COMPACTNESS OF SCALAR-FLAT CONFORMAL METRICS ON LOW-DIMENSIONAL MANIFOLDS WITH CONSTANT MEAN CURVATURE ON BOUNDARY

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ABSTRACT. We concern C^2-compactness of the solution set of the boundary Yamabe problem on smooth compact Riemannian manifolds with boundary provided that their dimensions are 4, 5 or 6. By conducting a quantitative analysis of a linear equation associated with the problem, we prove that the trace-free second fundamental form must vanish at possible blow-up points of a sequence of blowing-up solutions. Applying this result and the positive mass theorem, we deduce the C^2-compactness for all 4-manifolds (which may be non-umbilic). For the 5-dimensional case, we also establish that a sum of the second-order derivatives of the trace-free second fundamental form is non-negative at possible blow-up points. We use in an essential way this fact to obtain the C^2-compactness for all 5-manifolds. Finally, we show that the C^2-compactness on 6-manifolds is true if the trace-free second fundamental form on the boundary never vanishes.

1. INTRODUCTION

Let (M, g) be an N-dimensional ($N \geq 3$) smooth compact Riemannian manifold with boundary ∂M. Let also Δ_g be the Laplace-Beltrami operator on M, $R[g]$ the scalar curvature on M, ν the inward normal vector to ∂M, and $H[g]$ be the mean curvature of ∂M. In [22], Escobar asked if (M, g) can be conformally deformed to a scalar-flat manifold with boundary of constant mean curvature. This problem, which we will call the boundary Yamabe problem, can be understood as a generalization of the Riemann mapping theorem and is equivalent to finding a positive smooth solution to a nonlinear boundary value problem with critical exponent

$$\begin{cases} L_g U = 0 & \text{in } M, \\ B_g U = Q(M, \partial M) U^{\frac{N}{N-2}} & \text{on } \partial M. \end{cases} \tag{1.1}$$

Here L_g is the conformal Laplacian and B_g is the associated conformal boundary operator defined as

$$L_g = -\Delta_g + \frac{N-2}{4(N-1)} R[g] \quad \text{and} \quad B_g = -\frac{\partial}{\partial \nu} + \frac{N-2}{2} H[g],$$

and $Q(M, \partial M)$ is a constant whose sign is determined by the conformal structure of M.

Weak solutions to (1.1) correspond to critical points of the functional

$$Q(U) = \frac{\int_M (|\nabla_g U|_g^2 + \frac{N-2}{4(N-1)} R[g] U^2) dv_g + \int_{\partial M} H[g] U^2 dv_h}{(\int_{\partial M} |U|^{\frac{2(N-1)}{N-2}} dv_h)^{\frac{N-2}{N-1}}}$$

defined for an element U in the Sobolev space $H^1(M)$ with $U \neq 0$ on ∂M, where ∇_g represents the gradient on (M, g), h is the restriction of the metric g on ∂M, and dv_g and dv_h are the volume form on M and on ∂M, respectively. Escobar [22] proved that the Sobolev quotient

$$Q(M, \partial M) = \inf \left\{ Q(U) : U \in H^1(M),\, U \neq 0 \text{ on } \partial M \right\}$$

attains its minimizer if $Q(M, \partial M) < Q(\mathbb{B}^N, \partial \mathbb{B}^N)$ where the unit ball $\mathbb{B}^N = \{x \in \mathbb{R}^N : |x| < 1\}$ is endowed with the Euclidean metric. This is analogous to the observation of Aubin [7] for the classical Yamabe problem.

Thanks to the effort of several researchers, the existence of a solution to (1.1) is now well-established: Escobar [22, 24], Marques [42, 43], Almaraz [1] and Chen [12] found a minimizer of the functional Q for almost all manifolds. By applying the barycenter technique of Bahri and Coron, Mayer and Ndiaye [34] covered all the remaining cases. Regularity property of (1.1) was investigated by Cherrier [13].

Concerning multiplicity of solutions to (1.1), the only interesting case is when $Q(M, \partial M) > 0$. If $Q(M, \partial M) < 0$, the conformal covariance of the operators L_g and B_g shows that (1.1) has only one solution. If $Q(M, \partial M) = 0$, it is a linear equation and its solution is unique up to positive multiplicative constants. On the other hand, the case that M is conformally equivalent to the unit ball $\mathbb{B}^N$ (so that $Q(M, \partial M) = Q(\mathbb{B}^N, \partial \mathbb{B}^N) > 0$) is special, and the solution set of (1.1) was completely classified thanks to the works of Escobar [21] and Li and Zhu [39]; see Subsection 2.2.

In about two decades, several results on $C^2(M)$-compactness of the solution set of (1.1) appeared under the assumption that $Q(M, \partial M) > 0$. Felli and Ould Ahmedou [25, 26] deduced compactness results for locally conformally flat manifolds and 3-manifolds provided that their boundaries are umbilic. Very recently, the umbilicity condition was lifted for 3-manifolds by Almaraz et al. [5]. If the dimension N of the manifold M satisfies $N \geq 7$ and the trace-free second-fundamental form on ∂M is nonzero everywhere, the result of Almaraz [2] shows that the $C^2(M)$-compactness continues to hold. If either $N > 8$ and the Weyl tensor of M never vanishes on ∂M, or $N = 8$ and the Weyl tensor of ∂M never vanishes on ∂M, the $C^2(M)$-compactness is still true for manifolds M with umbilic boundary, as shown by Ghimenti and Micheletti [27].

Compactness results for other boundary Yamabe-type problems can be found in Han and Lin [30], Djadli et al. [17, 18], Disconzi and Khuri [16], and so on. By using the compactness property, Cádenas and Sierra [11] yielded uniqueness of solutions to (1.1) for some manifolds whose metrics are non-degenerate.

As far as the authors know, compactness results on (1.1) have been known only for manifolds with boundary of dimension $N = 3$ or $N \geq 7$, unless manifolds are locally conformally flat. The main purpose of this paper is to treat all manifolds with boundary of dimension $N = 4$ and 5, and generic manifolds with boundary of dimension $N = 6$.

Theorem 1.1. *For $N = 4, 5, 6$, let (M, g) be an N-dimensional smooth compact Riemannian manifold with boundary ∂M such that $Q(M, \partial M) > 0$ and M is not conformally equivalent to the unit ball $\mathbb{B}^N$. If $N = 6$, we also assume that the trace-free second-fundamental form never vanishes on ∂M. Then, for any $\varepsilon_0 > 0$ small, there exists a constant $C > 1$ depending only on M, g and ε_0 such that*

$$C^{-1} \leq U \leq C \quad on \ M \quad and \quad \|U\|_{C^2(M)} \leq C$$

for any solution $U \in H^1(M)$ to

$$\begin{cases} L_g U = 0 & in \ M, \\ B_g U = Q(M, \partial M) U^p & on \ \partial M \end{cases} \tag{1.2}$$

with $p \in [1 + \varepsilon_0, \frac{N}{N-2}]$.

The transversality argument shows that if $N \geq 4$, the set of metrics on M whose trace-free second fundamental form on ∂M vanishes nowhere is open and dense in the space of all Riemannian manifolds on M. This justifies the terminology 'generic' used above. Also, as can be observed in Theorem 1.1, we will deal with a slightly generalized equation (1.2) instead of (1.1).

