2π-periodic self-similar solutions for the anisotropic affine curve shortening problem

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Abstract

We study the existence of 2π-periodic positive solutions of the equation

\[ u_{\theta\theta} + u = \frac{a(\theta)}{u^3}, \]

where \( a(\theta) \) is a positive smooth 2π-periodic function. A priori estimates and sufficient conditions for the existence of solutions of the equation are established.

1 Introduction and statement of the results

We are concerned in this paper with the equation

\[ u_{\theta\theta} + u = \frac{a(\theta)}{u^3}, \quad \theta \in S^1, \tag{1.1} \]

where \( a(\theta) \) is a positive smooth function on \( S^1 = \mathbb{R}/2\pi\mathbb{Z} \). Equation (1.1) arises from the study of the generalized curve shortening problem, which can be derived as follows. Consider the following generalized curve shortening problem

\[ \frac{\partial \gamma}{\partial t} = \Phi(\theta) |k|^{\sigma-1} k N, \quad \sigma > 0, \quad \theta \in S^1, \tag{1.2} \]

where \( \gamma(\cdot, t) \) is a planar curve, \( k(\cdot, t) \) is its curvature with respect to the unit normal \( N \), and \( \Phi \) is a positive function depending on the normal angle \( \theta \) of the curve.

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problem has been extensively studied in the last two decades. See [1-7, 15, 16, 18-19, 21-25, 32]. Assuming that $\gamma(\cdot, t)$ is convex, then we can use the normal angle $\theta$ to parameterize $\gamma$, and equation (1.2) is equivalent to

$$\frac{\partial w}{\partial t} = \frac{-\Phi'(\theta)}{(w_{\theta\theta} + w)^{\sigma}} \quad \theta \in S^1,$$

(3.3)

where $w(\theta, t)$ is the support function of $\gamma(\cdot, t)$. A self-similar solution is of the form $w(\theta, t) = \xi(t)u(\theta)$, which means that the shape of the curves does not change during the evolution governed by (1.2). Such solutions are important in understanding the long time behaviors and the structure of singularities of (1.2). It is rather easy to see that $\xi(t)u(\theta)$ is a self-similar solution if and only if $u$ satisfies

$$u_{\theta\theta} + u = \frac{a(\theta)}{w^{p+1}}, \quad \theta \in S^1$$

(4.4)

with $a(\theta) = \Phi'(\theta)$, $p + 1 = \frac{1}{\sigma}$ and $|\xi(t)|^{\sigma-1}\xi(t)\xi'(t) = -C$, where $C$ is a positive constant. The case $\sigma = \frac{1}{2}$ or equivalently, $p = 2$, (1.2) is called the affine curve shortening problem, and equation (4.4) becomes (1.1). Thus a solution of equation (1.1) is a self-similar solution of the anisotropic affine curve shortening problem. Note that in general $a(\theta)$ can only be assumed to be $2\pi$-periodic. Equation (4.4) also appears in image processing [32], 2-dimensional $L^p$-Minkowski problem [8, 27] and other problems [25].

Equation (1.1) is a special case of

$$u'' + f(\theta, u) = 0$$

(5.5)

where $f : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}$ is continuous, $T$-periodic in the first variable and has a singularity of repulsive type near the origin. The existence of periodic solutions of (5.5) had been studied by many people. Using the Poincaré-Birkhoff fixed point theorem, the following result was proved by del Pino, Manásevich and Montero in [17]. Let $\{\mu_n\}_{n=0}^\infty$ be the eigenvalues of

$$u'' + \mu u = 0$$

with $T$-periodic boundary conditions:

$$u(0) = u(T), \quad u'(0) = u'(T),$$

that is, $\mu_n = (\frac{2\pi n}{T})^2$, $n = 0, 1, \ldots$. If $f$ satisfies

$$\frac{c'}{s'} \leq -f(\theta, s) \leq \frac{c}{s'}, \quad \forall s \in (0, \delta),$$

(1.1)

for some positive constants $c, c', \delta$ and $\nu \geq 1$, and there exists a nonnegative integer $n$ such that

$$\frac{\mu_n}{4} < \liminf_{s \rightarrow +\infty} \frac{f(\theta, s)}{s} \leq \limsup_{s \rightarrow +\infty} \frac{f(\theta, s)}{s} < \frac{\mu_{n+1}}{4}$$

(2.2)
uniformly in \( \theta \in [0, T] \), then problem (1.5) possesses at least one \( T \)-periodic positive solution. This result gives a Fredholm alternative-like result for the problem

\[
u'' + \lambda u = \frac{a(\theta)}{u^\nu},
\]

(1.6)

which means that (1.6) possesses a \( T \)-periodic solution if \( \lambda \neq \frac{m^2}{4} \) for all \( n = 0, 1, \ldots \). Thus for \( T = 2\pi \), problem (1.6) has at least one \( 2\pi \)-periodic solution for \( \lambda \) satisfying

\[
\frac{n^2}{4} < \lambda < \frac{(n+1)^2}{4}, \quad n = 0, 1, \ldots
\]

Equation (1.4) with \( p \neq 2 \) has been studied by many authors. When \( a \equiv 1 \), all solutions of (1.4) can be classified. See Abresch and Langer [1] in the case of \( p = 0 \) and B. Andrews [5] in the case of general \( p \). When \( a \) is \( 2\pi \)-periodic, Dohmen and Giga [18], Dohmen, Giga and Mizoguchi [19] studied the \( p \leq 1 \) case. Matano and Wei [29] proved that (1.4) is solvable if \( 0 \leq p < 7, p \neq 2 \) and \( a \) is \( 2\pi \)-periodic.

These results do not cover the affine case, that is, equation (1.1). Indeed, the situation for the affine case is quite different. It is known that there are some obstructions for the existence and one can’t get a priori estimates for the solutions of (1.1) without additional assumptions on \( a \) due to the invariance of the problem. To see this, let us consider its simplest form

\[
u_{\theta\theta} + u = \frac{1}{u^3}, \quad \theta \in \mathbb{S}^1.
\]

(1.7)

Equation (1.7) is invariant under the action of the special linear group \( SL(2, \mathbb{R}) \). For any matrix

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

in \( SL(2, \mathbb{R}) \), we have an “affine diffeomorphism” on \( \mathbb{S}^1 \) given by

\[
(cos \theta, sin \theta) \mapsto \frac{(cos \theta, sin \theta)A^T}{\| (cos \theta, sin \theta)A^T \|} = (cos \tilde{\theta}, sin \tilde{\theta}),
\]

that is,

\[
\tan \tilde{\theta} = \frac{c + d \tan \theta}{a + b \tan \theta}, \quad ad - bc = 1.
\]

For any function \( u \) in \( \mathbb{S}^1 \), define

\[
u_{\tilde{\theta}}(\tilde{\theta}) = \cos \tilde{\theta} \sqrt{(a \tan \tilde{\theta} - c)^2 + (b \tan \tilde{\theta} - d)^2} u(\theta).
\]

Then \( u_{\tilde{\theta}}(\tilde{\theta}) \) is a solution of (1.7) if and only if \( u(\theta) \) is a solution of (1.7). Starting with the trivial solution \( u \equiv 1 \), one can show that all solutions of (1.7) are given by a 2-parameter family of functions

\[
u_{\varepsilon, t}(\theta) = (\varepsilon^2 \cos^2(\theta - t) + \varepsilon^{-2} \sin^2(\theta - t))^{\frac{1}{2}},
\]

(1.8)
for $(\varepsilon, t) \in (0, 1] \times [0, \pi)$. Thus the solutions of (1.7) are not bounded. The group $SL(2, \mathbb{R})$ plays an important role as that of the conformal group $Conf(S^2)$ in the Nirenberg’s problem in geometry, which has been extensively studied and many significant results have been obtained. See [9-14, 25, 27, 32, 34] and the references therein.

In [2], Ai, Chou and Wei considered the solvability of problem (1.1) under the assumption that $a$ is $\pi$-periodic. After scaling, in this case the problem is equivalent to the equation (1.6) with $T = \pi$, $\lambda = \frac{\mu_n}{4} = 1$ and $\nu = 3$. Let

$$B(\theta) = \int_0^{\pi} \frac{a(\theta + t) - a(\theta) - 2^{-1}a'(\theta)\sin 2t}{\sin^2 t} dt = \int_0^{\pi} \frac{(a'(\theta + t) - a'(\theta))\sin 2t}{\sin^2 t} dt.$$

A function $a$ is called $B$-nondegenerate if $B(\theta)$ never vanishes at any critical point $\theta$ of the function $a$. They proved that if $a$ is a positive, $B$-nondegenerate and $C^2$-function of period $\pi$, then one can get a priori estimates for all $\pi$-periodic solutions of (1.1). Moreover, if the winding number of the map $G$ around the origin is not equal to $-1$, where

$$G(\theta) = (-B(\theta), a'(\theta)) \quad \theta \in [0, \pi),$$

then problem (1.1) has a $\pi$-periodic solution.

In this paper we study equation (1.1) for $2\pi$-periodic function $a$, which is more interesting and natural from geometric point of view. We will consider a slightly general form, that is, for a fixed $n \geq 2$, the existence of $n\pi$-periodic solutions of

$$u_{\theta\theta} + u = \frac{a(\theta)}{u^3}$$

(1.9)

where $a$ is $n\pi$-periodic. It is the same as the equation (1.6) with $T = \pi$, $\nu = 3$ and $\lambda = \frac{\mu_n}{4} = n^2$ by scaling.

Before we state our results, we comment on the difficulties in studying (1.9). A major problem is to study possible blow-ups. When $a(\theta)$ is $\pi$-periodic, since the blow-up sequence $u_{\varepsilon, \ell}$ (defined at (1.8)) is $\pi$-periodic, only single blow-up can occur. However, when $a$ is $n\pi$-periodic, there are $n$ possible blow-ups. We have to analyze the interaction between different blow-ups.

To state our main results, for any positive $C^2$-function $a(\theta)$, we define

$$A_n(\theta) = \sum_{j=1}^{n} \frac{a'(\theta + (j - 1)\pi)}{\sqrt{a(\theta + (j - 1)\pi)}}$$

(1.10)

and

$$B_n(\theta) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left( \sum_{j=1}^{n} a'(\theta + t + (j - 1)\pi) - \sum_{j=1}^{n} a'(\theta + (j - 1)\pi) \right) \sin 2t}{\sin^2 t} dt. \quad (1.11)$$
Following [2], a function \( a \) is called \( B_n \)-nondegenerate if \( B_n(\theta) \) does not vanish whenever \( A_n(\theta) = 0 \). Note that by definition, \( A_n(\theta) \) and \( B_n(\theta) \) are \( \pi \)-periodic.

Our first result is the a priori estimates.

**Theorem 1.1** Let \( a \) be a positive, \( C^2 \) and \( n\pi \)-periodic function. Suppose that \( a(\theta) \) is \( B_n \)-nondegenerate. Then there exists a constant \( C \) depending on \( \alpha \) only such that
\[
\frac{1}{C} \leq u \leq C
\]
for any \( n\pi \)-periodic solution \( u \) of (1.9).

As for the existence we have

**Theorem 1.2** Let \( a \) be a positive, \( C^2 \) and \( n\pi \)-periodic function. Suppose that
\[
\min_{A_n(\theta) = 0} B_n(\theta) > 0, \quad \text{or} \quad \max_{A_n(\theta) = 0} B_n(\theta) < 0,
\]
then equation (1.9) has an \( n\pi \)-periodic solution.

An example of \( a \) satisfying (1.13) is
\[
a(\theta) = (1 + b_1 \cos \theta + b_2 \cos 2\theta)^2,
\]
where \( b_1 \) and \( b_2 \) are some constants in \( (-\frac{1}{2}, \frac{1}{2}) \) to be determined later. It is easy to see that \( A_2(\theta) = -8b_2 \sin 2\theta \). Hence \( A_2(\theta) = 0 \) if and only if \( \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2} \) and
\[
a(\theta + t + \pi) + a(\theta + t) = \begin{cases} 
2(1 + b_2 \cos 2t)^2 + 2(b_1 \cos t)^2, & \theta = 0, \pi; \\
2(1 - b_2 \cos 2t)^2 + 2(b_1 \sin t)^2, & \theta = \frac{\pi}{2}, \frac{3\pi}{2}.
\end{cases}
\]

Then
\[
(a(\theta + t + \pi) + a(\theta + t))' = \begin{cases} 
4(1 + b_2 \cos 2t)(-2b_2 \sin 2t) - 2b_1^2 \sin 2t, & \theta = 0, \pi; \\
4(1 - b_2 \cos 2t)(2b_2 \sin 2t) + 2b_1^2 \sin 2t, & \theta = \frac{\pi}{2}, \frac{3\pi}{2}
\end{cases}
\]
and
\[
B_2(\theta) = \begin{cases} 
-4(b_1^2 + 4b_2)\pi - 8b_2^2 \pi, & \theta = 0, \pi; \\
4(b_1^2 + 4b_2)\pi - 8b_2^2 \pi, & \theta = \frac{\pi}{2}, \frac{3\pi}{2}
\end{cases}
\]
Thus (1.13) is satisfied if \( b_1^2 + 4b_2 = 0 \) and \( b_2 \neq 0 \).

