{B(x_0, r)} |\nabla u|^2 dx + \frac{r}{2} \int_{\partial B(x_0, r)} |\nabla u|^2 ds \\
+ n \int_{B(x_0, r)} h dx - r \int_{\partial B(x_0, r)} h ds - \frac{r}{\alpha - 1} \int_{\partial B(x_0, r)} u^{1-\alpha} ds \\
+ \int_{B(x_0, r)} u < x - x_0, \nabla h > = r \int_{\partial B(x_0, r)} (u_r)^2 ds,
\]
(2.1)

where \( u_r = \frac{\partial u}{\partial r} \). (Another equivalent derivation of (2.1) is by multiplying (1.1) with \((x - x_0) \cdot \nabla u\) and integrating over \( B(x_0, r) \).)

On the other hand, multiplying (1.1) by \( u \) and integrating over \( B(x_0, r) \) we find, for almost every \( 0 < r < r_0 \)
\[
\int_{B(x_0, r)} |\nabla u|^2 dx = \int_{\partial B(x_0, r)} uu_r ds - \int_{B(x_0, r)} u^{1-\alpha} dx + \int_{B(x_0, r)} h(x) udx.
\]
(2.2)

Taking the derivative of (2.2) with respect to \( r \), we get
\[
\int_{\partial B(x_0, r)} |\nabla u|^2 ds = \frac{d}{dr} \left[ \int_{\partial B(x_0, r)} uu_r ds \right] - \int_{\partial B(x_0, r)} u^{1-\alpha} ds + \int_{\partial B(x_0, r)} huds.
\]
(2.3)

Substituting \( \int_{B(x_0, r)} |\nabla u|^2 dx \) of (2.2) and \( \int_{\partial B(x_0, r)} |\nabla u|^2 ds \) of (2.3) into (2.1), we finally obtain
\[
\left( \frac{n}{\alpha - 1} + \frac{n - 2}{2} \right) \int_{B(x_0, r)} u^{1-\alpha} dx - \left( \frac{1}{2} + \frac{1}{\alpha - 1} \right) r \int_{\partial B(x_0, r)} u^{1-\alpha} ds \\
+ \left( n - \frac{n - 2}{2} \right) \int_{B(x_0, r)} h dx - \frac{r}{2} \int_{\partial B(x_0, r)} h ds + \frac{r}{2} \int_{\partial B(x_0, r)} \frac{d}{dr} \left[ \int_{\partial B(x_0, r)} uu_r ds \right] \\
- \frac{(n - 2)}{2} \int_{\partial B(x_0, r)} uu_r ds + \int_{B(x_0, r)} u < x - x_0, \nabla h > \\
= r \int_{\partial B(x_0, r)} (u_r)^2 ds.
\]
(2.4)

Rewriting (2.4), we have
\[
- \frac{(\alpha + 1)}{2(\alpha - 1)} \frac{d}{dr} \left[ r^{-\mu} \int_{B(x_0, r)} u^{1-\alpha} dx \right] + \frac{1}{2} r^{-\mu} \frac{d}{dr} \left[ \int_{\partial B(x_0, r)} uu_r ds \right] \\
+ \frac{(n + 2)}{2} r^{-(\mu + 1)} \int_{B(x_0, r)} h dx - \frac{1}{2} r^{-\mu} \int_{\partial B(x_0, r)} h ds \\
+ r^{-(\mu + 1)} \int_{B(x_0, r)} u < x - x_0, \nabla h > dx \\
= r^{-\mu} \int_{\partial B(x_0, r)} \left[ (u_r)^2 + \frac{n - 2}{2} r^{-1} uu_r \right] ds,
\]
(2.5)
where $\mu = \frac{(n-2)\alpha + (n+2)}{\alpha + 1}$. In the following, we denote $C$ for positive constants which may vary from line to line.