Our strategy follows the argument in the lecture note [44] of Schoen where he raised the question of C^2-compactness of the solution set of the classical Yamabe problem and resolved it for locally conformally flat manifolds. It has been further developed by Li and Zhu [40], Druet [19], Marques [41], Li and Zhang [37, 38] and Khuri et al. [31]. Furthermore, Li [35] and Li and Xiong [36] studied compactness results of the Q-curvature problem, which is the fourth-order analogue of the Yamabe problem.

Once Theorem 1.1 is established, one can deduce the existence of a solution to (1.1) by applying the standard Leray-Schauder degree argument as in [25, 30]. There also should exist the strong Morse inequality in our framework as in [31, Theorem 1.4].

We leave two more remarks for the theorem.

Remark 1.2. The key idea of our main theorem is to perform a fine analysis of associated linearized equations with (1.2) in proving that the trace-free second fundamental form must vanish at possible blow-up points of a sequence of blowing-up solutions. Interestingly, this process is somehow related to the way that Marques [43] constructed test functions in his existence theorem for (1.1) on low-dimensional manifolds with non-umbilic boundary. Indeed, his test functions consist of not only truncated bubbles but also some additive correction terms. This is a distinctive feature of the boundary Yamabe problem compared with the classical one.

Our argument can be be further applied in the following settings.

(1) Based on the existence results of Marques [42] and Almaraz [1] for (1.1) on manifolds with umbilic boundary, we expect that one can lower the threshold dimension 8 in the aforementioned compactness theorem of Ghimenti and Micheletti [27] to 6.

(2) As a matter of fact, the boundary Yamabe problem can be seen as the special case of the fractional Yamabe problem where the symbol of the differential operator is the same as that of the half-Laplacian. In [33], we proved that the solution set of the fractional Yamabe problem is C^2-compact on conformal infinities of asymptotically hyperbolic manifolds, under the assumptions that the dimension is sufficiently high and the second-fundamental form never vanishes. In view of our existence result [32], we expect that the compactness result holds for conformal infinities of dimension ≥ 4 as far as the same geometric condition is maintained.

(3) To examine stability issue under small perturbation of (1.1), Ghimenti et al. [28, 29] constructed blowing-up solutions when the linear perturbation of the mean curvature on the boundary is strictly positive everywhere; see also Deng et al. [15] where analogous results were derived in the setting of the fractional Yamabe problem. In building suitable approximation solutions, they had to analyze an associated linearized equation with (1.1) which is essentially the same as ours. Due to this reason, their results require some dimensional assumptions. Our method can allow one to treat lower-dimensional cases.

Remark 1.3. The proof of the main theorem shows that remarkable phenomena happen on 5-manifolds (M, g) with boundary.

(1) Given a point $y_0 \in M$, let G_{y_0} be the Green's function defined in (7.1). In Lemma 2.4 and the first paragraph of Section 7, we construct an asymptotically flat manifold $(M \setminus \{y_0\}, G_{y_0}^{\frac{4}{N-2}} g)$, which we call a conformal blow-up of (M, g). Corollary 7.4 reveals that its mass is involved with not only the Green's function G_{y_0} but also the trace-free second fundamental form on the boundary ∂M. Therefore, the mass carries global and local information simultaneously. This is in striking contrast with manifolds without boundary in that the mass of their conformal blow-ups depend only on the Green's function, namely, global information.

(2) In Subsection 5.2, we will see that the sign of the local information of mass is encoded in the $\epsilon^3 |\log \epsilon|$-order of the expansion of a local Pohozaev identity (2.9) with respect

to a small scaling parameter $\epsilon > 0$. This is totally different from the classical Yamabe problem. In the classical one, the order involving the logarithm contains meaningful geometric information only if the manifold is even-dimensional.

In [3], Almaraz constructed manifolds with umbilic boundary of dimension $N \geq 25$ on which the solution set of (1.1) is L^∞-unbounded (in particular, C^2-noncompact). In view of the full compactness result of Khuri et al. [31] and the non-compactness results of Brendle [8] and Brendle and Marques [10] for the classical Yamabe problem, a natural expectation is that the solution set of (1.1) is C^2-compact for all manifolds with boundary of dimension $N \leq 24$ under the validity of the positive mass theorem. However, although Schoen's argument in [44] works in principle and we develop several efficient methods for the boundary Yamabe problem in this paper, fully achieving this seems still a difficult task.

To establish the C^2-compactness result for general manifolds of high dimension, we must prove that the trace-less second fundamental form and the Weyl tensor vanish up to some high order at each blow-up point. This requires a very accurate pointwise estimate of blowing-up solutions, which can be achieved only if one has a good understanding of linearized equations. In the analysis on the classical Yamabe problem, Khuri et al. [31] observed that solutions of their linearized problems can be written explicitly in the form of rational functions. Unfortunately, the boundary Yamabe problem seems not to have a similar property.

On the other hand, we may also need a quite precise control of the Green's function G of the conformal Laplacian with Neumann boundary condition; see (7.1) of its definition. In our analysis, we only need a rough control of G (described in Lemma 2.4) as in the proof of the compactness theorem for 3-dimensional manifolds [5].

The rest of the paper is organized as follows:

- In Section 2, we recall some analytic and geometric tools which we need throughout the proof of Theorem 1.1. These include the expansion of the metric in Fermi coordinates, definition of the bubbles, a local Pohozaev's identity and the positive mass theorem on asymptotically flat manifolds with boundary.
- In Section 3, we characterize blow-up points of solutions to (1.2) and provide basic qualitative properties of solutions near blow-up points.
- In Section 4, we study a linearized equation associated with (1.2) arising from the first-order expansion of the metric. In order to treat low-dimensional manifolds, we need to understand its solution more precisely than higher-dimensional cases. For this aim, we decompose the solution into two pieces and analyze them quantitatively. This is one of the key parts of the proof. We also perform a refined blow-up analysis.
- In Section 5, we carry out the proof of the vanishing theorem of the trace-free second fundamental form at any isolated simple blow-up point. For 5-manifolds, we also establish that a sum of the second-order derivatives of the trace-free second fundamental form is non-negative at each isolated simple blow-up point. These results are based on the quantitative analysis of the linearized equation conducted in the previous section.
- In Section 6, employing the vanishing theorem, we prove a local Pohozaev sign condition that guarantees that every blow-up point is isolated simple.
- In Section 7, by applying the positive mass theorem, we conclude that the solution set of (1.2) is C^2-compact for every 4- and 5-manifold unless it is conformally equivalent to the unit ball. For 6-manifolds, we also show that the C^2-compactness of the solution set holds provided that the trace-free second fundamental form on the boundary never vanishes.
- In Appendix A, we provide technical arguments regarding the two pieces of the solutions to the linearized equation to (1.2).

To elucidate our method, we will omit most of the proofs of intermediate results which closely follow the corresponding ones in similar settings, leaving appropriate references instead.

Notations.

- Let $n = N - 1$. Moreover, for any $x \in \mathbb{R}^N_+ = \{(x_1, \cdots, x_n, x_N) \in \mathbb{R}^N : x_N > 0\}$, we denote $\bar{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$. We often identify $\bar{x} \in \mathbb{R}^n$ and $(\bar{x}, 0) \in \partial \mathbb{R}^N_+$.