Let
\[
G_n(\theta) = \left(-B_n(\theta), A_n(\theta)\right) : \mathbb{S}^1 \to \mathbb{R}^2,
\]
\[
\mathcal{G}_n(\theta) = \frac{G_n(\theta)}{|G_n(\theta)|}: \mathbb{S}^1 \to \mathbb{S}^1
\]

whose Brouwer degree \( \text{deg}(\mathcal{G}_n, \mathbb{S}^1) \) is well defined if \( a \) is \( B_n \)-nondegenerate. See the Appendix or [20] for definitions and properties. With the condition (1.13), we see that \( \text{deg}(\mathcal{G}_n, \mathbb{S}^1) = 0 \). (See (i) of the Appendix.)

The following result concerns with the case \( \text{deg}(\mathcal{G}_n, \mathbb{S}^1) \neq 0 \). For any fixed \( C^2 \)-function \( a(\theta) \), let

\[
\mathcal{A}_n(\theta) = \sum_{j=1}^{n} a'\left(\theta + (j - 1)\pi\right),
\]

\[
\mathcal{B}_n(\theta) = \int_{0}^{\pi} \frac{\left(\sum_{j=1}^{n} a'\left(\theta + t + (j - 1)\pi\right) - \sum_{j=1}^{n} a'(\theta + (j - 1)\pi)\right) \sin 2t}{\sin^2 t} dt,
\]

\[
\mathcal{C}_n(\theta) = (-\mathcal{B}_n(\theta), \mathcal{A}_n(\theta)).
\]

For \( a = 1 + \varepsilon a \) we have \( \mathcal{A}_n(\varepsilon)(\theta) \) and \( \mathcal{B}_n(\varepsilon)(\theta) \) given by (1.10) and (1.11). Then

\[
(-\mathcal{B}_n(\varepsilon)(\theta), \mathcal{A}_n(\varepsilon)(\theta)) = (-\mathcal{B}_n(\theta), \mathcal{A}_n(\theta))\varepsilon + O(\varepsilon^2).
\]

Thus for small \( \varepsilon, 1 + \varepsilon a \) is \( B_n \)-nondegenerate if \( \mathcal{C}_n(\theta) := (-\mathcal{B}_n(\theta), \mathcal{A}_n(\theta)) \neq (0, 0) \).

By homotopy invariance of degree, \( \text{deg}(\mathcal{G}_n, \mathbb{S}^1) = \text{deg}\left(\frac{\mathcal{C}_n}{|\mathcal{C}_n|}, \mathbb{S}^1\right) \).

**Theorem 1.3** Let \( a \) be a positive, \( C^2 \) and \( n\pi \)-periodic function such that for all \( \theta \), \( \mathcal{C}_n(\theta) = (-\mathcal{B}_n(\theta), \mathcal{A}_n(\theta)) \neq (0, 0) \) and \( \text{deg}(\frac{\mathcal{C}_n}{|\mathcal{C}_n|}, \mathbb{S}^1) \neq -2 \). Then the equation

\[
u_{\theta\theta} + u = \frac{1 + \varepsilon a(\theta)}{u^3}
\]

has an \( n\pi \)-periodic solution if \( \varepsilon \) is small.

It is not difficult to see that if \( n = 1 \), the map \( \mathcal{C}_n(\theta) \) is the same as the map \( G \) in [2] and it is \( \pi \)-periodic. In this case, we fix \( \varepsilon << 1 \) and consider the homotopy of \( a_s(\theta) = (1 - s)(1 + \varepsilon a(\theta)) + sa(\theta) \). For \( s \in [0, 1] \), the function \( a_s \) is always \( B \)-nondegenerate, and thus one can get uniform a priori estimates for all \( \pi \)-periodic solutions of

\[
u_{\theta\theta} + u = \frac{a_s(\theta)}{u^3}.
\]

Using the degree argument, one can solve the problem up to \( s = 1 \) if \( \text{deg}(\frac{\mathcal{C}_n}{|\mathcal{C}_n|}, \mathbb{R}/\pi\mathbb{Z}) = \text{winding number of } G+1 \neq 0 \). However, for \( n \geq 2 \), we do not know how to construct such a homotopy. The degree argument can only be used for small \( \varepsilon \). The general \( a \) case remains open.

We briefly sketch the idea of the proofs of our results. We only consider the case \( n = 2 \) since that of \( n \geq 3 \) is the same. To prove Theorems 1.1 and 1.2, we take a sequence \( \lambda_k \) such that \( \forall k, \lambda_k \neq 1, \lambda_k \to 1 \) as \( k \to \infty \) and consider the equation

\[
u_{\theta\theta} + \lambda_k u = \frac{a(\theta)}{u^3}, \quad \theta \in \mathbb{S}^1.
\]
According to [17], there is a \(2\pi\)-periodic solution \(u_k\). By careful analysis of blow-up, we can get asymptotic estimates of \(u_k\) and \(\lambda_k - 1\). Under the condition (1.13), these estimates ensure that \(u_k\) converges to a solution of (1.1) if \(\lambda_k \nearrow 1\) or \(\lambda_k \searrow 1\) as \(k \to \infty\). The proof of Theorem 1.3 follows from the Liapunov-Schmidt reduction and degree argument.

The paper is organized as follows. In section 2, some asymptotical estimates are given based on blow-up analysis, and in section 3 we give the necessary condition for the existence and sharp estimates. Theorem 1.1 and Theorem 1.2 are proved in section 4, and in section 5 we provide a proof of Theorem 1.3.

## 2 Preliminary Blow-up Analysis

Let \(a\) be a positive \(2\pi\)-periodic function, \(\lambda_k\) be a sequence such that \(\forall k, \lambda_k \neq 1, \lambda_k \to 1\) as \(k \to \infty\) and \(u_k\) be a \(2\pi\)-periodic solution of

\[
\dot{u}_{\theta\theta} + \lambda_k u = \frac{a(\theta)}{u^2}, \quad \theta \in [0, 2\pi].
\]

(2.1)

The aim of this section is to derive some asymptotical estimates of \(u_k\) and \(\lambda_k - 1\), which will lead to sharp estimates in the next section.

Here and after, unless otherwise stated, the letter \(C\) will always denote various generic constants which are independent of \(k\). We denote \(A \sim B\) if there exist two positive uniform constants \(C_1, C_2\) such that \(C_1 A \leq B \leq C_2 A\). \(C_k = o(1)\) means that \(\lim_{k \to +\infty} C_k = 0\).

We start with the following simple but useful lemma.

**Lemma 2.1** For any solution \(u\) of problem (2.1), defining \(F_k(\theta) = u_k^2 \theta + \lambda_k u_k^2 + \frac{a(\theta)}{u_k^2}\), then there is a constant \(C\) independent of \(k\) such that

\[
C^{-1}F_k(\theta_2) \leq F_k(\theta_1) \leq CF_k(\theta_2), \quad \forall \theta_1, \theta_2 \in [0, 2\pi].
\]

**Proof.** By equation (2.1), we can easily get that

\[
F_k'(\theta) = \frac{a'(\theta)}{u_k^2},
\]

which implies that

\[
|F_k'(\theta)| \leq C|F_k(\theta)|
\]

since \(a(\theta)\) is smooth and positive. Thus

\[
|\log F_k'| \leq C,
\]

which leads to

\[
C^{-1} \leq \frac{F_k(\theta_1)}{F_k(\theta_2)} \leq C, \quad \forall \theta_1, \theta_2 \in [0, 2\pi].
\]
Thus we finish the proof. □

From the above lemma, we know that $u_k$ is bounded from above if and only if it is bounded from below. Without loss of generality, we assume that $\min_{\theta \in [0, 2\pi]} u_k(\theta) \to 0$. Then $F_k(\theta) \to +\infty$ uniformly in $\theta$ as $k \to \infty$. This in particular implies that, as $k \to \infty$,

either $u_k(\theta)$ or $\frac{1}{u_k(\theta)} \to +\infty$ whenever $u_k(\theta) = 0$.

If $u_k'(\tau_k) = 0$ and $u_k(\tau_k) \to 0$, then by the equation (2.1) we see that $u_k''(\tau_k) = \frac{a(\tau_k)}{u_k(\tau_k)} - \lambda_k u_k(\tau_k) > 0$ and hence $\tau_k$ is a local minimum point. Similarly, if $u_k'(\theta_k) = 0$ and $u_k(\theta_k) \to +\infty$, then $u_k''(\theta_k) < 0$ and $\theta_k$ is a local maximum point. This also implies that the values of $u_k$ at any local maximum point of $u_k$ must approach $+\infty$, and the value of $u_k$ at any local minimum point of $u_k$ must approach 0.

Hence for $k >> 1$, the local minimum and maximum points of $u_k$ are isolated. Therefore they appear alternatively and satisfy

$$u_k(\tau_k) \sim \frac{1}{u_k(\theta_k)},$$

where $\tau_k$ and $\theta_k$ are local minimum and maximum points of $u_k$, respectively.

Let $\tau_k^1 < \tau_k^2$ be two consecutive local minimum and maximum points of $u_k$ and let $\theta_k \in [\tau_k^1, \tau_k^2]$ satisfy $u_k(\theta_k) = M_k = \max_{\theta \in [\tau_k^1, \tau_k^2]} u_k(\theta)$. Then $u_k(\tau_k^1), u_k(\tau_k^2) \to 0$ and $M_k \to \infty$ as $k \to \infty$. We have the following convergence result.

**Lemma 2.2** For $k \to \infty$ we have

$$\theta_k - \tau_k^1 \to \frac{\pi}{2},$$

$$\tau_k^2 - \tau_k^1 \to \pi,$$

$$\frac{u_k(\frac{\tau_k^2 - \tau_k^1}{\pi} \theta + \tau_k^1)}{M_k} \to \sin \theta \text{ uniformly in } [0, \pi].$$

In particular, $u_k$ has two minimum and two maximum points in $[0, 2\pi]$.

**Proof.** We first prove that $\theta_k - \tau_k^1 \leq \frac{\pi}{2}$. The idea is to compare with the case when $a$ is constant.

Let $m_k = u(\tau_k^1)$. Then $m_k \to 0$ and $M_k \sim \frac{1}{m_k}$. On the interval $[\tau_k^1, \theta_k]$, $u_k$ satisfies

$$\frac{C_1}{2u_k^3} \leq u_k'' + \lambda_k u_k \leq \frac{C_2}{2u_k^3}, \quad u_k' > 0 \text{ in } (\tau_k^1, \theta_k).

(2.5)

Set

$$H_C(u_k) = \lambda_k u_k^2 + \frac{C}{u_k^2}.$$

Then $(u_k')^2 + H_C(u_k)$ is strictly increasing over $[\tau_k^1, \theta_k]$ while $(u_k')^2 + H_{C_2}(u_k)$ is strictly decreasing over $[\tau_k^1, \theta_k]$.

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Let \( H_{C_2}(\hat{m}_k) = H_{C_2}(M_k) \) where \( \hat{m}_k < 1 \). Then since \( H_{C_2}(m_k) > H_{C_2}(M_k) = H_{C_2}(\hat{m}_k) \), we see that \( \hat{m}_k > m_k \) and \( \hat{m}_k \sim M_k^{-1} \sim m_k \). So there exists a unique \( \hat{\tau}_k^1 \in (\tau_k^1, \theta_k) \) such that \( u_k(\hat{\tau}_k^1) = \hat{m}_k \).