Since $u \in H^1(\Omega)$, it follows from Lemma 2.1 that there exists $C > 0$ such that $\|u\|_{L^\infty(B(x_0, r_0))} \leq C$. This and the facts that $|\nabla h| \leq b$ and $\left< x - x_0, \nabla h \right> \leq r |\nabla h|$ imply that

$$r^{-(\mu + 1)} \int_{B(x_0, r)} u < x - x_0, \nabla h > \, dx \leq Cr^{n-\mu}, \tag{2.6}$$

where $C = C(b, \|u\|_{L^\infty(B(x_0, r_0))}, n)$. On the other hand, we also know that

$$\left\{ \frac{(n + 2)}{2} r^{-(\mu + 1)} \int_{B(x_0, r)} h u \, dx \right\} \leq C r^{n-\mu-1}, \tag{2.7}$$

and

$$\left\{ \frac{1}{2} r^{-\mu} \int_{\partial B(x_0, r)} h u \, ds \right\} \leq C r^{n-\mu-1}, \tag{2.8}$$

where $C = C(a, \|u\|_{L^\infty(B(x_0, r_0))}, n)$.

Substituting (2.6), (2.7) and (2.8) into (2.5), we obtain

$$- \frac{\alpha + 1}{2(\alpha - 1)} \frac{d}{dr} \left[ r^{-\mu} \int_{B(x_0, r)} u^{1-\alpha} \, dx \right] + \frac{1}{2} r^{-\mu} \frac{d}{dr} \left[ \int_{\partial B(x_0, r)} u u_r \, ds \right] + C r^{n-\mu-1}$$

$$\geq r^{-\mu} \int_{\partial B(x_0, r)} \left[ |u_r|^2 + \frac{(n - 2)}{2} r^{-1} u u_r \right] \, ds, \tag{2.9}$$

where $C = C(a, b, \|u\|_{L^\infty(B(x_0, r_0))}, n)$.

Using the identity

$$\frac{d}{dr} \left[ \int_{\partial B(x_0, r)} u^2 \, ds \right] = 2 \int_{\partial B(x_0, r)} u u_r \, ds + (n - 1) \int_{\partial B(x_0, r)} u^2 r^{-1} \, ds \tag{2.10}$$

we have that

$$\frac{1}{2} \frac{d^2}{dr^2} \left[ r^{-\mu} \int_{\partial B(x_0, r)} u^2 \, ds \right] - \frac{d}{dr} \left[ r^{-\mu} \int_{\partial B(x_0, r)} u u_r \, ds \right]$$

$$= (n - \mu - 1) r^{-\mu} \int_{\partial B(x_0, r)} \left[ \frac{(n - 2 - \mu)}{2} r^{-2} u^2 + r^{-1} u u_r \right] \, ds, \tag{2.11}$$

Note that

$$r^{-\mu} \frac{d}{dr} \left[ \int_{\partial B(x_0, r)} u u_r \, ds \right] = \frac{d}{dr} \left[ r^{-\mu} \int_{\partial B(x_0, r)} u u_r \, ds \right] + \mu r^{-\mu-1} \int_{\partial B(x_0, r)} u u_r \, ds \tag{2.12}$$
Substituting (2.12) and (2.11) into (2.9), we obtain that
\[
- \frac{\alpha + 1}{2(\alpha - 1)} \frac{d}{dr} \left[ r^{-\mu} \int_{B(x_0, r)} u^{1-\alpha} dx \right] + \frac{1}{4} \frac{d^2}{dr^2} \left[ r^{-\mu} \int_{\partial B(x_0, r)} u^2 ds \right] + Cr^{n-\mu-1} \\
\geq r^{-\mu} \int_{\partial B(x_0, r)} \left[ (u_r)^2 + \frac{2n - 2\mu - 3}{2} r^{-1} uu_r \right] ds + \frac{1}{4} (n - \mu - 1)(n - \mu - 2)r^{-2}u^2 \right] ds
\]
which yields that
\[
(2.13) \quad - \frac{\alpha + 1}{2(\alpha - 1)} \frac{d}{dr} \left[ r^{-\mu} \int_{B(x_0, r)} u^{1-\alpha} dx \right] + \frac{1}{4} \frac{d^2}{dr^2} \left[ r^{-\mu} \int_{\partial B(x_0, r)} u^2 ds \right] \\
- \frac{1}{4} \frac{d}{dr} \left[ r^{-\mu-1} \int_{\partial B(x_0, r)} u^2 ds \right] + Cr^{n-\mu-1} \\
\geq r^{-\mu} \int_{\partial B(x_0, r)} \left[ (u_r)^2 + (n - \mu - 2)r^{-1} uu_r + \frac{1}{4} (n - \mu - 2)^2 r^{-2}u^2 \right] ds \\
= r^{-\mu} \int_{\partial B(x_0, r)} (u_r + \frac{1}{2} (n - \mu - 2) r^{-1} u)^2 ds \geq 0.
\]
Since \( n - \mu - 1 = \frac{2-3}{a+1} > -1 \) for \( \alpha > 1 \), we conclude from (2.13) that
\[
E_u(x_0, r) \equiv - \frac{\alpha + 1}{2(\alpha - 1)} r^{-\mu} \int_{B(x_0, r)} u^{1-\alpha} dx + \frac{1}{4} \frac{d}{dr} \left[ r^{-\mu} \int_{\partial B(x_0, r)} u^2 ds \right] \\
- \frac{1}{4} r^{-\mu-1} \int_{\partial B(x_0, r)} u^2 ds + C \int_0^r \xi^{n-\mu-1} d\xi
\]
is an increasing function of \( r \) for \( r \in (0, r_0) \). (Note that \( C = C(a, b, \|u\|_{L^\infty(B(x_0, r_0))}, n) \).)