- We will sometimes use $\partial_a = \frac{\partial}{\partial x_a}$, $\partial_{ab} = \frac{\partial^2}{\partial x_a \partial x_b}$, etc.

- Given $x \in \mathbb{R}^N_+$, $\bar{x} \in \mathbb{R}^n$ and $r > 0$, let $B^N_+(x, r)$ be the N-dimensional upper half-ball centered at x of radius r, and $B^n(\bar{x}, r)$ the n-dimensional ball centered at $\bar{x}$ of radius r. We often identify $B^n(\bar{x}, r)$ and $\partial B^N_+((\bar{x}, 0), r) \cap \partial \mathbb{R}^N_+$. Set $\partial_I B^N_+((\bar{x}, 0), r) = \partial B^N_+((\bar{x}, 0), r) \cap \mathbb{R}^N_+$.

- S represents a surface measure. Its subscript x or $\bar{x}$ denotes the dependent variables.

- $D^{1,2}(\mathbb{R}^N_+)$ is the homogeneous Sobolev space in $\mathbb{R}^N_+$ defined as

$$D^{1,2}(\mathbb{R}^N_+) = \left\{ U \in L^{\frac{2N}{N-2}}(\mathbb{R}^N_+) : \nabla U \in L^2(\mathbb{R}^N_+) \right\}.$$

- $|\mathbb{S}^{n-1}|$ is the surface area of the unit $(n-1)$-sphere $\mathbb{S}^{n-1}$.

- The metric h on the boundary ∂M of the Riemannian manifold (M, g) is the restriction of the metric g to ∂M.

- For any $y \in \partial M$ and $r > 0$ small, $B_g(y, r)$ and $B_h(y, r)$ stand for the geodesic half-ball on (M, g) and the geodesic ball on $(\partial M, h)$, respectively. Also, d_g is the distance function on (M, g).

- The Einstein summation convention for repeated indices is adopted throughout the paper. Unless otherwise stated, the indices i, j, k, l, m and s always range over values from 1 to n, while a, b, c and d take values from 1 to N. Also, δ_{ab} is the Kronecker delta.

- We denote by $R_{abcd}[g]$ the full Riemannian curvature tensor on (M, g), by $R_{ab}[g]$ the Ricci curvature tensor on M, and by $R[g]$ the scalar curvature on M. The quantities $R_{ijkl}[h]$, $R_{ij}[h]$ and $R[h]$ are the corresponding curvatures defined on the boundary $(\partial M, h)$.

- We write by $\mathrm{II}[g]$ the second fundamental form of ∂M, by $H[g] = \frac{1}{n} h^{ij} \mathrm{II}_{ij}[g]$ the mean curvature on ∂M, and by $\pi[g] = \mathrm{II}[g] - Hg$ the trace-free second fundamental form of ∂M. Furthermore, $\|\pi[g]\|^2 = h^{ik} h^{jl} \pi_{ij}[g] \pi_{kl}[g]$ stands for the square of its norm.

- For an r-tensor T, we write

$$\mathrm{Sym}_{i_1 \cdots i_r} T_{i_1 \cdots i_r} = \frac{1}{r!} \sum_{\sigma \in S_r} T_{i_{\sigma(1)} \cdots i_{\sigma(r)}}$$

where S_r is the symmetric group over a set of r symbols.

- For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$,

$$|\alpha| = \sum_{i=1}^n \alpha_i, \ \alpha! = \prod_{i=1}^n \alpha_i! \quad \text{and} \quad \frac{\partial}{\partial x_\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}. \tag{1.3}$$

β, β' and β'' also denote multi-indices.

- The letter C denotes a generic positive constant that may vary from line to line.

2. Preliminaries

2.1. **Metric expansion and conformal coordinates.** Fix a point $y_* \in \partial M$. For any $y \in \partial M$ near y_*, let $\bar{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ be normal coordinates on ∂M (centered at y_*) of y. Denote by $\nu(y)$ the inward normal vector to ∂M at y. We say that $x = (\bar{x}, x_N) \in \mathbb{R}^N_+$ are Fermi coordinates on M (centered at y_*) of the point $\exp_y(x_N \nu(x)) \in M$.

In Lemma 2.2 of Marques [42], the following expansion of the metric g near y_* was given.

Lemma 2.1. *In Fermi coordinates centered at $y_* \in M$, it holds that*

$$g_{ij}(x) = \delta_{ij} + A_{ij}(x) + O(|x|^4),$$

$g_{iN}(x) = 0$ *and* $g_{NN}(x) = 1$, *where*

$$A_{ij}(x) = -2\mathrm{II}_{ij}[g]x_N - \frac{1}{3}R_{ikjl}[h]x_kx_l - 2\mathrm{II}_{ij,k}[g]x_kx_N + (-R_{iNjN}[g] + \mathrm{II}_{is}[g]\mathrm{II}_{sj}[g])x_N^2$$

$$- \frac{1}{6}R_{ikjl,m}[h]x_kx_lx_m + \left(-\mathrm{II}_{ij,kl}[g] + \frac{2}{3}\mathrm{Sym}_{ij}(R_{iksl}[h]\mathrm{II}_{sj}[g])\right)x_kx_lx_N$$

$$+ \left(-R_{iNjN,k}[g] + 2\mathrm{Sym}_{ij}(\mathrm{II}_{is,k}[g]\mathrm{II}_{sj}[g])\right)x_kx_N^2$$

$$+ \frac{1}{6}\left(-2R_{iNjN,N}[g] + 8\mathrm{Sym}_{ij}(\mathrm{II}_{is}[g]R_{jNsN}[g])\right)x_N^3.$$

Every tensor in the expansion is evaluated at y_* and commas denote covariant differentiation.

The next lemma describes the existence of conformal coordinates. Refer to Propositions 3.1 and 3.2 of [42].

Lemma 2.2. *For a given point $y_* \in M$ and an integer $\kappa \geq 2$, there exists a metric $\tilde{g}$ on M conformal to g such that*

$$\det \tilde{g}(x) = 1 + O(|x|^\kappa) \tag{2.1}$$

in $\tilde{g}$-Fermi coordinates centered at y_*. In particular,

$$H[g] = H_{,i}[g] = R_{ij}[h] = 0 \quad and \quad R_{NN}[g] = -\|\pi[g]\|^2 \quad at\ y_*. \tag{2.2}$$

Moreover, $\tilde{g}$ can be written as $\tilde{g} = \omega^{\frac{4}{N-2}}g$ for some positive smooth function w on ∂M such that $w(y_*) = 1$ and $\nabla w(y_*) = 0$.