Over \([\tau_k^1, \hat{\tau}_k^1]\), by equation (2.1), \( u_k''(\theta) \sim m_k^{-3} \). Hence \( u_k'(\theta) \sim m_k^{-3}(\theta - \tau_k^1) \) for \( \theta \in [\tau_k^1, \hat{\tau}_k^1] \). Therefore

\[
u_k(\hat{\tau}_k^1) - u_k(\tau_k^1) = \int_{\tau_k^1}^{\hat{\tau}_k^1} u'(\theta)d\theta \sim m_k^{-3}(\hat{\tau}_k^1 - \tau_k^1)^2 \]

which implies that

\[
\hat{\tau}_k^1 - \tau_k^1 = O(m_k^2). \tag{2.6}
\]

On the other hand, we have

\[
(u_k'(\theta))^2 \geq H_{C_2}(M_k) - H_{C_2}(u_k), \quad \forall \theta \in [\tau_k^1, \theta_k].
\]

Hence

\[
\theta_k - \hat{\tau}_k^1 \leq \int_{\tilde{m}_k}^{M_k} \frac{du}{\sqrt{H_{C_2}(M_k) - H_{C_2}(u)}}. \tag{2.7}
\]

Let \( T_k = \int_{\tilde{m}_k}^{M_k} \frac{du}{\sqrt{H_{C_2}(M_k) - H_{C_2}(u)}} \). Then we note that \( T_k \) is the length between consecutive minimum and maximum points of the equation

\[
v'' + \lambda_k v = \frac{C_2}{v^3}, \quad v(0) = \tilde{m}_k, \quad v(T_k) = M_k, \quad v'(\theta) > 0 \text{ for } \theta \in (0, T_k) \tag{2.8}
\]

since \( v'(\theta) = \sqrt{H_{C_2}(M_k) - H_{C_2}(v)} \). On the other hand, by rescaling of \( \tilde{v}(\tilde{\theta}) = (\frac{C_2}{\lambda_k})^{-\frac{1}{3}} v(\frac{\tilde{\theta}}{\sqrt{\lambda_k}}) \), \( \tilde{v} \) satisfies

\[
\tilde{v}'' + \tilde{\theta} = \frac{1}{\tilde{v}^3}. \tag{2.9}
\]

By (1.8), all solutions of (2.9) have period \( \pi \). This implies that

\[
T_k = \int_{\tilde{m}_k}^{M_k} \frac{du}{\sqrt{H_{C_2}(M_k) - H_{C_2}(u)}} = \frac{\pi}{2\sqrt{\lambda_k}}. \tag{2.10}
\]

So

\[
\lim_{k \rightarrow +\infty} (\theta_k - \hat{\tau}_k^1) = \lim_{k \rightarrow +\infty} (\theta_k - \hat{\tau}_k^1 + \hat{\tau}_k^1 - \tau_k^1) \leq \frac{\pi}{2}. \tag{2.11}
\]

Similarly, we can prove that

\[
\lim_{k \rightarrow +\infty} (\tau_k^2 - \theta_k) \leq \frac{\pi}{2}, \quad \lim_{k \rightarrow +\infty} (\tau_k^2 - \tau_k^1) \leq \pi. \tag{2.12}
\]

Now we let \( \tilde{u}_k = \frac{u_k(\frac{\tau_k^2 - \theta_k}{\pi} + \frac{\tau_k^1 - \tau_k^1}{\pi})}{M_k} \). Then it follows from equation (2.1) that \( \tilde{u}_k \) satisfies

\[
\tilde{u}_{k, \theta} + (\tau_k^2 - \tau_k^1)^2 \lambda_k \tilde{u}_k = \frac{2a(\tau_k^2 - \tau_k^1)}{\pi M_k^2} \frac{\theta - \tau_k^1}{\tilde{u}_k^{(2)}(\pi)}. \tag{2.13}
\]
Integration by parts gives that
\[
(\frac{\tau_k^2 - \tau_k^1}{\pi})^2 \lambda_k \int_0^\pi \ddot{u}_k^2 \, d\theta - \int_0^\pi \ddot{u}_{k,\theta} d\theta = (\frac{\tau_k^2 - \tau_k^1}{\pi M_k^2}) \int_0^\pi a(\frac{\tau_k^2 - \tau_k^1}{\pi} \theta + \tau_k^1) \ddot{u}_k^2 \, d\theta \geq 0
\]

since \(\ddot{u}_{k,\theta}(0) = \ddot{u}_{k,\theta}(\pi) = 0\). Hence
\[
\int_0^\pi \ddot{u}_{k,\theta}^2 \, d\theta \leq (\frac{\tau_k^2 - \tau_k^1}{\pi})^2 \lambda_k \int_0^\pi \ddot{u}_k^2 \, d\theta. \tag{2.14}
\]

Using the fact that \(0 < \tilde{u}_k(\theta) \leq \max_{\theta \in [0, \pi]} \tilde{u}_k(\theta) = 1\) and (2.14), we deduce that \(\ddot{u}_k\) is bounded in \(H^1([0, \pi])\). Thus we can assume \(\ddot{u}_k \rightharpoonup \ddot{u}\) weakly in \(H^1([0, \pi])\) and \(\tau = \lim_{k \to \infty} (\tau_k^2 - \tau_k^1)\). Letting \(k \to \infty\), one gets
\[
\int_0^\pi \ddot{u}_\theta^2 \, d\theta \leq \frac{\tau^2}{\pi^2} \int_0^\pi \ddot{u}^2 \, d\theta. \tag{2.15}
\]

By the embedding theorem, \(\ddot{u}_k \to \ddot{u}\) in \(C([0, \pi])\), and so
\[
\ddot{u}(0) = \lim_{k \to \infty} \frac{u_k(\tau_k^1)}{M_k} = 0, \quad \ddot{u}(\pi) = \lim_{k \to \infty} \frac{u_k(\tau_k^2)}{M_k} = 0
\]

and \(\ddot{u} \in H_0^1([0, \pi])\).

On the other hand, by Wirtinger’s Inequality, we know that
\[
\int_0^\pi \ddot{u}^2 \, d\theta \leq \int_0^\pi \ddot{u}_\theta^2 \, d\theta, \tag{2.16}
\]

and equality holds if and only if \(\ddot{u} = C \sin \theta\).

Combining (2.15) and (2.16), we see that \(\tau \geq \pi\). But according to the previous argument, \(\tau \leq \pi\) and hence \(\tau = \pi\) and the equality at (2.16) holds. Therefore, \(\ddot{u} = C \sin \theta\). The assumption that \(\max_{\theta \in [0, \pi]} \ddot{u}_k(\theta) = 1\) yields that \(\max_{\theta \in [0, \pi]} \ddot{u}(\theta) = 1\). Hence \(\ddot{u} = \sin \theta\) and \(\ddot{u}_k \to \sin \theta\) in \(C^1([0, \pi])\). This proves (2.4). The proof of Lemma 2.2 is completed.

The following so-called Pohozaev’s Identity will be used frequently in the rest of the paper.

**Lemma 2.3** Let \(a\) be a positive, \(C^2\) and \(2\pi\)-periodic function. Then we have
\[
\int_0^{2\pi} \frac{a'(\theta) + 4(1 - \lambda_k)u^3u'}{u^2} (1 + \cos 2\theta) d\theta = 0 \tag{2.17}
\]

and
\[
\int_0^{2\pi} \frac{a'(\theta) + 4(1 - \lambda_k)u^3u'}{u^2} \sin 2\theta d\theta = 0 \tag{2.18}
\]

for any \(2\pi\)-periodic solution \(u\) of (2.1).
Proof. For any solution $u$ of (2.1), we have
\[
\left(\frac{u'^2}{2}\right)'' + 4\left(\frac{u'^2}{2}\right)' = \frac{(a(\theta) + (1 - \lambda_k)u^4)' - u^2}{u^2}
\]
and the lemma follows from an integration over $[0, 2\pi]$. \hfill \Box

Let $\varepsilon_k = u_k(\tau_k) = \min_{\theta \in [0, 2\pi]} u_k(\theta) \to 0$ and let $\theta_k$ be the next local maximum point of $u_k$. Then $M_k = u_k(\theta_k) \to \infty$. The above lemmas lead to the following estimates.

**Proposition 2.4** There exists a uniform positive constant $C$ such that
\[
|\lambda_k - 1| \leq C\varepsilon_k^2.
\]

**Proof.** We first prove
\[
\int_0^{2\pi} \frac{1}{u_k^2} d\theta \leq C. \tag{2.21}
\]
To this end, let $\overline{u}_k(\theta) = u_k\left(\frac{2(\theta - \tau_k)}{\pi} + \tau_k\right)$, $\theta \in [0, \frac{\pi}{2}]$ and $\overline{\lambda}_k = (\frac{2(\theta_k - \tau_k)}{\pi})^2 \lambda_k$. Then $\overline{u}_k$ satisfies
\[
\overline{u}_{k,\theta} + \overline{\lambda}_k \overline{u}_k = \left(\frac{2(\theta_k - \tau_k)}{\pi}\right)^2 \frac{a(\theta_k - \tau_k)}{\overline{u}_k^3} \left(2\overline{u}_k - \tau_k\right)
\]
and
\[
\left(\frac{\overline{u}_k^2}{2}\right)'' + 4\left(\frac{\overline{u}_k^2}{2}\right)' = \left(\frac{2(\theta_k - \tau_k)}{\pi}\right)^3 \frac{a'(\theta_k - \tau_k)}{\overline{u}_k^2} \left(2\overline{u}_k - \tau_k\right) + 4(1 - \overline{\lambda}_k)\overline{u}_k\overline{u}_k'. \tag{2.23}
\]
By virtue of $\overline{u}_k(0) = 0$ and $\overline{u}_k'(\frac{\pi}{2}) = 0$ we obtain
\[
\int_0^{\frac{\pi}{2}} \left(\frac{\overline{u}_k^2}{2}\right)'' + 4\left(\frac{\overline{u}_k^2}{2}\right)'\sin 2\theta d\theta = 0. \tag{2.24}
\]
Consequently, from (2.23) and (2.24) we deduce that
\[
\left(\frac{2(\theta_k - \tau_k)}{\pi}\right)^3 \int_0^{\frac{\pi}{2}} a'(\theta_k - \tau_k) \overline{u}_k\sin 2\theta d\theta = 4(\overline{\lambda}_k - 1) \int_0^{\frac{\pi}{2}} \overline{u}_k\overline{u}_k' \sin 2\theta d\theta
\]
\[
= -4(\overline{\lambda}_k - 1) \int_0^{\frac{\pi}{2}} \overline{u}_k^2 \cos 2\theta d\theta. \tag{2.25}
\]
It follows from Lemma 2.2 that $\exists C > 0$ such that
\[
|\int_0^{\frac{\pi}{2}} \overline{u}_k^2 d\theta| \leq C|\int_0^{\frac{\pi}{2}} \overline{u}_k^2 \cos 2\theta d\theta|.
\]
Hence for small $\delta$, 

\[
\bar{x}_k - 1 \int_0^\frac{\pi}{\delta} \frac{\tau_k^2}{u_k^2} d\theta \leq C \left| \int_0^\frac{\pi}{\delta} d\left( \frac{2(\theta_k - \tau_k)}{\pi} \theta + \tau_k \right) \frac{1}{u_k^2} d\theta \right| \\
\leq C(\int_0^\delta f_k(\theta) d\theta + \int_\frac{\delta}{\pi}^\frac{\pi - \delta}{\pi} f_k(\theta) d\theta + \int_\frac{\pi - \delta}{\pi}^\frac{\pi}{\delta} f_k(\theta) d\theta) \\
\leq C\delta \int_0^\frac{\pi}{\delta} \frac{1}{u_k^2} d\theta + C(\delta)M_k^{-2}
\]

where $f_k(\theta) = \frac{e^{(2(\theta_k - \tau_k))\sin \theta}}{u_k^2}$, $C(\delta)$ is a constant depending on $\delta$ and in the last inequality we have used (2.4).

On the other hand, from (2.22) and (2.26) we get 

\[
\int_0^\frac{\pi}{\delta} (\tau_k^2 - \frac{\tau_k^2}{u_k^2}) d\theta \\
= \left( \frac{2(\theta_k - \tau_k)}{\pi} \right)^2 \int_0^\frac{\pi}{\delta} \frac{(2(\theta_k - \tau_k)\theta + \tau_k)}{u_k^2} d\theta + (1 - \bar{x}_k) \int_0^\frac{\pi}{\delta} \frac{\tau_k^2}{u_k^2} d\theta \\
\geq \left( \frac{2(\theta_k - \tau_k)}{\pi} \right)^2 \int_0^\frac{\pi}{\delta} \frac{(2(\theta_k - \tau_k)\theta + \tau_k)}{u_k^2} d\theta - C\delta \int_0^\frac{\pi}{\delta} \frac{1}{u_k^2} d\theta - C(\delta)M_k^{-2} \\
\geq \frac{\min_{\theta \in S^1} a(\theta)}{2} \int_0^\frac{\pi}{\delta} \frac{1}{u_k^2} d\theta - C
\]

since $\delta$ is small. The left hand side of (2.27) can be estimated as follows. Reflecting the function $u_k$ with respect to $\theta = \frac{\pi}{2}$, we get a new function defined on $[0, \pi]$, still denoted by $u_k$. Then $u_k(0) = u_k(\pi)$. By Sobolev Inequality (See Proposition 1.3 of [2]) we have that 

\[
\int_0^\pi \frac{1}{u_k^2} d\theta \int_0^\pi (u_k^2 - u_k^2 d\theta) \leq \pi^2.
\]

Since $u_k$ is symmetric with respect to $\theta = \frac{\pi}{2}$, we see that 

\[
\int_0^\pi \frac{1}{u_k^2} d\theta \int_0^\pi (u_k^2 - u_k^2 d\theta) \leq \frac{\pi^2}{4}. 
\]

(2.28)

Combining (2.27) with (2.28) we obtain 

\[
\int_0^\frac{\pi}{\delta} \frac{1}{u_k^2} d\theta \left( \frac{\min_{\theta \in S^1} a(\theta)}{2} \int_0^\frac{\pi}{\delta} \frac{1}{u_k^2} d\theta - C \right) \leq \frac{\pi^2}{4}.
\]

(2.29)

Consequently, 

\[
\int_0^\frac{\pi}{\delta} \frac{1}{u_k^2} d\theta \leq C 
\]

(2.30)
and
\[ \int_{\tau_k}^{\theta_k} \frac{1}{u_k^2} d\theta \leq \frac{2(\theta_k - \tau_k)}{\pi} \int_0^{\frac{\pi}{2}} \frac{1}{u_k^2} d\theta \leq C. \] (2.31)

Similarly, let \( \tau_k^l \) be the local minimum point of \( u_k \) next to \( \theta_k \), we have
\[ \int_{\tau_k}^{\tau_k^l} \frac{1}{u_k^2} d\theta \leq C. \] (2.32)

It follows from (2.31) and (2.32) that
\[ \int_{\tau_k}^{2\pi+\tau_k} \frac{1}{u_k^2} d\theta \leq C. \] (2.33)

The same argument yields
\[ \int_{\tau_k}^{2\pi+\tau_k} \frac{1}{u_k^2} d\theta \leq C. \] (2.34)

(2.21) follows from (2.33) and (2.34) since \( u_k \) has two local minimum and maximum points on \([0, 2\pi)\).