Next we obtain another formulation of \( E_u(x_0, r) \). First we have
\[
(2.15) \quad \frac{d}{dr} \left[ r^{-\mu} \int_{\partial B(x_0, r)} u^2 ds \right] = (n-\mu-1)r^{-\mu-1} \int_{\partial B(x_0, r)} u^2 ds + 2r^{-\mu} \int_{\partial B(x_0, r)} uu_r ds.
\]
Then by (2.2)
\[
(2.16) \quad \int_{\partial B(x_0, r)} uu_r ds = \int_{B(x_0, r)} |\nabla u|^2 dx + \int_{B(x_0, r)} u^{1-\alpha} dx - \int_{B(x_0, r)} h(x) u dx.
\]
Substituting (2.15) and (2.16) into (2.14), we obtain an equivalent formulation of \( E_u(x_0, r) \):
\[
E_u(x_0, r) = - \frac{1}{(\alpha - 1)} r^{-\mu} \int_{B(x_0, r)} u^{1-\alpha} dx + \frac{1}{2} r^{-\mu} \int_{B(x_0, r)} |\nabla u|^2 dx
\]
\[(2.17) \quad - \frac{1}{(\alpha + 1)} r^{-\mu - 1} \int_{\partial B(x_0, r)} u^2 ds - \frac{1}{2} r^{-\mu} \int_{B(x_0, r)} h u dx + C \int_0^r \xi^{n-\mu - 1} d\xi.\]

All the derivatives in the above expressions are to be understood in the sense of distributions. Now we obtain the following lemmas.

**Lemma 2.2.** If \( u \in \mathcal{E}_\alpha \) is a nonnegative finite energy stationary solution of (1.1), then \( E_u(x_0, r) \), defined at (2.14), is an increasing function of \( r \) for \( r \in (0, r_0) \), where \( B(x_0, 2r_0) \subset \Omega \).

**Lemma 2.3.** \( E_u(x_0, r) \) is a continuous function of \( x_0 \in \Omega \) and \( r > 0 \).

**Proof.** The proof is similar to that of Lemma 2 of [Pa]. \( \square \)

### 3. Hausdorff Dimension Estimate

In this section we will prove Theorem 1.1. For any fixed \( \epsilon > 0 \) sufficiently small and \( u \in \mathcal{E}_\alpha \) being a finite energy stationary solution of (1.1), by Lemma 2.2, one easily sees that if \( x_0 \in \Sigma \), there are two cases for \( x_0 \):

(i) \( \lim_{r \to 0^+} E_u(x_0, r) \geq -\epsilon \),

(ii) \( \lim_{r \to 0^+} E_u(x_0, r) < -\epsilon \).

For the first case, we have the following lemma.