2.2. **Bubbles in the Euclidean half-space.** Assume that $N \geq 3$. For $\lambda > 0$ and $\xi \in \mathbb{R}^n$, let a bubble $W_{\lambda,\xi}$ be a function defined as

$$W_{\lambda,\xi}(x) = \frac{\lambda^{\frac{N-2}{2}}}{(|\bar{x} - \xi|^2 + (x_N + \lambda)^2)^{\frac{N-2}{2}}} \quad \text{for } x \in \mathbb{R}_+^N, \tag{2.3}$$

which is an extremal function of the Sobolev trace inequality $D^{1,2}(\mathbb{R}_+^N) \hookrightarrow L^{\frac{2(N-1)}{N-2}}(\mathbb{R}^n)$; see Escobar [20]. According to Li and Zhu [37], any solution to the boundary Yamabe problem on $\mathbb{R}_+^N$

$$\begin{cases} -\Delta U = 0 & \text{in } \mathbb{R}_+^N, \\ U > 0 & \text{in } \mathbb{R}_+^N, \\ -\dfrac{\partial U}{\partial x_N} = (N-2)U^{\frac{N}{N-2}} & \text{on } \mathbb{R}^n \end{cases} \tag{2.4}$$

must be a bubble. Note that a sequence $\{W_{\frac{1}{n},0}\}_{n \in \mathbb{N}}$ of bubbles exhibits a blow-up phenomenon as $n \to \infty$, and in particular, the family of all bubbles is not $L^\infty(\mathbb{R}_+^N)$-bounded. Furthermore, Dávila et al. [14] proved that the solution space of the linear problem

$$\begin{cases} -\Delta\Phi = 0 & \text{in } \mathbb{R}_+^N, \\ -\dfrac{\partial \Phi}{\partial x_N} = Nw_{\lambda,\xi}^{\frac{2}{N-2}}\Phi & \text{on } \mathbb{R}^n, \\ \|\Phi(\cdot,0)\|_{L^\infty(\mathbb{R}^n)} < \infty, \end{cases}$$

where $w_{\lambda,\xi}(\bar{x}) = W_{\lambda,\xi}(\bar{x},0)$ on $\mathbb{R}^n$, is spanned by

$$Z_{\lambda,\xi}^1 = \frac{\partial W_{\lambda,\xi}}{\partial \xi_1}, \cdots, Z_{\lambda,\xi}^n = \frac{\partial W_{\lambda,\xi}}{\partial \xi_n} \quad \text{and} \quad Z_{\lambda,\xi}^0 = -\frac{\partial W_{\lambda,\xi}}{\partial \lambda};$$

refer also to Lemma 2.1 of [2].

2.3. **Conformally invariant equations.** Let $\delta = \frac{N}{N-2} - p \geq 0$. It turns out that it is more convenient to deal with the following form of the equation

$$\begin{cases} L_g U = 0 & \text{on } M, \\ B_g U = (N-2) f^{-\delta} U^p & \text{on } \partial M \end{cases} \tag{2.5}$$

than (1.2). Indeed, by the conformal covariance property of the operators L_g and B_g, the metric $\tilde{g} = \omega^{\frac{4}{N-2}} g$ conformal to g and the function $\widetilde{U} = \omega^{-1} U > 0$ on M satisfy

$$\begin{cases} L_{\tilde{g}} \widetilde{U} = 0 & \text{on } M, \\ B_{\tilde{g}} \widetilde{U} = (N-2) \tilde{f}^{-\delta} \widetilde{U}^p & \text{on } \partial M \end{cases} \tag{2.6}$$

where $\tilde{f} = \omega f$. Obviously, it is an equation of the same type as (2.5).

We will study a sequence $\{U_m\}_{m \in \mathbb{N}}$ of solutions to (2.5) with suitable choices of the exponents $p = p_m \in [1 + \varepsilon_0, \frac{N}{N-2}]$ and $\delta = \delta_m = \frac{N}{N-2} - p_m$, the metric $g = g_m$ on M and the smooth positive function $f = f_m$ on ∂M. Although we postpone their specific description to Section 3, we stress that our choices will induce that $p_m \to p_0$, $g_m \to g_0$ in $C^4(M, \mathbb{R}^{N \times N})$ and $f_m \to f_0 > 0$ in $C^2(\partial M)$ as $m \to \infty$, and g_0 is a metric on M.

2.4. **Pohozaev's identity.** In the analysis of blowing-up solutions, we shall rely on the following version of local Pohozaev's identity. For its derivation, see Proposition 3.1 of [2].

Lemma 2.3. *Assume that $N \geq 3$. Let $U \in H^1(B_+^N(0, \rho_1))$ be a solution to*

$$\begin{cases} -\Delta U = Q & \text{in } B_+^N(0, \rho_1), \\ -\dfrac{\partial U}{\partial x_N} + \dfrac{N-2}{2} HU = f U^p & \text{on } B^n(0, \rho_1) \end{cases}$$

where $p \in [1, \frac{N}{N-2}]$, $Q \in L^\infty(B_+^N(0, \rho_1))$ and $H, f \in C^1(B^n(0, \rho_1))$. For any $\rho \in (0, \rho_1)$, we define

$$\mathcal{P}'(U, \rho) = \int_{\partial_I B_+^N(0, \rho)} \left[-\left(\frac{N-2}{2}\right) U \frac{\partial U}{\partial \nu} - \frac{\rho}{2} |\nabla U|^2 + \rho \left| \frac{\partial U}{\partial \nu} \right|^2 \right] dS_x \tag{2.7}$$

and

$$\mathcal{P}(U, \rho) = \mathcal{P}'(U, \rho) + \frac{\rho}{p+1} \int_{\partial B^n(0, \rho)} f U^{p+1} dS_{\bar{x}} \tag{2.8}$$

where ν is the inward unit normal vector with respect to $\partial_I B_+^N(0, \rho)$. Then we have

$$\begin{aligned} \mathcal{P}(U, \rho) = &-\int_{B_+^N(0, \rho)} Q \left[x_a \partial_a U + \left(\frac{N-2}{2}\right) U \right] dx \\ &+ \frac{N-2}{2} \int_{B^n(0, \rho)} H \left[x_i \partial_i U + \left(\frac{N-2}{2}\right) U \right] U d\bar{x} \\ &- \frac{1}{p+1} \int_{B^n(0, \rho)} x_i \partial_i f U^{p+1} d\bar{x} + \left(\frac{N-1}{p+1} - \frac{N-2}{2}\right) \int_{B^n(0, \rho)} f U^{p+1} d\bar{x} \end{aligned} \tag{2.9}$$

for all $\rho \in (0, \rho_1)$.

2.5. **Positive mass theorem.** In [4], Almaraz et al. introduced the mass of N-dimensional asymptotically flat manifolds with non-compact boundary and proved the associated positive mass theorem for arbitrary manifolds of dimension $3 \leq N \leq 7$ and spin manifolds of dimension $N \geq 3$. In [5], Almaraz et al. used the positive mass theorem to describe the asymptotic behavior of the Green's function of the conformal Laplacian on a smooth compact Riemannian manifold (M, g) with boundary in terms of the mass.

The version of the positive mass theorem which we will apply in this paper is summarized in the following lemma. This is a combination of Theorem 1.3 of [4] and Proposition 3.5 of [5].