Now we can prove the estimate (2.20). Using the identity (2.18) we see that
\[ 4|\lambda - 1| \left| \int_0^{2\pi} u_k(\theta) u_k'(\theta) \sin 2\theta d\theta \right| \leq \left| \int_0^{2\pi} \frac{d'(\theta) \sin 2\theta}{u_k^2} \right| \leq C. \] (2.35)

It is easy to see from Lemma 2.2 that
\[ \left| \int_0^{2\pi} u_k(\theta) u_k'(\theta) \sin 2\theta d\theta \right| = \left| \int_0^{2\pi} u_k^2(\theta) \cos 2\theta d\theta \right| \geq CM_k^2. \] (2.36)

Inserting (2.36) into (2.35), we lead to
\[ |\lambda - 1| \leq \frac{C}{M_k^2} \leq C\varepsilon_k^2. \]

Thus (2.20) is proved.

Let \( \varepsilon_k = \min_{\theta \in [0, 2\pi]} u_k(\theta) = u_k(\tau_k) \to 0 \) and
\[ U_\varepsilon(\theta) = (\varepsilon^2 \cos^2 \theta + \varepsilon^{-2} \sin^2 \theta)^{\frac{1}{2}}. \]

We define a transformation
\[ \theta = \tau_k + \psi_k(y) = \tau_k + \int_y^{\psi_k(y)} \frac{1}{U_{\varepsilon_k}^{-1}(\tau)} d\tau, \] (2.37)

where \( \varepsilon_k = (a(\tau_k))^{-\frac{1}{4}} \varepsilon_k \). It induces a rule of transformation of the equation (2.1) as follows: let
\[ v_k(y) = U_{\varepsilon_k}^{-1}(y) u_k(\tau_k + \psi_k(y)) = U_{\varepsilon_k}^{-1}(\theta - \tau_k) u_k(\theta), \]

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using
\[
\frac{dy}{d\theta} = U_{\varepsilon_k}^{-2}(\theta - \tau_k), \quad \text{and}
\]
\[
\frac{d^2y}{d\theta^2} = (\varepsilon_k^2 - \varepsilon_k^{-2}) \sin 2(\theta - \tau_k) U_{\varepsilon_k}^{-4}(\theta - \tau_k),
\]

one can verify that
\[
\frac{d^2v_k}{dy^2} + v_k = \frac{a(\tau_k + \psi_k)}{v_k^4} + (1 - \lambda_k) \frac{v_k}{U_{\varepsilon_k}^{-4}(\theta - \tau_k)}.
\] (2.38)

Using the estimate (2.20), we can prove the following result on the asymptotical behavior of \( u_k \).

**Proposition 2.5** Let \( a \) be a positive, \( C^2 \) and \( 2\pi \)-periodic function and let \( u_k \) be a solution of (2.1) such that \( \varepsilon_k = \min_{\theta \in [0, 2\pi]} u_k(\theta) = u_k(\tau_k) \to 0 \) and \( \tau_k \to \theta_0 \). Then the functions \( v_k, v_{k,y}, \frac{1}{v_k} \) are bounded. That is, \( \exists C > 0 \) such that
\[
\frac{1}{C} \leq v_k(y) \leq C, \quad |v_{k,y}(y)| \leq C, \quad y \in [0, 2\pi],
\] (2.39)

which implies
\[
C_1 U_{\varepsilon_k}^{-1}(\theta - \tau_k) \leq u_k(\theta) \leq C_2 U_{\varepsilon_k}^{-1}(\theta - \tau_k), \quad \theta \in [0, 2\pi].
\] (2.40)

Moreover, for any \( 0 < \alpha < 1 \)
\[
v_k \to v^\infty \quad \text{in} \quad C^{1,\alpha}([0, 2\pi]), \quad k \to \infty,
\] (2.41)

where the function \( v^\infty \) is given by
\[
v^\infty(y) = \begin{cases} a^{\frac{1}{2}}(\theta_0), & y \in [-\frac{\pi}{2}, \frac{\pi}{2}], \\ (a^{\frac{1}{2}}(\theta_0) \sin^2 y + a(\theta_0 + \pi)a^{-\frac{1}{2}}(\theta_0) \cos^2 y)^{\frac{1}{2}}, & y \in [\frac{\pi}{2}, \frac{3\pi}{2}]. \end{cases}
\] (2.42)

**Proof.** For simplicity of notations we assume \( \tau_k = 0 \) and \( \theta_0 = 0 \). Let
\[
\tilde{F}_k(y) = \frac{1}{2} \left( v_{k,y}^2 + v_k^2 + \frac{a(\psi_k(y))}{v_k^2} \right).
\]

Then
\[
\frac{d\tilde{F}_k}{dy} = v_{k,y} \left( v_{k,y} + v_k - \frac{a(\psi_k)}{v_k^3} \right) + \frac{a(\psi_k)}{2v_k^2} = (1 - \lambda_k) \frac{v_k v_{k,y}}{U_{\varepsilon_k}^{-4}(y)} + \frac{a(\psi_k(y))}{2v_k^2 U_{\varepsilon_k}^{-4}(y)}.
\] (2.43)
Using Proposition 2.4, we have
\[
\left| \frac{1 - \lambda_k}{U^2_{\varepsilon'_k - 1}(y)} \right| \leq C \frac{\varepsilon'_k^2}{\varepsilon'_k \cos^2 y + \varepsilon'_k \sin^2 y} \leq C. \tag{2.44}
\]
Combining (2.43) and (2.44) we get that
\[
\left| \frac{(1 - \lambda_k)v_k y}{U^2_{\varepsilon'_k - 1}(y)} \right| \leq \frac{C|v_k v_{k,y}|}{\varepsilon'_k \cos^2 y + \varepsilon'_k \sin^2 y} \leq C|\tilde{F}_k|\frac{1}{\varepsilon'_k \cos^2 y + \varepsilon'_k \sin^2 y}
\]
and
\[
\left| a_k(\psi_k(y)) \right| \leq \frac{Ca(\psi_k(y))}{v_k^2(\varepsilon'_k - 2 \cos^2 y + \varepsilon'_k \sin^2 y)} \leq \frac{C|\tilde{F}_k|}{\varepsilon'_k \cos^2 y + \varepsilon'_k \sin^2 y}
\]
Thus
\[
\left| \frac{d\tilde{F}_k}{dy} \right| \leq C \frac{1}{\varepsilon'_k \cos^2 y + \varepsilon'_k \sin^2 y}. \tag{2.45}
\]
Since
\[
\int_{0}^{2\pi} \frac{dy}{\varepsilon'_k \cos^2 y + \varepsilon'_k \sin^2 y} = \int_{0}^{2\pi} d\theta = 2\pi,
\]
from (2.45) we derive that
\[
C^{-1} \tilde{F}_k(y_2) \leq \tilde{F}_k(y_1) \leq C \tilde{F}_k(y_2), \quad \text{for any } y_1, y_2.
\]
In particular, since \( \tilde{F}_k(0) \leq C \), this implies that
\[
C^{-1} \leq \tilde{F}_k(y) \leq C,
\]
that is,
\[
v_k(y) \leq C, \quad |v_{k,y}(y)| \leq C \quad \text{and} \quad \frac{1}{v_k} \leq C, \tag{2.46}
\]
which concludes (2.39) and (2.40).

From (2.39) a better estimate for \( \lambda_k - 1 \) can be derived:
\[
|\lambda_k - 1| \leq C\varepsilon'_k \log \frac{1}{\varepsilon'_k}. \tag{2.47}
\]
In fact, from (2.35), we have that
\[
|\lambda_k - 1| \int_{0}^{2\pi} u_k^2 d\theta \leq \int_{0}^{2\pi} \frac{d(\theta) \sin 2\theta}{u_k^2} |d\theta|
\]
\[
\leq C \int_{0}^{2\pi} \frac{\sin 2\theta}{u_k^2} d\theta \leq C \int_{0}^{2\pi} \frac{|\sin 2\theta|}{U^2_{\varepsilon'_k}(\theta)} d\theta
\]
\[
\leq C \varepsilon'_k \log \frac{1}{\varepsilon'_k} \leq C \varepsilon'_k \log \frac{1}{\varepsilon'_k},
\]
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which proves (2.47).

Let $f_k = (1 - \lambda_k)\frac{v_k}{\varepsilon_k^{2p-1}(y)}$. From (2.47), we see that for $p > 1$,

$$\int_0^{2\pi} |f_k|^p dy \leq C|1 - \lambda_k|^p \int_0^{2\pi} \frac{1}{U_{\varepsilon_k}^{2p-1}} dy \leq C(\varepsilon_k^4 \log \frac{1}{\varepsilon_k})^{p-4p} \rightarrow 0$$

(2.48)

where we have used the following estimate

$$\int_0^{2\pi} \frac{1}{(\varepsilon_k^{-2} \cos^2 y + \varepsilon_k^{12} \sin^2 y)^{2p}} dy \leq C \varepsilon_k^{2-4p}.$$

Standard regularity shows that $\{v_k\}$ converges to some $v^\infty$ in $C^{1,\alpha}$-norm for any $\alpha > 0$. Away from $y = \frac{\pi}{2}, \frac{3\pi}{2}$, $U_{\varepsilon_k}^4(y)$ is bounded from below, so $v^\infty$ is $C^2$ if $y \neq \frac{\pi}{2}, \frac{3\pi}{2}$ and satisfies

$$v_{yy}^\infty + v^\infty = \frac{a(0)}{(v^\infty)^3}, \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

(2.49)

and

$$v_{yy}^\infty + v^\infty = \frac{a(\pi)}{(v^\infty)^3}, \quad y \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

(2.50)

Hence

$$v^\infty(y) = a^\frac{1}{2}(0) \quad \text{if} \quad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

since $v^\infty(0) = \lim_{k \to \infty} v_k(0) = a^\frac{1}{2}(0)$ and $(v^\infty)'(0) = \lim_{k \to \infty} v_k'(0) = 0$. It follows that

$$v^\infty\left(\frac{\pi}{2}\right) = v^\infty(0) = a^\frac{1}{2}(0), \quad (v^\infty)'\left(\frac{\pi}{2}\right) = (v^\infty)'(0) = 0.$$

Therefore, from (2.50) we get

$$v^\infty(y) = \left(a^\frac{1}{2}(0) \sin^2 y + a(\pi)a^{-\frac{1}{2}}(0) \cos^2 y\right)^{\frac{1}{2}}, \quad y \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right].$$

□

**Remark:** Under the same conditions as those of Proposition 2.5, the following estimates can be obtained:

$$\|v_k - v^\infty\|_{C^1([0,2\pi])} \leq C \varepsilon_k^2 |\log \varepsilon_k|,$$

(2.51)

where the function $v^\infty_k$ is given by

$$v^\infty_k(y) = \begin{cases} a^\frac{1}{2}(\tau_k), & y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \\ \left(a^\frac{1}{2}(\tau_k) \sin^2 y + a(\tau_k + \pi)a^{-\frac{1}{2}}(\tau_k) \cos^2 y\right)^{\frac{1}{2}}, & y \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]. \end{cases}$$