**Lemma 3.1.** There exists \( \epsilon^* > 0 \) such that if \( \lim_{r \to 0^+} E_u(x_0, r) \geq -\epsilon^* \), then

\[(3.1) \quad \lim_{r \to 0^+} r^{-\mu} \int_{B(x_0, r)} |\nabla u|^2 dx \geq \epsilon^*.\]

**Proof.** The monotonicity of \( E_u(x_0, r) \) on \( r \) implies that, if \( \lim_{r \to 0^+} E_u(x_0, r) \geq -\epsilon \), for some \( \epsilon > 0 \), there exists \( 0 < r_0 < R \) such that for \( 0 < r < r_0 \),

\[ E_u(x_0, r) \geq -\epsilon. \]

It follows from the second formulation (2.17) of \( E_u(x_0, r) \) that

\[
- \frac{1}{(\alpha + 1)} r^{-\mu} \int_{B(x_0, r)} u^{1+\eta} dx + \frac{1}{2} r^{-\mu} \int_{B(x_0, r)} |\nabla u|^2 dx \\
- \frac{1}{(\alpha + 1)} r^{-\mu - 1} \int_{\partial B(x_0, r)} u^2 ds - \frac{1}{2} r^{-\mu} \int_{B(x_0, r)} h u dx + C \int_0^r \xi^{n-\mu - 1} d\xi \\
\geq -\epsilon.
\]
Suppose that \( \lim_{r \to 0^+} r^{-\mu} \int_{B(x_0, r)} |\nabla u|^2 \, dx < \epsilon \). Then
\[
- \frac{1}{(\alpha - 1)} \lim_{r \to 0^+} \left[ r^{-\mu} \int_{B(x_0, r)} u^{1-\alpha} \, dx \right] \\
+ \frac{1}{2} \lim_{r \to 0^+} \left[ r^{-\mu} \int_{B(x_0, r)} |\nabla u|^2 \, dx \right] \\
- \frac{1}{(\alpha + 1)} \lim_{r \to 0} \left[ r^{-\mu-1} \int_{\partial B(x_0, r)} u^2 \, ds \right] \\
\geq -\epsilon
\]
which implies that
\[
\frac{1}{\alpha - 1} \lim_{r \to 0^+} \left[ r^{-\mu} \int_{B(x_0, r)} u^{1-\alpha} \, dx \right] \\
+ \frac{1}{\alpha + 1} \lim_{r \to 0} \left[ r^{-\mu-1} \int_{\partial B(x_0, r)} u^2 \, ds \right] \leq \frac{3\epsilon}{2}.
\]
This shows that there exists \( 0 < r_1 < r_0 \) such that for \( 0 < r < r_1 \),
\[
r^{-\mu} \int_{B(x_0, r)} u^{1-\alpha} \, dx \leq 2(\alpha - 1)\epsilon, \tag{3.2}
\]
\[
r^{-\mu-1} \int_{\partial B(x_0, r)} u^2 \, ds \leq 2(\alpha + 1)\epsilon. \tag{3.3}
\]
It follows from (3.3) that for \( 0 < r < r_1 \),
\[
\int_{B(x_0, r)} u^2 \, dx \leq C\epsilon r^{\mu+2}, \tag{3.4}
\]
where \( C = C(\alpha, n) \). Thus, we derive from (3.2), (3.4) and the Hölder inequality that
\[
Cr^n = \int_{B(x_0, r)} u^{-2(\alpha-1)/(\alpha+1)} u^{2(\alpha-1)/(\alpha+1)} \, dx \\
\leq \left( \int_{B(x_0, r)} u^{1-\alpha} \, dx \right)^{2/(\alpha+1)} \left( \int_{B(x_0, r)} u^2 \, dx \right)^{(\alpha-1)/(\alpha+1)} \\
\leq C\epsilon r^{2\alpha^2(\mu+2)/(\alpha+1)} \\
= C\epsilon r^n,
\]
which is a contradiction if we choose \( \epsilon > 0 \) sufficiently small. Thus, we conclude that the \( \epsilon^* > 0 \) as mentioned in the lemma exists and
\[
\lim_{r \to 0^+} r^{-\mu} \int_{B(x_0, r)} |\nabla u|^2 \, dx \geq \epsilon^*,
\]
which finishes the proof of the lemma. \( \Box \)
To study the second case, we need the following Poincaré type inequality from [JL]. (See Theorem 2.1 of [JL]. Note that $n - 2 < \mu < n$.)