Lemma 2.4. *For $3 \le N \le 7$, let (M, g) be an N-dimensional smooth compact Riemannian manifold with boundary, and y_0 be an arbitrarily fixed point on M. Suppose that we have the metric expansion*

$$g_{ab}(x) = \delta_{ab} + A_{ab}(x) + O(|x|^{2d+2}), \quad d = \left\lfloor \frac{N-2}{2} \right\rfloor \tag{2.10}$$

with

$$A_{iN}(x) = A_{NN}(x) = 0, \quad A_{ij}(x) = O(|x|^{d+1}), \quad \mathrm{trace}(A(x)) = O(|x|^{2d+2}) \tag{2.11}$$

in Fermi coordinates centered at y_0. Assume also that G is a smooth positive function on $M \setminus \{y_0\}$ such that

$$G(x) = |x|^{2-N} + \phi(x) \tag{2.12}$$

in the same coordinates, where ϕ is a smooth function on $M \setminus \{y_0\}$ satisfying

$$\phi(x) = O(|x|^{d+3-N} |\log |x||) \quad as \ |x| \to 0. \tag{2.13}$$

Then the manifold $(M \setminus \{y_0\}, G^{\frac{4}{N-2}} g)$ is asymptotically flat with the mass

$$m_0 = \lim_{\rho \to 0} \mathcal{I}(y_0, \rho) \ge 0 \tag{2.14}$$

where

$$
\begin{aligned}
\mathcal{I}(y_0, \rho) =\ & \frac{4(N-1)}{N-2} \int_{\partial_I B_+^N(0,\rho)} \left(|x|^{2-N} \partial_a G(x) - \partial_a |x|^{2-N} G(x) \right) \frac{x_a}{|x|} \, dS_x \\
& - \int_{\partial_I B_+^N(0,\rho)} \left(\rho^{3-2N} x_a \partial_b A_{ab}(x) - 2N \rho^{1-2N} x_a x_b A_{ab}(x) \right) dS_x.
\end{aligned}
\tag{2.15}
$$

Furthermore, if M is not conformally equivalent to the standard unit ball in $\mathbb{R}^N$,

$$R\!\left[G^{\frac{4}{N-2}} g \right] \ge 0 \quad on \ M \setminus \{y_0\} \quad and \quad H\!\left[G^{\frac{4}{N-2}} g \right] \ge 0 \quad on \ \partial M \setminus \{y_0\}, \tag{2.16}$$

then $m_0 > 0$.

The integral expression $\mathcal{I}$ for the mass was introduced by Brendle and Chen [9]. In Lemma 7.3, we will examine the relationship between the integral $\mathcal{I}$ and the function $\mathcal{P}'$ defined in (2.7), after choosing the function G concretely.

3. Basic properties of blow-up

3.1. Characterization of blow-up points.
We recall the notion of blow-up, isolated blow-up and isolated simple blow-up. By virtue of Proposition 3.2, it is enough to consider when the blow-up occurs near a point on the boundary. The version we will use here is identical to those in [2, 5].

Definition 3.1. Pick a small number $\rho_1 > 0$ such that g_m-Fermi coordinates centered at $y \in \partial M$ is well-defined in the closed geodesic half-ball $\overline{B_+^N(y, \rho_1)} \subset M$ for every $m \in \mathbb{N}$ and $y \in \partial M$.

(1) $y_0 \in \partial M$ is called a blow-up point of a sequence $\{U_m\}_{m \in \mathbb{N}}$ in $H^1(M)$ if there exists a sequence of points $\{y_m\}_{m \in \mathbb{N}} \subset \partial M$ such that y_m is a local maximum of $U_m|_{\partial M}$ satisfying that $U_m(y_m) \to \infty$ and $y_m \to y_0$ as $m \to \infty$. For the sake of brevity, we will often say that $y_m \to y_0$ is a blow-up point of $\{U_m\}_{m \in \mathbb{N}}$.

(2) $y_0 \in \partial M$ is an isolated blow-up point of $\{U_m\}_{m \in \mathbb{N}}$ if y_0 is a blow-up point such that

$$U_m(y) \le C d_{g_m}(y, y_m)^{-\frac{1}{p_m - 1}} \quad \text{for any } y \in M \setminus \{y_m\}, \ d_{g_m}(y, y_m) < \rho_2$$

for some $C > 0$ and $\rho_2 \in (0, \rho_1]$.

(3) Let $\overline{U}_m$ be a weighted spherical average of U_m, i.e.,

$$\overline{U}_m(\rho) = \rho^{\frac{1}{p_m-1}} \left(\frac{\int_{\partial_I B_+^N(y_m,\rho)} U_m \, dS_{g_m}}{\int_{\partial_I B_+^N(y_m,\rho)} dS_{g_m}} \right), \quad \rho \in (0, \rho_1). \tag{3.1}$$

We say that an isolated blow-up point y_0 of $\{U_m\}_{m\in\mathbb{N}}$ is simple if there exists a number $\rho_3 \in (0, \rho_2]$ such that $\overline{U}_m$ possesses exactly one critical point in the interval $(0, \rho_3)$ for large $m \in \mathbb{N}$.

Hereafter, we always assume that $U_m \in H^1(M)$ is a solution to (2.5) with $p = p_m$, $g = g_m$ and $f_m = 1$ for each $m \in \mathbb{N}$. For simplicity, we will just say that $\{U_m\}_{m\in\mathbb{N}}$ is a sequence of solutions to (2.5). We also assume that $y_m \to y_0 \in \partial M$ is a blow-up point of $\{U_m\}_{m\in\mathbb{N}}$. Set $M_m = U_m(y_m)$ and $\epsilon_m = M_m^{-(p_m-1)}$ for each $m \in \mathbb{N}$. Obviously, $M_m \to \infty$ and $\epsilon_m \to 0$ as $m \to \infty$.

Choose a suitable positive smooth function ω_m on M so that the metric $\tilde{g}_m = \omega_m^{\frac{4}{N-2}} g_m$ on M satisfies properties depicted in Lemma 2.2 where y_* is replaced with y_m. Then $\widetilde{U}_m = \omega_m^{-1} U_m$ is a solution to (2.6) with $\tilde{g} = \tilde{g}_m$ and $\tilde{f} = \tilde{f}_m = \omega_m f_m$, and a sequence $\{\tilde{g}_m\}_{m\in\mathbb{N}}$ of the metrics converges to a metric $\tilde{g}_0$ in $C^4(M, \mathbb{R}^{N\times N})$ as $m \to \infty$. We shall often use $x \in \mathbb{R}_+^N$ to denote $\tilde{g}_m$-Fermi coordinates centered at y_m so that $\widetilde{U}_m$ can be regarded as a function in $\mathbb{R}_+^N$ near the origin.

3.2. Basic properties of blowing-up solutions.

Firstly, we study asymptotic behavior of a sequence $\{U_m\}_{m\in\mathbb{N}}$ of solutions to (2.5) near blow-up points. It can be proved as in e.g. Proposition 1.1 of [30] or Proposition 3.2 of [25].

Proposition 3.2. *Assume that $N \geq 3$ and $p \in [1 + \varepsilon_0, \frac{N}{N-2}]$. Given arbitrary small $\varepsilon_1 > 0$ and large $R > 0$, there are constants $C_0, C_1 > 0$ depending only on $(M^N, g), \varepsilon_0, \varepsilon_1$ and R such that if $U \in H^1(M)$ is a solution to (1.2) with the property that $\max_M U \geq C_0$, then $\frac{N}{N-2} - p < \varepsilon_1$ and $U|_{\partial M}$ possesses local maxima $y_{01}, \cdots y_{0\mathcal{N}} \in \partial M$ for some integer $\mathcal{N} = \mathcal{N}(U) \geq 1$, for which the following statements hold:*

(1) *It is valid that*

$$\overline{B_h(y_{0m_1}, \rho_{m_1})} \cap \overline{B_h(y_{0m_2}, \rho_{m_2})} = \emptyset \quad for \; 1 \leq m_1 \neq m_2 \leq \mathcal{N}$$

where $\rho_m = RU(y_{0m})^{-(p-1)}$.