(2.52)
By definition, \( v_k^\infty \in C^1([-\pi, \pi]) \) and is uniformly bounded from above and below. Observe that, \( v_k^\infty \) satisfies

\[
\frac{d^2 v_k^\infty}{dy^2} + v_k^\infty = \begin{cases} \frac{a(\tau_k)}{(v_k^\infty)^3}, & \text{for } y \in [-\frac{\pi}{2}, \frac{\pi}{2}], \\ \frac{a(\tau_k + \pi)}{(v_k^\infty)^3}, & \text{for } y \in [\frac{\pi}{2}, \frac{3\pi}{2}]. \end{cases}
\]  

(2.53)

Combining (2.38) and (2.53), we have for \( y \in [-\frac{\pi}{2}, \frac{\pi}{2}] \)

\[
\frac{d^2 (v_k - v_k^\infty)}{dy^2} + (v_k - v_k^\infty) = \frac{a(\tau_k + \psi_k) - a(\tau_k)}{v_k^3} + \frac{a(\tau_k)}{v_k^3} - \frac{a(\tau_k)}{(v_k^\infty)^3} + \frac{(1 - \lambda_k)v_k}{U_{\varepsilon_k^1}(y)}
\]

\[
= \frac{a(\tau_k + \psi_k) - a(\tau_k)}{v_k^3} + c_k(y)(v_k - v_k^\infty) + \frac{(1 - \lambda_k)v_k}{U_{\varepsilon_k^1}(y)},
\]

(2.54)

where \( \varepsilon_k' = (a(\tau_k))^{-\frac{1}{2}} \varepsilon_k \), and \( c_k \) is a bounded function due to (2.39) and the definition of \( v_k^\infty \). In fact, \( c_k(y) = -3 + o(1) \). Thus, the function \( \varphi(y) := v_k - v_k^\infty \) satisfies

\[
\frac{d^2 \varphi}{dy^2} + \tilde{c}_k \varphi = \tilde{f}, \quad \varphi(-\frac{\pi}{2}) = \varphi'(-\frac{\pi}{2}) = 0
\]

(2.55)

where \( \tilde{c}_k \) is a bounded function and

\[
\tilde{f} = \frac{a(\tau_k + \psi_k) - a(\tau_k)}{v_k^3} + \frac{(1 - \lambda_k)v_k}{U_{\varepsilon_k^1}(y)}.
\]

Let \( \tilde{h}_i, i = 1, 2 \) be the two fundamental solutions of the linear equation \( \frac{d^2 \varphi}{dy^2} + \tilde{c}_k \varphi = 0 \) such that \( \tilde{h}_1(0) = 1, \tilde{h}'_1(0) = 0, \tilde{h}_2(0) = 0, \tilde{h}'_2(0) = 1 \). Since \( \tilde{c}_k \) is bounded, it is easy to see that \( \| \tilde{h}_i \|_{C^1([-\frac{\pi}{2}, \frac{\pi}{2}])} \leq C, i = 1, 2 \). By the method of variation of parameters (since \( \varphi(\frac{\pi}{2}) = \varphi'(\frac{\pi}{2}) = 0 \)), we have

\[
\varphi(y) = -\tilde{h}_1(y) \int_{-\frac{\pi}{2}}^y \tilde{h}_2 \tilde{f} + \tilde{h}_2(y) \int_{-\frac{\pi}{2}}^y \tilde{h}_1 \tilde{f}
\]

(2.56)

and hence

\[
\| v_k - v_k^\infty \|_{C^1([-\frac{\pi}{2}, \frac{\pi}{2}])} \leq C \| \tilde{f} \|_{L^1([-\frac{\pi}{2}, \frac{\pi}{2}])}.
\]

(2.57)

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Note that

\[
\int_{-\frac{3\pi}{2}}^{\frac{3\pi}{2}} |a(\tau_k + \psi_k(y)) - a(\tau_k)| dy \\
= \int_{-\frac{3\pi}{2}}^{\frac{3\pi}{2}} |a(\theta) - a(\tau_k)| \frac{a^2(\tau_k)\varepsilon_k}{a(\tau_k)\sin^2(\theta - \tau_k) + \varepsilon_k^4 \cos^2(\theta - \tau_k)} d\theta \\
= \int_{-\frac{3\pi}{2}}^{\frac{3\pi}{2}} |a(\theta + \tau_k) - a(\tau_k)| \frac{a^2(\tau_k)\varepsilon_k^2}{a(\tau_k)\sin^2 \theta + \varepsilon_k^4 \cos^2 \theta} d\theta \\
= \frac{1}{\sin|\theta|} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{|a(\theta + \tau_k) - a(\tau_k)|}{a(\tau_k)\sin^2 \theta + \varepsilon_k^4 \cos^2 \theta} d\theta \\
\leq C\varepsilon_k^2 |\log \varepsilon_k|.
\]

On the other hand, similar to (2.48), it is easy to see that

\[
\int_{-\frac{3\pi}{2}}^{\frac{3\pi}{2}} \frac{(1 - \lambda_k)v_k}{U_{\varepsilon_k-1}(y)} dy \leq C\varepsilon_k^2 |\log \varepsilon_k|. 
\]

(2.59)

Combining (2.58) and (2.59), we obtain

\[
\|v_k - v_k^\infty\|_{C^1([\frac{-\pi}{2},\frac{3\pi}{2}])} \leq C\varepsilon_k^2 |\log \varepsilon_k|. 
\]

(2.60)

For \( y \in [\frac{-\pi}{2}, \frac{3\pi}{2}] \), \( v_k^\infty \) satisfies

\[
\frac{d^2v_k^\infty}{dy^2} + v_k^\infty = \frac{a(\tau_k + \pi)}{(v_k^\infty)^3}. 
\]

Using the similar arguments as before, we can obtain

\[
\|v_k - v_k^\infty\|_{C^1([\frac{-\pi}{2},\frac{3\pi}{2}])} \leq C\varepsilon_k^2 |\log \varepsilon_k|. 
\]

(2.61)

Now the estimate (2.51) is a consequence of (2.60) and (2.61).

3 Sharp blow-up estimates

In this section we will use Pohozaev Identities to get a sharp estimate of \( \lambda_k - 1 \). Let \( a(\theta) \) be a \( 2\pi \)-periodic positive \( C^2 \) function and \( \lambda_k \to 1 \), and let \( u_k \) be a \( 2\pi \)-periodic solution of

\[
u_{k,\theta\theta} + \lambda_k u_k = \frac{a(\theta)}{u_k^3}, \quad \theta \in [0,2\pi].
\]

(3.1)

The main result of this section is
Proposition 3.1 Assume that \( \min_{\theta \in [0,2\pi]} u_k(\theta) = u_k(\tau_k) = \varepsilon_k \to 0 \) and \( \tau_k \to \theta_0 \).

Then
\[
A_2(\theta_0) = \frac{a'(\theta_0)}{\sqrt{a(\theta_0)}} + \frac{a'(\pi + \theta_0)}{\sqrt{a(\pi + \theta_0)}} = 0,
\]
and
\[
\lambda_k - 1
\]
\[
= \frac{\varepsilon_k^4}{2\pi a^2(0)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( a'(\theta + \theta_0) + a'(\theta + \pi + \theta_0) - a'(\theta_0) - a'(\pi + \theta_0) \right) \sin 2\theta \, d\theta + o(\varepsilon_k^4)
\]
\[
= \frac{\varepsilon_k^4}{2\pi a^2(0)} B_2(\theta_0) + o(\varepsilon_k^4).
\]

Proof. For simplicity of notations, we assume \( \tau_k = 0 \). This can be achieved by translation. Then \( \theta_0 = 0 \). By (2.17) and Proposition 2.4 we get that
\[
\int_{0}^{2\pi} \frac{a'(\theta)}{u_k^2(\theta)} (\cos 2\theta + 1) \, d\theta = 4 \int_{0}^{2\pi} (\lambda_k - 1) u_k u_k' (\cos 2\theta + 1) \, d\theta
\]
\[
= 4(\lambda_k - 1) \int_{0}^{2\pi} u_k^2 \sin 2\theta \, d\theta
\]
\[
= 4(\lambda_k - 1) \int_{0}^{2\pi} \left( \frac{u_k}{M_k} \right)^2 \sin 2\theta \, d\theta \cdot M_k^2
\]
\[
= 4(\lambda_k - 1) \left( \int_{0}^{2\pi} \sin^2 \theta \, d\theta + o(1) \right) M_k^2
\]
\[
= o(1 - \lambda_k |M_k^2|) = o(1).
\]

On the other hand, as \( k \to +\infty \), using the change of variable \( \theta = \psi_k(y) \) we have
\[
\int_{0}^{2\pi} \frac{a'(\theta)}{u_k^2(\theta)} (\cos 2\theta + 1) \, d\theta
\]
\[
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\theta)}{u_k^2(\theta)} (\cos 2\theta + 1) \, d\theta
\]
\[
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\psi_k(y))}{u_k^2(\psi_k(y))} \frac{2 \cos^2 y}{\cos^2 y + a^{-1}(\tau_k) \varepsilon_k^4 \sin^2 y} \, dy
\]
\[
= 2 \left( \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(0)}{v_k(\psi_k(y))^2} \, dy + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{a'(\pi)}{(v_k(\psi_k(y)))^2} \, dy \right) + o(1)
\]
\[
= 2\pi \left( \frac{a'(0)}{\sqrt{a(0)}} + \frac{a'(\pi)}{\sqrt{a(\pi)}} \right) + o(1)
\]
\]
(3.5)

by Lebesgue Dominated Convergence Theorem and Proposition 2.5. This proves (3.2).

The proof of (3.3) is more involved.
By (2.18) we see that
\[
\int_0^{2\pi} \frac{a_i'(\theta)}{u_k^2(\theta)} \sin 2\theta d\theta = -4(\lambda_k - 1) \int_0^{2\pi} u_k^2(\theta) \cos 2\theta d\theta. \tag{3.6}
\]
From \(u_k(\psi_k(y))(\varepsilon_k' - 2) \cos^2 y + \varepsilon_k'^2 \sin^2 y)^{\frac{1}{2}} \rightarrow v^\infty(y)\) in \(C^1\) we have
\[
|u_k(\psi_k(y))(\varepsilon_k' - 2) \cos^2 y + \varepsilon_k'^2 \sin^2 y)^{\frac{1}{2}} - v^\infty(y)| = o(1). \tag{3.7}
\]
Hence
\[
\int_0^{2\pi} u_k^2(\theta) \cos 2\theta d\theta = \int_0^{2\pi} \left( (v^\infty)^2(\psi_k^{-1}(\theta)) + o(1) \right) (\varepsilon_k' \cos^2 \theta + \varepsilon_k'^2 \sin^2 \theta) \cos 2\theta d\theta
\]
\[
=-\varepsilon_k^{-2}a^{\frac{1}{2}}(\tau_k) \int_{-\frac{2\pi}{\pi}}^{\frac{2\pi}{\pi}} (v^\infty)^2(\psi_k^{-1}(\theta)) \sin^2 \theta \cos 2\theta d\theta + o(\varepsilon_k^{-2})
\]
\[
=\varepsilon_k^{-2}a^{\frac{1}{2}}(0) \int_{-\frac{2\pi}{\pi}}^{\frac{2\pi}{\pi}} (v^\infty)^2(\psi_k^{-1}(\theta)) \sin^2 \theta \cos 2\theta d\theta + o(\varepsilon_k^{-2}). \tag{3.8}
\]
For \(\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]\), we have \(\psi_k^{-1}(\theta) \in [-\frac{\pi}{2}, \frac{\pi}{2}]\), so \((v^\infty)^2(\psi_k^{-1}(\theta)) = a^{\frac{1}{2}}(0)\) and
\[
a^{\frac{1}{2}}(0) \int_{-\frac{2\pi}{\pi}}^{\frac{2\pi}{\pi}} (v^\infty)^2(\psi_k^{-1}(\theta)) \sin^2 \theta \cos 2\theta d\theta = -\frac{\pi}{4} a(0). \tag{3.9}
\]
Similarly, if \(\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]\), we have \(\psi_k^{-1}(\theta) \in [\frac{\pi}{2}, \frac{3\pi}{2}]\) and
\[
(v^\infty)^2(\psi_k^{-1}(\theta)) = a^{\frac{1}{2}}(0) \sin^2 \psi_k^{-1}(\theta) + a(\pi) a^{-\frac{1}{2}}(0) \cos^2 \psi_k^{-1}(\theta)
\]
\[
= \frac{\varepsilon_k'^4 a(\pi) a^{-\frac{1}{2}}(0) \cos^2 \theta + a^{\frac{1}{2}}(0) \sin^2 \theta}{\varepsilon_k'^4 \cos^2 \theta + \sin^2 \theta}.
\]
Then
\[
a^{\frac{1}{2}}(0) \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (v^\infty)^2(\psi_k^{-1}(\theta)) \sin^2 \theta \cos 2\theta d\theta
\]
\[
=a^{\frac{1}{2}}(0) \int_{-\frac{2\pi}{\pi}}^{\frac{2\pi}{\pi}} \frac{\varepsilon_k'^4 a(\pi) a^{-\frac{1}{2}}(0) \cos^2 \theta + a^{\frac{1}{2}}(0) \sin^2 \theta}{\varepsilon_k'^4 \cos^2 \theta + \sin^2 \theta} \sin^2 \theta \cos 2\theta d\theta \tag{3.10}
\]
\[
=a(0) \int_{\pi}^{2\pi} \sin^2 \theta \cos 2\theta d\theta + o(1)
\]
\[
=-\frac{\pi}{4} a(0) + o(1).
\]
Substituting (3.9) and (3.10) into (3.8), we get
\[
\int_0^{2\pi} u_k^2(\theta) \cos 2\theta d\theta = -\frac{\pi}{2} a(0) \varepsilon_k^{-2} + o(\varepsilon_k^{-2}).
\]  
\[\text{(3.11)}\]