**Lemma 3.2.** Let $B_\ell$ be any ball in $\mathbb{R}^n$ with radius $\ell$, and $T \subset B_\ell$ be a $\mathcal{H}^\mu$-measurable set, such that

$$
\mathcal{H}^\mu(T) \geq \theta_1 \ell^\mu,
$$
and that for any $x \in \mathbb{R}^n$, and $r > 0$,

$$
\mathcal{H}^\mu(T \cap B_r(x)) \leq \theta_2 r^\mu
$$
holds. Then for any $u \in H^1(B_\ell)$ such that $T \subset \Sigma$, where $\Sigma$ is defined in (1.2), we have

$$
\int_{B_\ell} u^2 \leq c(n, \mu) \frac{\theta_2^2}{\theta_1} \ell^2 \int_{B_\ell} |\nabla u|^2.
$$

The following lemma plays an important role.

**Lemma 3.3.** Let $T \subset \Sigma$ be as in Lemma 3.2 and (3.5), (3.6) hold. Let $u \in \mathcal{E}_\alpha$ be a finite energy stationary solution of (1.1) and $x_0 \in T$. Then, for $0 < 2r < d(x_0, \partial\Omega)$ sufficiently small,

$$
r^{-\mu-1} \int_{\partial B(x_0, r)} u^2 ds \leq Cr^{-\mu} \int_{B(x_0, r)} \left[ |\nabla u|^2 + u^{1-\alpha} - hu \right] dx
$$
where $C = C(n, \alpha) > 0$.

**Proof.** Without loss of generality, we assume that $x_0 = 0$. Define $F(r) = \int_{B(0, r)} u^2 dx$. Then $F(r) \to 0$ as $r \to 0$. We use polar coordinates $(t, \theta)$ on $B(0, r)$. Let $G(t) = \int_{\partial B(0, t)} u^2(t, \theta) d\theta$ for $0 < t < r$. Then

$$
\frac{dG}{dt}(t) = 2 \int_{\partial B(0, t)} u(t, \theta) u_t(t, \theta) d\theta
$$

$$
= 2t^{1-n} \int_{\partial B(0, t)} uu_t ds
$$

$$
= 2t^{1-n} \int_{B(0, t)} \left[ |\nabla u|^2 + u(1-\alpha) - hu \right] dx
$$

$$
\geq 0,
$$
where we are using (2.2) and the fact that

$$
\int_{B(0, t)} [u^{1-\alpha} - hu] dx > 0,
$$
for $t > 0$ sufficiently small. To obtain (3.9), we use the Young’s inequality to see that

$$
(3.10) \quad |B(0, t)| \leq d_1 \int_{B(0, t)} u \, dx + d_2 \int_{B(0, t)} u^{1-\alpha} \, dx,
$$

where $d_1 = d_1(\alpha) > 0$, $d_2 = d_2(\alpha) > 0$. Since $\lim_{t \to 0^+} \frac{\int_{B(0, t)} u \, dx}{|B(0, t)|} = 0$, for any $0 < \epsilon < \frac{1}{20d_1}$, there is $t_0 = t_0(\epsilon) > 0$ such that for $0 < t < t_0$, we have

$$
(3.11) \quad \int_{B(0, t)} u \, dx \leq \epsilon |B(0, t)|.
$$

This and (3.10) imply that

$$
\int_{B(0, t)} u^{1-\alpha} \, dx \geq \frac{1}{2d_2} |B(0, t)|.
$$

This and (3.11) imply (3.9) (noting that $\|h\|_{L^\infty(\Omega)} \leq a$). Note that the derivative $u_t$ in the computations is to be understood in the sense of distribution. Since $\int_{B(0, t)} |\nabla u|^2 \, dx$, $\int_{B(0, t)} u^{1-\alpha} \, dx$ and $\int_{B(0, t)} h(x)u(x) \, dx$ are continuous functions of $0 < t < r$ (see Lemma 2.3), we have that $\frac{d^2}{dt^2}(t)$ is a continuous function of $0 < t < r$. Thus, $G \in C^1(0, r)$ is an increasing function. This also implies that $\frac{d^2}{dt^2}F(t)$ is a continuous function for $0 < t < r$.