(2) *For each $m = 1, \cdots, \mathcal{N}$, we have*

$$\left\| U(y_{0m})^{-1} U \left(U(y_{0m})^{-(p-1)} \cdot \right) - W_{1,0} \right\|_{C^2(\overline{B_+^N(0,2R)})} \leq \varepsilon_1$$

in g-Fermi coordinates centered in y_m.

(3) *It holds that*

$$U(y) \, d_h(y, \{y_{01}, \cdots, y_{0\mathcal{N}}\})^{\frac{1}{p-1}} \leq C_1 \quad for \; y \in M.$$

Secondly, we discuss behavior of a sequence of solutions $\{U_m\}_{m\in\mathbb{N}}$ to (2.5) near isolated blow-up points. The next lemma can be proved as in e.g. Proposition 1.4 of [30] or Lemma 2.6 of [25].

Lemma 3.3. *Let $y_m \to y_0 \in \partial M$ be an isolated blow-up point of a sequence $\{U_m\}_{m\in\mathbb{N}}$ of solutions to (2.5). In addition, suppose that $\{R_m\}_{m\in\mathbb{N}}$ and $\{\tau_m\}_{m\in\mathbb{N}}$ are arbitrary sequences of positive numbers such that $R_m \to \infty$ and $\tau_m \to 0$ as $m \to \infty$. Then $p_m \to \frac{N}{N-2}$ as $m \to \infty$, and $\{U_\ell\}_{\ell\in\mathbb{N}}$ and $\{p_\ell\}_{\ell\in\mathbb{N}}$ have subsequences $\{U_{\ell_m}\}_{m\in\mathbb{N}}$ and $\{p_{\ell_m}\}_{m\in\mathbb{N}}$ such that*

$$\left\| \epsilon_{\ell_m}^{\frac{1}{p_{\ell_m}-1}} U_{\ell_m}(\epsilon_{\ell_m} \cdot) - W_{1,0} \right\|_{C^2(\overline{B_+^N(0,R_m)})} \leq \tau_m \tag{3.2}$$

in g_m-Fermi coordinates centered in y_m and $R_m \epsilon_{\ell_m} \to 0$ as $m \to \infty$.

Therefore, we can select $\{R_m\}_{m\in\mathbb{N}}$ and $\{U_{\ell_m}\}_{m\in\mathbb{N}}$ satisfying (3.2) and $R_m\epsilon_{\ell_m} \to 0$. In order to simplify notations, we will use $\{U_m\}_{m\in\mathbb{N}}$ instead of $\{U_{\ell_m}\}_{m\in\mathbb{N}}$, and so on.

The following result is a simple consequence of Lemma 3.3 with the selection $\tau_m = \frac{1}{2}w_{1,0}(R_m)$. Its proof is given in Corollary 3.6 of [33].

Corollary 3.4. *Suppose that $y_m \to y_0 \in \partial M$ is an isolated blow-up point of a sequence $\{U_m\}_{m\in\mathbb{N}}$ of solutions to (2.5).*

(1) If $\{\widetilde{U}_m\}_{m\in\mathbb{N}}$ is a sequence of solutions to (2.6) constructed as in Subsection 3.1, then $y_m \to y_0 \in \partial M$ is an isolated blow-up point of $\{\widetilde{U}_m\}_{m\in\mathbb{N}}$.

(2) The function $\overline{U}_m$ in (3.1) has exactly one critical point in the interval $(0, R_m\epsilon_m)$ for large $m \in \mathbb{N}$. In particular, if the isolated blow-up point $y_0 \in \partial M$ of $\{U_m\}_{m\in\mathbb{N}}$ is also simple, then $\overline{U}'_m(r) < 0$ for all $r \in [R_m\epsilon_m, r_3)$; see Definition 3.1 (3).

Thirdly, we examine how a sequence $\{U_m\}_{m\in\mathbb{N}}$ of solutions to (2.5) behaves near isolated simple blow-up points. See Proposition 4.3 of [2] for its proof.

Proposition 3.5. *Assume that $N \geq 3$ and $y_m \to y_0 \in \partial M$ is an isolated simple blow-up point of a sequence $\{U_m\}_{m\in\mathbb{N}}$ of solutions to (2.5), and $\{\widetilde{U}_m\}_{m\in\mathbb{N}}$ is a sequence of solutions to (2.6) constructed as in Subsection 3.1. Then there exists $C > 0$ and $\rho_4 \in (0, \rho_3)$ independent of $m \in \mathbb{N}$ such that*

$$M_m \left|\nabla^\ell \widetilde{U}_m(x)\right| \leq C|x|^{-(N-2+\ell)} \quad in \ \left\{x \in \mathbb{R}^N_+ : 0 < |x| \leq \rho_4\right\} \tag{3.3}$$

for $\ell = 0, 1, 2$ and

$$M_m\widetilde{U}_m(x) \geq C^{-1}G_m(x) \quad in \ \left\{x \in \mathbb{R}^N_+ : R_m\epsilon_m \leq |x| \leq \rho_4\right\}$$

in $\tilde{g}_m$-Fermi coordinate system centered at y_m. Here, G_m is the Green's function satisfying

$$\begin{cases} L_{g_m}G_m = 0 & in \ B^N_+(0, \rho_4), \\ B_{g_m}G_m = \delta_0 & on \ B^n(0, \rho_4), \\ G_m = 0 & on \ \partial_I B^N_+(0, \rho_4), \\ \lim_{|x|\to 0}|x|^{N-2}G_m(x) = 1, \end{cases}$$

and δ_0 is the Dirac measure centered at $0 \in \mathbb{R}^N_+$. Also,

$$M_m^{\delta_m} = M_m^{\frac{N}{N-2}-p_m} \to 1 \quad as \ m \to \infty. \tag{3.4}$$

4. Linear problems and refined blow-up analysis

4.1. **Linear problems.** In this subsection, we study the linear problem

$$\begin{cases} -\Delta\Psi = 2\epsilon\pi_{ij}x_N\partial_{ij}W_{1,0} & in \ \mathbb{R}^N_+ = \mathbb{R}^n \times (0,\infty), \\ -\lim_{x_N\to 0}\dfrac{\partial\Psi}{\partial x_N} = Nw_{1,0}^{\frac{2}{N-2}}\Psi & on \ \mathbb{R}^n. \end{cases} \tag{4.1}$$

which arises from the first-order expansion of the metric on M; see Lemma 2.1. Here, $\epsilon > 0$ is a small parameter, $W_{1,0}$ is the function defined in (2.3), $w_{1,0}(\bar{x}) = W_{1,0}(\bar{x}, 0)$ for $\bar{x} \in \mathbb{R}^n$, and π is a trace-free symmetric 2-tensor (that is, $n \times n$-matrices).