Now we estimate the left hand side of (3.6). It follows from (3.7) that
\[
\int_{-\pi}^{\pi} \frac{a'(\theta)}{u_k^2(\theta)} \sin 2\theta d\theta = \int_{-\pi}^{\pi} \frac{a'(\theta)}{u_k^2(\theta) (3\varepsilon_k^2 \cos^2 \theta + \varepsilon_k^{-2} \sin^2 \theta)^{-1} - (\varepsilon_k^2 \cos^2 \theta + \varepsilon_k^{-2} \sin^2 \theta)} \sin 2\theta d\theta
\]
\[
= \int_{-\pi}^{\pi} \frac{1}{(u_k^\infty)^2(\psi_k^{-1}(\theta))} + O(\varepsilon_k^2 |\log \varepsilon_k|) \frac{\varepsilon_k^2 a'(\theta) \sin 2\theta}{\varepsilon_k^2 \cos^2 \theta + \sin^2 \theta} d\theta
\]
\[
= \int_{-\pi}^{\pi} \frac{\varepsilon_k^2 a'(\theta) \sin 2\theta}{\varepsilon_k^2 (u_k^\infty)^2(\psi_k^{-1}(\theta))} \frac{\varepsilon_k^4 \cos^2 \theta + \sin^2 \theta}{\varepsilon_k^4 \cos^2 \theta + \sin^2 \theta} d\theta + o(\varepsilon_k^2)
\]
\[
= \int_{-\pi}^{\pi} \frac{\varepsilon_k^2 a'(\theta) \sin 2\theta}{\varepsilon_k^2 a(\tau_k)} \frac{\varepsilon_k^4 \cos^2 \theta + \sin^2 \theta}{\varepsilon_k^4 \cos^2 \theta + \sin^2 \theta} d\theta + o(\varepsilon_k^2)
\]
\[
= \frac{1}{a(0)} \int_{-\pi}^{\pi} \frac{a'(\theta) - a'(0)}{\sin^2 \theta} \sin 2\theta d\theta + o(\varepsilon_k^2),
\]  
\[\text{(3.12)}\]

where we have used (2.51).

Similarly, we have
\[
\int_{-\pi}^{\pi} \frac{a'(\theta) \sin 2\theta}{u_k^2} d\theta
\]
\[
= \int_{-\pi}^{\pi} \frac{1}{(u_k^\infty)^2(\psi_k^{-1}(\theta))} \varepsilon_k^2 a'(\theta) \sin 2\theta d\theta + o(\varepsilon_k^2)
\]
\[
= \int_{-\pi}^{\pi} \frac{a'(\theta) \sin 2\theta}{\varepsilon_k^2 a(\tau_k) a^{-\frac{1}{2}}(\tau_k) \cos^2 \theta + a^{\frac{1}{2}}(\tau_k) \sin^2 \theta} d\theta + o(\varepsilon_k^2)
\]
\[
= \int_{-\pi}^{\pi} \frac{a'(\theta) - a'(\pi)}{\sin \theta} \sin \theta \sin 2\theta d\theta + o(\varepsilon_k^2)
\]
\[
= \frac{1}{a(\tau_k)} \int_{-\pi}^{\pi} \frac{a'(\theta) - a'(\pi)}{\sin^2 \theta} \sin 2\theta d\theta + o(\varepsilon_k^2)
\]
\[
= \frac{1}{a(0)} \int_{-\pi}^{\pi} \frac{a'(\theta + \pi) - a'(\pi)}{\sin^2 \theta} \sin 2\theta d\theta + o(\varepsilon_k^2).
\]  
\[\text{(3.13)}\]
Combining (3.6) and (3.13) we can obtain that
\[
\int_0^{2\pi} \frac{a'(\theta)\sin 2\theta}{u_k^2} d\theta = \frac{\varepsilon_k^4}{a(0)} \int_0^{\pi} \frac{a' (\theta) - a'(0) + a'(\theta + \pi) - a'(\theta + \pi)}{\sin^2 \theta} \sin 2\theta d\theta + o(\varepsilon_k^2).
\]
This completes the proof of Proposition 3.1. \(\square\)

4 Proofs of Theorem 1.1 and Theorem 1.2

In this section, we prove Theorem 1.1 and Theorem 1.2. We only discuss the case \(n = 2\) since for \(n > 2\), the proof is the same.

**Proof of Theorem 1.1:** This is an immediate consequence of Proposition 3.1. Indeed, if there is a sequence \(u_k\) of
\[
u_{k,\theta} + u_k = \frac{a(\theta)}{u_k^3}, \quad \theta \in \mathbb{S}^1
\]such that \(\varepsilon_k = \min_{\theta \in \mathbb{S}^1} u_k(\theta) = u_k(\tau_k) \to 0\). Let \(\tau_k \to \theta_0\), then by Proposition 3.1, we have
\[
A_2(\theta_0) = \frac{a'(\theta_0)}{\sqrt{a(\theta_0)}} + \frac{a'(\pi + \theta_0)}{\sqrt{a(\pi + \theta_0)}} = 0
\]
and
\[
\frac{\varepsilon_k^4}{2\pi a^2(\theta_0)} \int_0^{\pi} \frac{a'(\theta + \theta_0) + a'(\theta + \theta_0 + \pi) - a'(\theta_0) - a'(\theta_0 + \pi)}{\sin^2 \theta} \sin 2\theta d\theta + o(\varepsilon_k^2)
\]
\[
= \frac{\varepsilon_k^4}{2\pi a^2(\theta_0)} B_2(\theta_0) + o(\varepsilon_k^4) = 0.
\]
Hence
\[
B_2(\theta_0) = 0.
\]
This contradicts the assumption that \(a\) is \(B_2\)-nondegenerate and proves Theorem 1.1. \(\square\)

**Proof of Theorem 1.2:** Without loss of generality we suppose \(\min_{A_2(\theta) = 0} B_2(\theta) > 0\). We take a sequence \(\lambda_k < 1\) and \(\lambda_k \to 1\). According to [17], there is a \(2\pi\)-periodic solution \(u_k\) of
\[
u_{k,\theta} + \lambda_k u_k = \frac{a(\theta)}{u_k^3},
\]
If \(\varepsilon_k = \min_{\theta \in \mathbb{S}^1} u_k(\theta) = u_k(\tau_k) \to 0, \tau_k \to \theta_0\), then by Proposition 3.1 we have \(A_2(\theta_0) = 0\) and
\[
\lambda_k - 1 = \frac{\varepsilon_k^4}{2\pi a^2(\theta_0)} B_2(\theta_0) + o(\varepsilon_k^4).
\]
So \(B_2(\theta_0) < 0\), which is impossible. Thus the sequence \(u_k\) are uniformly bounded from below and above. By taking a limit, we obtain a solution \(u\) of (1.1). \(\square\)
5 Proof of Theorem 1.3

In this section, we will use Liapunov-Schmidt reduction and degree theory to prove Theorem 1.3. Similar approach has been used by Rey-Wei ([30], [31]) and Wei-Xu ([34]). In particular, we shall follow the argument in Section 4 of [34]. For simplicity we only give the proof of 2π-periodic case and that of $n > 2$ is similar and hence it is omitted.

Let $\varepsilon$ be a small positive number and

$$S[u] := u_{\theta\theta} + u - \frac{1 + \varepsilon a(\theta)}{u^3}. \quad (5.1)$$

In order to prove Theorem 1.3, we need to find a solution of $S[u] = 0$. In the following we consider $S[u] = 0$ as a perturbation of

$$u_{\theta\theta} + u = \frac{1}{u^3}. \quad (5.2)$$

It is known that all solutions of (5.2) are given by

$$U_{\Lambda,t}(\theta) = (\Lambda^2 \cos^2(\theta - t) + \Lambda^{-2} \sin^2(\theta - t))^{\frac{1}{2}},$$

where $(\Lambda, t) \in (0, 1] \times S^1$. Note that when $\Lambda = 1$, $U_{\Lambda,t} \equiv 1$ which is independent of $t$.

We are going to find a 2π-periodic solution $u$ of $S[u] = 0$ having the form

$$u(\theta) = U_{\Lambda,t}(\theta) + \phi(\theta), \quad (5.3)$$

where $\phi(\theta)$ is relatively small.

Substituting (5.3) into the equation (5.1), we obtain

$$S[U_{\Lambda,t} + \phi] = S[U_{\Lambda,t}] + L_{\Lambda,t}[\phi] + N[\phi], \quad (5.4)$$

where

$$S[U_{\Lambda,t}] = \frac{d^2 U_{\Lambda,t}}{d\theta^2} + U_{\Lambda,t} - \frac{1 + \varepsilon a(\theta)}{U_{\Lambda,t}^3},$$

$$L_{\Lambda,t}[\phi] = \frac{d^2 \phi}{d\theta^2} + \phi + \frac{3\phi}{U_{\Lambda,t}^3},$$

and

$$N[\phi] = -\left(\frac{1 + \varepsilon a(\theta)}{(U_{\Lambda,t} + \phi)^3} - \frac{1 + \varepsilon a(\theta)}{U_{\Lambda,t}^3} + \frac{3\phi}{U_{\Lambda,t}^3}\right).$$

Since $L_{\Lambda,t}$ is a second order ODE operator, the kernel of $L$ is two-dimensional. It is easy to see that

$$Z_1 = \frac{\Lambda \cos^2(\theta - t) - \Lambda^{-3} \sin^2(\theta - t)}{U_{\Lambda,t}}, \quad Z_2 = \frac{\sin(2(\theta - t))}{U_{\Lambda,t}}. \quad (5.5)$$
are orthogonal and satisfy \( L_{\Lambda,t}[Z_1] = L_{\Lambda,t}[Z_2] = 0 \). Thus we deduce that
\[
\{ \phi \in C^2(\mathbb{S}^1) | L_{\Lambda,t}[\phi] = 0 \} = \text{span}\{Z_1, Z_2\}.
\]

For later purpose, we need to consider another two kernels:
\[
\tilde{Z}_1 = \cos(2t)a_{\Lambda}Z_1 - \sin(2t)b_{\Lambda}Z_2, \quad \tilde{Z}_2 = \cos(2t)b_{\Lambda}Z_2 + \sin(2t)a_{\Lambda}Z_1
\]
where \( a_{\Lambda} = \frac{1}{\sqrt{\int_0^{2\pi} Z_1^2}}, \ b_{\Lambda} = \frac{1}{\sqrt{\int_0^{2\pi} Z_2^2}} \).

It is easy to see that \( \int_0^{2\pi} \tilde{Z}_1^2 = 1, \int_0^{2\pi} \tilde{Z}_2^2 = 1, \int_0^{2\pi} \tilde{Z}_1\tilde{Z}_2 = 0, \) and \( \text{span}\{\tilde{Z}_1, \tilde{Z}_2\} = \text{span}\{Z_1, Z_2\} \). The reason for choosing \( \tilde{Z}_1, \tilde{Z}_2 \) instead of \( Z_1, Z_2 \) will be clear later. But we note that when \( \Lambda = 1, \tilde{Z}_1 = \frac{1}{\sqrt{\pi}} \cos(2\theta), \tilde{Z}_2 = \frac{1}{\sqrt{\pi}} \sin(2\theta) \) (which are independent of \( t \)).