Now we consider the function $F(r)$. By making a Taylor expansion of $F(r)$ at $r$, we obtain that

$$
0 = \int_{B(0, r)} u^2 \, dx - \left( \int_{\partial B(0, r)} u^2 \, ds \right) r + \left[ \int_0^1 \eta \left( (n-1)\xi_n^{n-2} \int_{\partial B(0, \xi)} u^2(\xi, r) \, d\xi \right) d\theta \right. \\
+ 2 \int_{\partial B(0, \xi)} u \xi \, ds \right] r^2 \\
= \int_{B(0, r)} u^2 \, dx - \left( \int_{\partial B(0, r)} u^2 \, ds \right) r + \left[ \int_0^1 \eta \left( (n-1)\xi_n^{n-2} G(\xi) \right) \\
+ 2 \int_{B(0, \xi)} (|\nabla u|^2 + u(u^{-\alpha} - h)) \, dx \right] r^2,
$$

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where $\xi_0 = \eta r$ and $0 < \eta < 1$. Since $G(t)$ is an increasing function, we obtain that
\[
\begin{align*}
 r \left[ \int_{\partial B(0, r)} u^2 ds \right] & \leq \int_{B(0, r)} u^2 dx + r^2 \left[ \int_0^1 \eta \left( (n-1)\eta n^{-2}r^n - 2G(r) \right) + 2 \int_{B(0, r)} (|\nabla u|^2 + u(u^{-\alpha} - h)) dx \right] d\eta \\
& \leq \int_{B(0, r)} u^2 dx + \frac{(n-1)}{n} \int_{\partial B(0, 1)} u^2(r, \theta) d\theta + r^2 \int_{B(0, r)} (|\nabla u|^2 + u(u^{-\alpha} - h)) dx \\
& = \int_{B(0, r)} u^2 dx + \frac{(n-1)}{n} \int_{\partial B(0, r)} u^2 ds + r^2 \int_{B(0, r)} (|\nabla u|^2 + u(u^{-\alpha} - h)) dx,
\end{align*}
\]
where we use the fact that for $r$ sufficiently small,
\[
(3.12) \quad \int_{B(0, r)} [u^{1-\alpha} - hu] dx \geq \int_{B(0, \xi_0)} [u^{1-\alpha} - hu] dx.
\]
We explain a little on the proof of (3.12). It follows from (3.9) that for $r$ sufficiently small, $J(r) := \int_{B(0, r)} u^{1-\alpha} - hu dx > 0$. This, the continuity of $\int_{B(0, r)} u^{1-\alpha} dx$; $\int_{B(0, r)} hu dx$ and $\lim_{r \to 0^+} J(r) = 0$ imply that $J(r)$ is increasing. Therefore, (3.12) holds. Thus, we see
\[
\frac{1}{n} \int_{\partial B(0, r)} u^2 ds \leq \int_{B(0, r)} u^2 dx + r^2 \int_{B(0, r)} (|\nabla u|^2 + u^{1-\alpha} - hu) dx.
\]
This and Lemma 3.2 imply that
\[
(3.13) \quad r^{-\alpha - 1} \int_{\partial B(0, r)} u^2 ds \leq C r^{-\alpha} \int_{B(0, r)} (|\nabla u|^2 + u^{1-\alpha} - hu) dx,
\]
where $C = C(n)$. This completes the proof.

Now we consider the second case:
\[
\lim_{r \to 0^+} E_u(x_0, r) < -\epsilon^*
\]
for $\epsilon^* > 0$ being given in Lemma 3.1.