Proposition 4.1. *Suppose that $N \geq 3$. There exists a smooth solution Ψ to (4.1) and a constant $C > 0$ depending only on N such that*

$$\left|\nabla^\ell\Psi(x)\right| \leq C\epsilon\left(\max_{i,j=1,\cdots,n}|\pi_{ij}|\right)\frac{1}{1+|x|^{N-3+\ell}} \quad in \ \mathbb{R}^N_+ \tag{4.2}$$

for any $\ell \in \mathbb{N} \cup \{0\}$,

$$\Psi(0) = \frac{\partial\Psi}{\partial x_1}(0) = \cdots = \frac{\partial\Psi}{\partial x_n}(0) = 0 \quad and \quad \int_{\mathbb{R}^n}w_{1,0}^{\frac{N}{N-2}}\Psi d\bar{x} = 0. \tag{4.3}$$

Proof. Pick a smooth function $\chi : [0, \infty) \to [0, 1]$ such that $\chi(t) = 1$ on $[0, 1]$ and 0 in $[2, \infty)$. Set also $\chi_\Lambda(t) = \chi(\frac{t}{\Lambda})$ for any $\Lambda > 0$. In Proposition 5.1 of [2], it was proved that for each $\Lambda > 0$, there exists a smooth function Ψ_Λ to

$$\begin{cases} -\Delta\Psi = 2\epsilon\pi_{ij}\chi_\Lambda(|x|)x_N\partial_{ij}W_{1,0} & \text{in } \mathbb{R}^N_+, \\ -\lim_{x_N\to 0}\dfrac{\partial\Psi}{\partial x_N} = Nw_{1,0}^{\frac{2}{N-2}}\Psi & \text{on } \mathbb{R}^n \end{cases} \tag{4.4}$$

satisfying (4.2)-(4.3) for some constant $C > 0$ depending only on N (thereby being independent of $\Lambda > 0$).

Now, we choose a sequence $\{\Lambda_m\}_{m\in\mathbb{N}}$ of positive increasing numbers which diverges to ∞. By the standard elliptic estimates, we may assume that the sequence $\{\Psi_{\Lambda_m}\}_{m\in\mathbb{N}}$ of solutions to (4.4) with $\Lambda = \Lambda_m$ converges to a smooth solution Ψ to (4.1) in $C^2_{\text{loc}}(\mathbb{R}^N_+)$. In particular, Ψ satisfies (4.2)-(4.3). $\qquad\square$

Remark 4.2. If $N \geq 5$, we infer from (4.2) that $\Psi \in D^{1,2}(\mathbb{R}^N_+)$. In this case, one can argue as in Proposition 4.1 of [33] to deduce the above proposition. Also, (4.1), (4.3) and the condition $\text{trace}(\pi) = 0$ imply

$$\int_{\mathbb{R}^N_+}\nabla\Psi\cdot\nabla W_{1,0}dx = 0.$$

For a better understanding of the function Ψ, we decompose it into two pieces: The first part Φ is a rational function with parameters $a_1, a_2 \in \mathbb{R}$ whose Laplacian is the same as that of Ψ in $\mathbb{R}^N_+$, whose precise form is given in Lemma 4.3. The second part Ξ is a harmonic function with prescribed boundary condition, which is described in Lemma 4.5. The proof of the lemmas are postponed until Appendix A.

Lemma 4.3. *Suppose that $N \geq 4$. Given any $a_1, a_2 \in \mathbb{R}$, let*

$$\Phi(x) = \frac{\epsilon\pi_{ij}x_ix_j}{(|\bar{x}|^2 + (x_N+1)^2)^{\frac{N}{2}}}\left[\left(\frac{N-2}{2}\right)(x_N-1)\right.$$
$$\left. + \frac{a_1(x_N+1)}{(|\bar{x}|^2 + (x_N+1)^2)^2} + \frac{a_2}{|\bar{x}|^2 + (x_N+1)^2}\right] \tag{4.5}$$

in $\mathbb{R}^N_+$. Then it is a solution of

$$-\Delta\Phi = 2\epsilon\pi_{ij}x_N\partial_{ij}W_{1,0} \quad \text{in } \mathbb{R}^N_+. \tag{4.6}$$

Remark 4.4. The function Φ in (4.5) and the correction term ψ_ϵ defined in Page 387 of Marques [43] share a similar pointwise behavior. However, Φ have two degrees of freedom on the coefficients, while ψ_ϵ has only one.

Lemma 4.5. *Suppose that $N \geq 4$. The function $\Xi = \Psi - \Phi$ satisfies*

$$\begin{cases} -\Delta\Xi = 0 & \text{in } \mathbb{R}^N_+, \\ -\lim_{x_N\to 0}\dfrac{\partial\Xi}{\partial x_N} = Nw_{1,0}^{\frac{2}{N-2}}\Xi + q & \text{on } \mathbb{R}^n \end{cases} \tag{4.7}$$

where

$$q(\bar{x}) = \frac{\epsilon\pi_{ij}x_ix_j}{(|\bar{x}|^2 + 1)^{\frac{N}{2}}}\left[\frac{N-2}{2} + a_1\left\{\frac{1}{(|\bar{x}|^2 + 1)^2} - \frac{4}{(|\bar{x}|^2 + 1)^3}\right\} - \frac{2a_2}{(|\bar{x}|^2 + 1)^2}\right] \tag{4.8}$$

on $\mathbb{R}^n$.

We prove an auxiliary lemma that comes from the mountain pass structure of the fractional Yamabe problem in $\mathbb{R}^N_+$. It will used in the proof of Proposition 5.1 for $N = 5$ and 6.

Lemma 4.6. *For $N \geq 5$, it holds that $\Xi \in D^{1,2}(\mathbb{R}^N_+)$ and*

$$\int_{\mathbb{R}^N_+} |\nabla\Xi|^2 dx - N \int_{\mathbb{R}^n} w_{1,0}^{\frac{2}{N-2}} \Xi^2 d\bar{x} \geq 0. \tag{4.9}$$

Proof. By (4.2) and (4.5), we readily observe that $\Xi \in D^{1,2}(\mathbb{R}^N_+)$.

Testing Ξ in (2.4) and $W_{1,0}$ in (4.7) gives

$$(N-2) \int_{\mathbb{R}^n} w_{1,0}^{\frac{N}{N-2}} \Xi d\bar{x} = \int_{\mathbb{R}^N_+} \nabla\Xi \cdot \nabla W_{1,0} dx$$

$$= N \int_{\mathbb{R}^n} w_{1,0}^{\frac{N}{N-2}} \Xi d\bar{x} + \int_{\mathbb{R}^n} q w_{1,0} d\bar{x} = N \int_{\mathbb{R}^n} w_{1,0}^{\frac{N}{N-2}} \Xi d\bar{x}$$

where the last equality holds owing to the condition that $\text{trace}(\pi) = 0$. Thus

$$\int_{\mathbb{R}^N_+} \nabla\Xi \cdot \nabla W_{1,0} dx = \int_{\mathbb{R}^n} w_{1,0}^{\frac{N}{N-2}} \Xi d\bar{x} = 0. \tag{4.10}$$

One can now argue as in the proof of Lemma 4.5 of [15] to deduce the validity of (4.9). Here we provide a more direct proof.