For \( h \in C(\mathbb{S}^1) \), we consider the following linear problem:
\[
\begin{cases}
L_{\Lambda,t}[\phi] = h + c_1\tilde{Z}_1 + c_2\tilde{Z}_2, \quad \theta \in \mathbb{S}^1, \\
\int_0^{2\pi} \phi(\theta)\tilde{Z}_1d\theta = \int_0^{2\pi} \phi(\theta)\tilde{Z}_2d\theta = 0,
\end{cases}
\]
where \( (c_1, c_2) \in \mathbb{R}^2 \). By Fredholm Alternative theorem, we know that (5.7) is solvable if and only if \( (c_1, c_2) \) satisfies
\[
\begin{cases}
\int_0^{2\pi} h\tilde{Z}_1d\theta + c_1\int_0^{2\pi} \tilde{Z}_1^2d\theta + c_2\int_0^{2\pi} \tilde{Z}_2d\theta = 0, \\
\int_0^{2\pi} h\tilde{Z}_2d\theta + c_1\int_0^{2\pi} \tilde{Z}_1\tilde{Z}_2d\theta + c_2\int_0^{2\pi} \tilde{Z}_2^2d\theta = 0.
\end{cases}
\]
(5.8)

Since \( \int_0^{2\pi} \tilde{Z}_1\tilde{Z}_2d\theta = 0 \), \( (c_1, c_2) \) is uniquely determined by the following
\[
c_1 = \int_0^{2\pi} h\tilde{Z}_1d\theta, \quad c_2 = \int_0^{2\pi} h\tilde{Z}_2d\theta.
\]
(5.9)

Moreover, if (5.8) is satisfied, then the solution is unique and there is a positive constant \( C \) which depends on the lower bound of \( \Lambda \) only such that
\[
\|\phi\|_{C(\mathbb{S}^1)} \leq C\|h\|_{C(\mathbb{S}^1)}
\]
(5.10)

and
\[
|c_1| + |c_2| \leq C\|h\|_{C(\mathbb{S}^1)}.
\]
(5.11)

The estimate (5.10) is a consequence of the fact that \( h \mapsto \phi \) is a bounded linear operator from \( C(\mathbb{S}^1) \to C(\mathbb{S}^1) \), and (5.11) follows from (5.8).

After solving the linear problem, now we can solve the nonlinear problem:
\[
\begin{cases}
L_{\Lambda,t}[\phi] = -S[U_{\Lambda,t}] - N[\phi] + c_1\tilde{Z}_1 + c_2\tilde{Z}_2, \quad \theta \in \mathbb{S}^1, \\
\int_0^{2\pi} \phi(\theta)\tilde{Z}_1d\theta = \int_0^{2\pi} \phi(\theta)\tilde{Z}_2d\theta = 0,
\end{cases}
\]
(5.12)

for some coefficients \( c_1 \) and \( c_2 \). Namely we have
Lemma 5.1 For $\Lambda_0 > 0$, there exist $\varepsilon_0 > 0$ and $C$ which is independent of $\varepsilon$ such that for any $\varepsilon < \varepsilon_0$, $\Lambda_0 \leq \Lambda \leq 1$ and $t \in \mathbb{S}^1$, problem (5.12) has a unique solution $(\phi, c_1, c_2) = (\phi_{\Lambda, t, \varepsilon}, c_{\Lambda, t, \varepsilon}, c_{\Lambda, t, \varepsilon})$ satisfying
\[ \|\phi_{\Lambda, t, \varepsilon}\|_{C(\mathbb{S}^1)} \leq C\varepsilon. \] (5.13)

Moreover, the maps $(\Lambda, t) \to \phi_{\Lambda, t, \varepsilon}$ and $(\Lambda, t) \to (c_1(\Lambda, t, \varepsilon), c_2(\Lambda, t, \varepsilon))$ are continuous.

When $\Lambda = 1$, $\phi_{1, t, \varepsilon}$ and $(c_1(\Lambda, t, \varepsilon), c_2(\Lambda, t, \varepsilon))$ are independent of $t$.

Proof. We will use the contraction mapping principle to prove the lemma. To this end, we write the first equation of problem (5.12) in its equivalent form:
\[ \phi = A(-S[U_{\Lambda, t}] - N[\phi]) := B[\phi]. \] (5.14)

For a positive constant $\varepsilon_1 \leq \frac{\Lambda_0}{2}$, define a convex set in $C(\mathbb{S}^1)$ by
\[ C := \left\{ \phi \mid \phi \text{ is } 2\pi - \text{periodic}, \|\phi\|_{C(\mathbb{S}^1)} \leq \varepsilon_1, \int_0^{2\pi} \phi(\theta) \bar{Z}_1 d\theta = \int_0^{2\pi} \phi(\theta) \bar{Z}_2 d\theta = 0 \right\}. \]

It follows from the mean value theorem that
\[ \|N[\phi]\|_{C(\mathbb{S}^1)} \leq C(\Lambda_0)(\varepsilon_1 \frac{\varepsilon a}{U_{\Lambda, t}} \|\phi\|_{C(\mathbb{S}^1)} + \|\phi\|^2_{U_{\Lambda, t}}) \leq C(\Lambda_0)(\varepsilon_1 \varepsilon + \varepsilon_1^2), \quad \forall \phi \in C. \] (5.15)

Since $S[U_{\Lambda, t}] = -\frac{e a}{U_{\Lambda, t}}$, we have that for $\phi, \phi_1 \in C$,
\[ \|B[\phi]\|_{C(\mathbb{S}^1)} \leq C(\Lambda_0)(\|S[U_{\Lambda, t}]\|_{C(\mathbb{S}^1)} + \|N[\phi]\|_{C(\mathbb{S}^1)}) \leq C(\Lambda_0)(\varepsilon + \varepsilon_1 \varepsilon + \varepsilon_1^2) \] (5.16)

and
\[ \|B[\phi_1] - B[\phi]\|_{C(\mathbb{S}^1)} \leq C(\Lambda_0)\|N[\phi_1] - N[\phi]\|_{C(\mathbb{S}^1)} \leq C(\Lambda_0)(\varepsilon + \|\phi_1\|_{C(\mathbb{S}^1)} + \|\phi\|_{C(\mathbb{S}^1)})\|\phi_1 - \phi\|_{C(\mathbb{S}^1)} \] (5.17)

\[ \leq C(\Lambda_0)(\varepsilon + 2\varepsilon_1)\|\phi_1 - \phi\|_{C(\mathbb{S}^1)}. \]

Letting $\varepsilon_0 = \frac{\varepsilon_1}{4C(\Lambda_0)}$, $\varepsilon_1 < \frac{1}{4C(\Lambda_0)}$, then (5.16) and (5.17) imply that the operator $B$ is a contraction mapping from $C$ to $C$. Hence $C$ has a unique fixed point $\phi_{\Lambda, t, \varepsilon} \in C$ and
\[ \|\phi_{\Lambda, t, \varepsilon}\|_{C(\mathbb{S}^1)} = \|B[\phi_{\Lambda, t, \varepsilon}]\|_{C(\mathbb{S}^1)} \leq 2C(\Lambda_0)\varepsilon + C(\Lambda_0)\|\phi_{\Lambda, t, \varepsilon}\|^2_{C(\mathbb{S}^1)}, \]
that is
\[ \|\phi_{\Lambda,t}\|_{C(S^1)} \leq C\varepsilon. \]
The continuity of \( \phi_{\Lambda,t,\varepsilon}, (c_1(\Lambda, t, \varepsilon), c_2(\Lambda, t, \varepsilon)) \) on parameters \( \Lambda, t \) also follows from the contraction mapping theorem.

When \( \Lambda = 1, U_{\Lambda,t} = 1 \), \( \tilde{Z}_1 = \frac{1}{\sqrt{\varepsilon}} \cos(2\theta), \tilde{Z}_2 = \frac{1}{\sqrt{\varepsilon}} \sin(2\theta) \) are all independent of \( t \) and hence \( \phi_{1,t,\varepsilon}, c_1(1, t, \varepsilon), c_2(1, t, \varepsilon) \) are also independent of \( t \). Hence Lemma 5.1 holds.

The proof of Theorem 1.3 will be finished if for \( \varepsilon \leq \varepsilon_0 \) we can find some \( (\Lambda, t) \in (0,1] \times S^1 \) such that \( (c_1(\Lambda, t, \varepsilon), c_2(\Lambda, t, \varepsilon)) = 0 \) in problem (5.12). This will be accomplished by degree theory. In order to use degree theory we need the asymptotic expansions of \( c_1(\Lambda, t, \varepsilon) \) and \( c_2(\Lambda, t, \varepsilon) \) as \( \Lambda \to 0 \).

According to (5.9), \( c_1(\Lambda, t, \varepsilon) \) and \( c_2(\Lambda, t, \varepsilon) \) satisfy
\[
\begin{align*}
\frac{c_1(\Lambda, t, \varepsilon)}{2\pi} &= \int_0^{2\pi} (S[U_{\Lambda,t}] + N[\phi])\tilde{Z}_1 d\theta, \\
\frac{c_2(\Lambda, t, \varepsilon)}{2\pi} &= \int_0^{2\pi} (S[U_{\Lambda,t}] + N[\phi])\tilde{Z}_2 d\theta.
\end{align*}
\]
(5.18)

Let
\[
\begin{align*}
\overline{c}_1(\Lambda, t, \varepsilon) &= \frac{\int_0^{2\pi} S[U_{\Lambda,t}] Z_1 d\theta}{\int_0^{2\pi} Z_1^2 d\theta}, \\
\overline{c}_2(\Lambda, t, \varepsilon) &= \frac{\int_0^{2\pi} S[U_{\Lambda,t}] Z_2 d\theta}{\int_0^{2\pi} Z_2^2 d\theta}.
\end{align*}
\]
(5.19)

Then we have

**Lemma 5.2** For \( \Lambda \to 0 \) we have
\[
\overline{c}_1(\Lambda, t, \varepsilon) = \varepsilon(\frac{\overline{B}_2(t)}{2\pi} \Lambda^5 + o(\Lambda^5))
\]
(5.20)
and
\[
\overline{c}_2(\Lambda, t, \varepsilon) = \varepsilon(-\frac{\overline{A}_2(t)}{4} + o(1))
\]
(5.21)
where \( \overline{A}_2 \) and \( \overline{B}_2 \) are defined in (1.15) and (1.16) respectively.

**Proof.** From the definition of \( U_{\Lambda,t} \), we have
\[
\int_0^{2\pi} Z_1^2 d\theta = \int_0^{2\pi} \frac{\cos^2(\theta - t) - \Lambda^{-3} \sin^2(\theta - t)}{\Lambda^2 \cos^2(\theta - t) + \Lambda^{-2} \sin^2(\theta - t)} d\theta
\]
\[
= \Lambda^{-2} \int_0^{2\pi} (\Lambda^2 \cos^2 \theta + \Lambda^{-2} \sin^2 \theta) d\theta + 4\Lambda^{-2} \int_0^{2\pi} \frac{\cos^4 \theta - \cos^2 \theta}{\Lambda^2 \cos^2 \theta + \Lambda^{-2} \sin^2 \theta} d\theta = \pi \Lambda^{-1} + O(1)
\]
(5.22)

and
\[
\frac{1}{\varepsilon} \int_0^{2\pi} S[U_{\Lambda,t}] Z_1 d\theta = -\int_0^{2\pi} a(\theta + t) \frac{\Lambda \cos^2 \theta - \Lambda^{-3} \sin^2 \theta}{(\Lambda^2 \cos^2 \theta + \Lambda^{-2} \sin^2 \theta)^2} d\theta.
\]
(5.23)
Let
\[ \theta = \psi_\Lambda(y) = \int_0^y \frac{d\tau}{\Lambda^2 \cos^2 \tau + \Lambda^2 \sin^2 \tau}. \]

Then \( \psi : S^1 \to S^1 \) is a diffeomorphism and
\[
\frac{\cos^2 \theta}{(\Lambda^2 \cos^2 \theta + \Lambda^2 \sin^2 \theta)^2} d\theta = \frac{1}{\Lambda^2} \cos^2 y dy, \tag{5.24}
\]
\[
\frac{\sin^2 \theta}{(\Lambda^2 \cos^2 \theta + \Lambda^2 \sin^2 \theta)^2} d\theta = \Lambda^2 \sin^2 y dy, \tag{5.25}
\]
\[
\sin 2\theta \frac{d\theta}{\Lambda^2 \cos^2 \theta + \Lambda^2 \sin^2 \theta} = \frac{\sin 2y}{\Lambda^2 \cos^2 y + \Lambda^2 \sin^2 y} dy. \tag{5.26}
\]