**Lemma 3.4.** If $x_0 \in \Sigma$ and $\lim_{r \to 0^+} E_u(x_0, r) < -\epsilon^*$, then
\[
(3.14) \quad \lim_{r \to 0^+} \left[ \frac{1}{(\alpha - 1)} r^{-\mu} \int_{B(x_0, r)} u^{1-\alpha} dx + \frac{1}{(\alpha + 1)} r^{-\mu - 1} \int_{\partial B(x_0, r)} u^2 ds \right] \geq \epsilon^*/2.
\]

**Proof.** This follows from the second formulation (2.17) of $E_u(x_0, r)$.

Finally we are in the position to prove Theorem 1.1.
We shall show $\mathcal{H}^\mu(\Sigma) = 0$. We prove it by contradiction. Suppose $\mathcal{H}^\mu(\Sigma) > 0$ (possibly with infinite measure). Then since $\Sigma$ is a Souslin set, Theorem 5.6 and its proof in [Fa] say that, there is a closed subset $T \subset \Sigma$, with $0 < \mathcal{H}^\mu(T) < \infty$, and for some constant $\theta > 0$,

$$\mathcal{H}^\mu(T \cap B_r(x)) \leq \theta r^\mu$$

holds for any $x \in \mathbb{R}^n$, $r > 0$.

Let $\epsilon^* > 0$ be given in Lemma 3.1 and $x_0 \in T$. Then either

$$\lim_{r \to 0^+} E_u(x_0, r) \geq -\epsilon^*$$

or

$$\lim_{r \to 0^+} E_u(x_0, r) < -\epsilon^*.$$

In the first case, we have by Lemma 3.1

$$\lim_{r \to 0^+} r^{-\mu} \int_{B(x_0, r)} |\nabla u|^2 dx \geq \epsilon^*.$$

In the second case, we have (3.14). By Lemma 3.3 we have

$$\lim_{r \to 0^+} r^{-\mu} \int_{B(x_0, r)} \left[ |\nabla u|^2 + u^{1-\alpha} \right] dx \geq C\epsilon^*$$

for some constant $C = C(n, \alpha) > 0$, since

$$|r^{-\mu} \int_{B(x_0, r)} hudx| \leq Cr^{n-\mu}$$

and $n - \mu > 0$.

In conclusion, we have proved that there exists $\epsilon^* > 0$ such that if $x_0 \in T$, then

$$\lim_{r \to 0^+} r^{-\mu} \int_{B(x_0, r)} \left[ |\nabla u|^2 + u^{1-\alpha} \right] dx \geq C\epsilon^*,$$

for some $C = C(n, \alpha) > 0$. This implies that there exists $\delta_0 > 0$ sufficiently small such that for $0 < r < \delta_0$,

$$r^{-\mu} \int_{B(x_0, r)} \left[ |\nabla u|^2 + u^{1-\alpha} \right] dx \geq \frac{C}{2}\epsilon^*.$$

Then for any $0 < \delta < \frac{\delta_0}{10}$ and for any $U$ open, such that $T \subset U$,

$$\left\{ B_r(x) : x \in T, 0 < r < \frac{1}{2}\delta, B_r(x) \subset U \text{ and } r^{-\mu} \int_{B(x, r)} \left[ |\nabla u|^2 + u^{1-\alpha} \right] dx \geq \frac{C}{2}\epsilon^* \right\}$$

is a finite covering of $T$. Hence, by Vitali covering lemma, there is a pairwise disjoint subcollection $\{B_{x_k}(x_k)\}_{k=1}^\infty$, such that $T \subset \bigcup_{k=1}^\infty B_{\delta x_k}(x_k)$. Hence, it follows
from (3.16) that
\[
\mathcal{H}_5^\mu(T) \leq C(\mu) \sum_{k=1}^{\infty} (5r_k)^\mu
\]
\[
\leq C(n, \mu, \theta) \sum_{k=1}^{\infty} \int_{B_{r_k}(x_k)} \left[ |\nabla u|^2 + u^{1-\alpha} \right] dx
\]
\[
\leq C(n, \mu, \theta) \int_{U} \left[ |\nabla u|^2 + u^{1-\alpha} \right] dx.
\]
Since \(\mathcal{H}^\mu(T) < \infty\), we can choose \(U\) with arbitrary small \(\mathcal{H}^\mu\)-measure so that the right hand side of the inequality can be arbitrarily small. Thus we have \(\mathcal{H}_5^\mu(T) = 0\). Letting \(\delta \to 0\), we conclude \(\mathcal{H}^\mu(T) = 0\), which gives the contradiction and completes the proof of Theorem 1.1. 

\[
\square
\]

REFERENCES


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