Inserting (5.24) and (5.25) into (5.23), as \( \Lambda \to 0 \), we get
\[
\frac{1}{\varepsilon} \int_0^{2\pi} S[U_\Lambda] Z_1 d\theta = - \frac{1}{\Lambda} \int_0^{2\pi} a(\psi_\Lambda(y) + t) \cos 2y dy
\]
\[
= \frac{1}{2\Lambda} \int_0^{2\pi} a'(\psi_\Lambda(y) + t) \psi'_\Lambda(y) \sin 2y dy
\]
\[
= \frac{\Lambda}{2} \int_0^{2\pi} a'(\theta + t) \frac{\sin 2\theta}{\Lambda^4 \cos^2 \theta + \sin^2 \theta} d\theta
\]
\[
= \frac{\Lambda}{2} \int_0^{2\pi} \left[ a'(\theta + t) - a'(t) \right] \frac{\sin 2\theta}{\Lambda^4 \cos^2 \theta + \sin^2 \theta} d\theta
\]
\[
+ \frac{\Lambda}{2} \int_0^{2\pi} \left[ a'(\theta + t) - a'(t + \pi) \right] \frac{\sin 2\theta}{\Lambda^4 \cos^2 \theta + \sin^2 \theta} d\theta
\]
\[
= \frac{\Lambda}{2} \int_0^{2\pi} \left[ a'(\theta + t) - a'(t) \right] \frac{\sin \theta \sin 2\theta}{\sin^2 \theta + \Lambda^4 \cos^2 \theta} d\theta
\]
\[
+ \frac{\Lambda}{2} \int_0^{2\pi} \left[ a'(\theta + t + \pi) - a'(t + \pi) \right] \frac{\sin \theta \sin 2\theta}{\sin^2 \theta + \Lambda^4 \cos^2 \theta} d\theta
\]
\[
= \frac{\Lambda}{2} B_2(t) + o(\Lambda)
\]  
where we have used (5.26), \( \int_0^{\pi} \frac{\sin 2\theta}{\Lambda^4 \cos^2 \theta + \sin^2 \theta} d\theta = 0 \) and the dominated convergence theorem. Hence it follows from (5.19), (5.22) and (5.27) that
\[
\tau_1(\Lambda, t, \varepsilon) = \varepsilon \left( \frac{B_2(t)}{2\pi} \Lambda^5 + o(\Lambda^5) \right). \tag{5.28}
\]
Similarly,
\[
\int_0^{2\pi} Z_2^2 d\theta = \int_0^{2\pi} \frac{(\sin 2(\theta - t))^2}{\Lambda^2 \cos^2(\theta - t) + \Lambda^{-2} \sin^2(\theta - t)} d\theta \\
= \int_0^{2\pi} \frac{4 \cos^2\theta \sin^2\theta}{\Lambda^2 \cos^2\theta + \Lambda^{-2} \sin^2\theta} d\theta \\
= 4\Lambda^2 \int_0^{2\pi} (\cos^2\theta - \frac{\Lambda^2 \cos^4\theta}{\Lambda^2 \cos^2\theta + \Lambda^{-2} \sin^2\theta}) d\theta \\
= 4\Lambda^2 \int_0^{2\pi} \cos^2\theta d\theta + O(\Lambda^4) \\
= 4\pi\Lambda^2 + O(\Lambda^4)
\]
and
\[
\frac{1}{\varepsilon} \int_0^{2\pi} S[U_{\Lambda, t}] Z_2 d\theta = -\int_0^{2\pi} a(\theta + t) \frac{\sin(2\theta)}{(\Lambda^2 \cos^2\theta + \Lambda^{-2} \sin^2\theta)^2} d\theta \\
= -\int_0^{2\pi} a(\psi(\psi(y) + t)) \sin 2y dy \\
= -\frac{1}{2} \int_0^{2\pi} \frac{da(\psi(y) + t)}{dy} (\cos 2y + 1) dy \\
= -\frac{1}{2} \int_0^{2\pi} a'(\psi(\psi(y) + t)) \frac{\cos 2y + 1}{\Lambda^{-2} \cos^2 y + \Lambda^2 \sin^2 y} dy \\
= -\Lambda^2 \left[ \int_{-\pi}^{\pi} a'(\psi(\psi(y) + t)) \frac{\cos^2 y}{\cos^2 y + \Lambda^2 \sin^2 y} dy \\
+ \int_{\pi}^{\frac{3\pi}{2}} a'(\psi(\psi(y) + t)) \frac{\cos^2 y}{\cos^2 y + \Lambda^2 \sin^2 y} dy \right] \\
= -\pi\Lambda^2 [a'(t) + a'(\pi + t) + o(1)] \\
= \pi\Lambda^2 [-\overline{A}_2(t) + o(1)]
\]
provided by \( \psi(y) \to 0 \) for \( y \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and \( \psi(y) \to \pi \) for \( y \in (\frac{\pi}{2}, \frac{3\pi}{2}) \). So from (5.19), (5.29) and (5.30) we have
\[
\overline{\sigma}_2(\Lambda, t, \varepsilon) = \varepsilon (-\frac{\overline{A}_2(t)}{4} + o(1)).
\]
This completes the proof of Lemma 5.2. \( \square \)

Now we fix \( \Lambda_0 < \Lambda_1 \) small enough such that for all \( \Lambda_0 \leq \Lambda \leq \Lambda_1 \)
\[
|\langle \overline{\sigma}_1(\Lambda, t, \varepsilon), \overline{\sigma}_2(\Lambda, t, \varepsilon) \rangle| \geq \frac{\varepsilon}{16\pi^2} |\overline{A}_2^2(t) + \overline{B}_2^2(t)\Lambda^{10}|^{\frac{1}{2}}, \quad \forall t \in S^1.
\]
Then for \( s \in [0, 1] \),
\[
(1 - s) \frac{\langle \overline{\sigma}_1(\Lambda, t, \varepsilon), \overline{\sigma}_2(\Lambda, t, \varepsilon) \rangle}{\varepsilon} + s \frac{\overline{B}_2(t)}{2\pi} \Lambda^5, -\frac{\overline{A}_2(t)}{4} \neq (0, 0).
\]

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Lemma 5.3 For $\varepsilon \to 0$, $\Lambda_0 \leq \Lambda$, we have

$$c_1(\Lambda, t, \varepsilon) = \cos(2t)\overline{c}_1(\Lambda, t, \varepsilon) - \sin(2t)\overline{c}_2(\Lambda, t, \varepsilon) + O(\varepsilon^2)$$  \hspace{1cm} (5.33)

and

$$c_2(\Lambda, t, \varepsilon) = \cos(2t)\overline{c}_2(\Lambda, t, \varepsilon) + \sin(2t)\overline{c}_1(\Lambda, t, \varepsilon) + O(\varepsilon^2).$$  \hspace{1cm} (5.34)

Proof. From (5.15) and the definitions of $c_1(\Lambda, t, \varepsilon)$ we have

$$\left| \int_0^{2\pi} (S(U_{\Lambda,t} + N[\phi])Z_1 d\theta \right| - \|c_1(\Lambda, t, \varepsilon)\| \leq \int_0^{2\pi} N[\phi] Z_1 d\theta$$

$$\leq C\varepsilon^2$$

for some constant $C$ depending on $\Lambda_0$. The proof of (5.34) is the same. \hfill \Box

Now we choose an $\varepsilon_0$ such that for $s \in [0,1]$ and $0 < \varepsilon < \varepsilon_0$, $\Lambda_0 \leq \Lambda \leq \Lambda_1$,

$$(1 - s)(\overline{c}_1(\Lambda, t, \varepsilon) \overline{c}_2(\Lambda, t, \varepsilon)) + s e^{2it}(\overline{c}_1(\Lambda, t, \varepsilon), \overline{c}_2(\Lambda, t, \varepsilon)) \neq (0,0). \hspace{1cm} (5.35)$$

This is possible by Lemma 5.3. Here we denote $e^{i\theta}(x, y) = ((\cos \theta)x - (\sin \theta)y, (\cos \theta)y + (\sin \theta)x)$.

Set $\Lambda = 1 - \lambda$. We see that $\lambda \to 1$ as $\Lambda \to 0$ and $\lambda = 0$ if and only if $\Lambda = 1$. Let $\Lambda_0 = 1 - \lambda_0$. Then we have continuous maps from $D(\lambda_0) = \{(X, Y) \in \mathbb{R}^2 \mid X^2 + Y^2 \leq \lambda_0^2 \}$ to $\mathbb{R}^2$ given by

$$G_\varepsilon(X, Y) = \frac{1}{\varepsilon}(c_1(\Lambda, t, \varepsilon), c_2(\Lambda, t, \varepsilon)), \quad \overline{G}_\varepsilon(X, Y) = \frac{1}{\varepsilon}e^{-2it}(\overline{c}_1(\Lambda, t, \varepsilon), \overline{c}_2(\Lambda, t, \varepsilon)),$$

where $(X, Y) = (\lambda \cos t, \lambda \sin t)$.

Note that $G_\varepsilon(0, 0)$ and $\overline{G}_\varepsilon(0, 0)$ are well-defined since when $\lambda = 0$, $c_1, c_2$ are independent of $t$.

Proof of Theorem 1.3: For $\varepsilon$ small enough, we have by Lemma 5.2

$$\tilde{G}_\varepsilon(X, Y) = e^{2it}(\frac{\tilde{B}_2(t)}{2\pi} + o(\Lambda^5), -\frac{\tilde{A}_2(t)}{4} + o(1)) : \mathbb{R}^2_{1-\Lambda_0} \to \mathbb{R}^2$$

where $\mathbb{R}^2_{1-\Lambda_0} = \{x^2 + y^2 \leq (1 - \Lambda_0)^2\}$.

Set $\delta = \frac{2\Lambda^5}{\pi}$. We obtain

$$\tilde{G}_\varepsilon(X, Y) = \frac{1}{4}e^{2it}(\delta \tilde{B}_2(t) + o(\delta), -\tilde{A}_2(t) + o(1)). \hspace{1cm} (5.36)$$
By non-degeneracy assumption on the function \(a\), we see that if \(\delta\) is sufficiently small, \(|\hat{G}_\varepsilon(X, Y)| > 0\) on \(\partial \mathbb{B}_1 \setminus (0) = \mathbb{S}^1_{1-\delta}\) By simple property of the degree theory (see Proposition 1.27 of [20]), we have

\[
\deg(\hat{G}_\varepsilon, \mathbb{B}_1 \setminus (0), 0) = \deg(\hat{G}_\varepsilon, |\hat{G}_\varepsilon|, \mathbb{S}^1_{1-\delta}).
\]

For any real numbers \(s > 0\) and \(\eta > 0\), we define the maps:

\[
G_{s, \eta}(X, Y) = e^{2it}(-s \hat{B}_2(t), \eta \hat{A}_2(t)).
\]

Since these maps never vanish for all \((X, Y) \in \mathbb{S}^1\), their degrees are well defined and they all have the same degrees which implies that

\[
\deg(G_{s, \eta}, \mathbb{S}^1_{1-\delta}) = \deg(\hat{G}_\varepsilon, \mathbb{S}^1_{1-\delta}) = \deg(e^{2it}\hat{G}_2, \mathbb{S}^1) = 2 + \deg(\hat{G}_2, \mathbb{S}^1) \neq 0,
\]

where we have identified the domains \(\mathbb{S}^1\) and \(\mathbb{S}^1_{1-\delta}\) for our map \(\hat{G}_2\) which is clearly true since they give the same values for \(\hat{G}_2\).

By the properties of the degree (see the Appendix), there exists \((\lambda, t) \in D(\lambda_0)\) such that \((c_1(\Lambda, t, \varepsilon), c_2(\Lambda, t, \varepsilon)) = (0, 0)\). This completes the proof of Theorem 1.3.

\(\square\)

Appendix: Definition and Properties of \(d(\psi, \mathbb{S}^{N-1})\)

In this appendix, we give the definition of the degree \(d(\psi, \mathbb{S}^{N-1})\) (where \(N \geq 2\) and collect some basic properties of the degree. All the materials in this appendix can be found in the book by Fonseca and Gambo [20].

Let \(\mathbb{B}^N\) denote the unit ball \(\{ |x| < 1 \} \subset \mathbb{R}^N\) and \(\mathbb{S}^{N-1} = \partial \mathbb{B}^N\) be the unit sphere in \(\mathbb{R}^N\). Let \(\phi : \mathbb{B}^N \to \mathbb{R}^N\) be a continuous function such that \(\phi(\mathbb{S}^{N-1}) \in \mathbb{R}^N \setminus \{0\}\). Then the Brouwer degree \(d(\phi, \mathbb{B}^N, 0)\) can be defined in the usual way (Definition 1.26 of [20]). Now we define

\[
\psi = \frac{\phi(x)}{||\phi(x)||} : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}.
\]

By [Proposition 1.27, p.24] of [20], we have

\[
(A1) \quad d(\phi, \mathbb{B}^N, 0) = d(\psi, \mathbb{S}^{N-1}, p) = d(\psi, \mathbb{S}^{N-1}), \quad \forall \ p \in \mathbb{S}^{N-1}.
\]

We recall the following properties of \(d(\psi, \mathbb{S}^{N-1})\) (Corollary 2.3 of [20]): Let \(\psi : \mathbb{S}^{N-1} \to \mathbb{S}^{N-1}\) be a continuous mapping. Then the following assertions are equivalent:

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(i) $\psi$ is not homotopic to a constant.

(ii) Every continuous extension $\phi$ of $\psi$: $\mathbb{B}^N \rightarrow \mathbb{R}^N$ admits a zero.

(iii) Every continuous extension $\phi$ of $\psi$: $\mathbb{B}^N \rightarrow \mathbb{R}^N$ verifies $d(\phi, \mathbb{B}^N, 0) = d(\psi, S^{N-1}) \neq 0$.

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