Define the energy functional J of (2.4) as

$$J(U) = \frac{1}{2} \int_{\mathbb{R}^N_+} |\nabla U|^2 dx - \frac{(N-2)^2}{2(N-1)} \int_{\mathbb{R}^n} U_+^{\frac{2(N-1)}{N-2}} d\bar{x} \quad \text{for } U \in D^{1,2}(\mathbb{R}^N_+)$$

and the Nehari manifold $\mathcal{M}$ associated with J as

$$\mathcal{M} = \left\{ U \in D^{1,2}(\mathbb{R}^N_+) \setminus \{0\} : \int_{\mathbb{R}^N_+} |\nabla U|^2 dx = (N-2) \int_{\mathbb{R}^n} U_+^{\frac{2(N-1)}{N-2}} d\bar{x} \right\}$$

where $U_+ = \max\{U, 0\}$. Then J is a functional of class C^2, $\mathcal{M}$ is a C^1-Hilbert manifold and $W_{1,0} \in \mathcal{M}$. Moreover, the tangent space $T_{W_{1,0}}\mathcal{M}$ of $\mathcal{M}$ at $W_{1,0}$ is

$$T_{W_{1,0}}\mathcal{M} = \left\{ U \in D^{1,2}(\mathbb{R}^N_+) : \int_{\mathbb{R}^N_+} \nabla W_{1,0} \cdot \nabla U dx = (N-1) \int_{\mathbb{R}^n} w_{1,0}^{\frac{N}{N-2}} U d\bar{x} \right\}.$$

In particular, (4.10) implies that $\Xi \in T_{W_{1,0}}\mathcal{M}$. By Theorem 1.1 of [20], $W_{1,0}$ is a minimizer of J in $\mathcal{M}$. Therefore

$$0 \leq \left. \frac{d^2 J(W_{1,0} + \varepsilon\Xi)}{d\varepsilon^2} \right|_{\varepsilon=0} = \int_{\mathbb{R}^N_+} |\nabla\Xi|^2 dx - N \int_{\mathbb{R}^n} w_{1,0}^{\frac{2}{N-2}} \Xi^2 d\bar{x},$$

which is (4.9). $\qquad\square$

4.2. **Refined blow-up analysis.** By using Proposition 4.1, we can analyze the ϵ_m-order behavior of a sequence $\{U_m\}_{m \in \mathbb{N}}$ of solutions to (2.5) near isolated simple blow-up points. Owing to Corollary 3.4 (i) and Lemma 2.2, $y_m \to y_0$ is an isolated blow-up point of a sequence $\{\widetilde{U}_m\}_{m \in \mathbb{N}}$ of solutions to (2.6) constructed in Subsection 3.1, and $M_m = \widetilde{U}_m(y_m)$.

Proposition 4.7. *Suppose that $N \geq 4$ and $y_m \to y_0 \in \partial M$ is an isolated simple blow-up point of $\{U_m\}_{m \in \mathbb{N}}$. Let Ψ_m be the solution of (4.1) with $\epsilon = \epsilon_m$ and $\pi = \pi[\tilde{g}_m](y_m)$, and*

$$\widetilde{V}_m(x) = \epsilon_m^{\frac{1}{p_m-1}} \widetilde{U}_m (\epsilon_m x) \quad \text{in } B^N_+(0, \rho_4 \epsilon_m^{-1}). \tag{4.11}$$

Then there exists $C > 0$ and $\rho_5 \in (0, \rho_4]$ independent of $m \in \mathbb{N}$ such that

$$\left| \nabla^\ell_{\bar{x}} \widetilde{V}_m - \nabla^\ell_{\bar{x}}(W_{1,0} + \Psi_m) \right|(x) \leq \frac{C\epsilon_m^2}{1 + |x|^{N-4+\ell}} \quad \text{in } B^N_+(0, \rho_5 \epsilon_m^{-1}) \tag{4.12}$$

for $\ell = 0, 1, 2$.

For $N \geq 5$, the proposition was proved in Proposition 6.1 of [2] and Proposition 4.2 of [33]. Also, a slight modification of the arguments in [2, 33] shows that it also holds for $N = 4$. Check Proposition 5.3 of [5] where its 3-dimensional version was derived.

5. QUANTITATIVE ANALYSIS ON THE TRACE-FREE SECOND FUNDAMENTAL FORM

5.1. **Vanishing theorem of the trace-free second fundamental form.** In the next proposition, we prove that the trace-free second fundamental form must vanish at each isolated simple blow-up point of blowing-up solutions when $N = 4, 5, 6$. An analogous result for $N \geq 7$ can be found in Theorem 7.1 of [2].

Proposition 5.1. *Suppose that $N = 4, 5, 6$ and $y_m \to y_0 \in \partial M$ is an isolated simple blow-up point of the sequence $\{U_m\}_{m \in \mathbb{N}}$ of the solutions to (2.5). If $\{\tilde{g}_m\}_{m \in \mathbb{N}}$ is a sequence of the metrics constructed in Subsection 3.1, then there exists $C > 0$ independent of $m \in \mathbb{N}$ such that*

$$\|\pi[\tilde{g}_m](y_m)\|^2 \leq \begin{cases} \dfrac{C}{|\log \epsilon_m|} & \text{for } N = 4, \\ C\epsilon_m |\log \epsilon_m| & \text{for } N = 5, \\ C\epsilon_m & \text{for } N = 6. \end{cases} \tag{5.1}$$

Particularly, $\pi[\tilde{g}_0](y_0) = 0$.

Let $\{\widetilde{U}_m\}_{m \in \mathbb{N}}$ be a sequence of solutions to (2.6) depicted in Subsection 3.1. By appealing $\tilde{g}_m$-Fermi coordinates on M centered at y_m, we regard $\widetilde{U}_m$ as a function defined near $0 \in \mathbb{R}^N_+$. For brevity, we write $\pi_m = \pi[\tilde{g}_m](y_m)$ for all $m \in \mathbb{N}$.

Denoting $\hat{g}_m = \tilde{g}_m(\epsilon_m \cdot)$ and $\hat{f}_m = \tilde{f}_m(\epsilon_m \cdot)$, we see from (2.6) that the function $\widetilde{V}_m$ introduced in (4.11) solves

$$\begin{cases} -\Delta \widetilde{V}_m = -\left[\dfrac{N-2}{4(N-1)}\right] \epsilon_m^2 R[\tilde{g}_m](\epsilon_m \cdot)\widetilde{V}_m + (\Delta_{\hat{g}_m} - \Delta)\widetilde{V}_m & \text{in } B_+^N(0, \rho_5 \epsilon_m^{-1}), \\ -\dfrac{\partial \widetilde{V}_m}{\partial x_N} + \left[\dfrac{N-2}{2}\right] \epsilon_m H[\tilde{g}_m](\epsilon_m \cdot)\widetilde{V}_m = (N-2)\hat{f}_m^{-\delta_m} \widetilde{V}_m^{p_m} & \text{on } B^n(0, \rho_5 \epsilon_m^{-1}). \end{cases}$$

Thus, employing Pohozaev's identity (2.9), one can write

$$\mathcal{P}\left(\widetilde{V}_m, \rho \epsilon_m^{-1}\right) = \mathcal{P}_{1m}\left(\widetilde{V}_m, \rho \epsilon_m^{-1}\right) + \frac{\delta_m}{p_m + 1}\mathcal{P}_{2m}\left(\widetilde{V}_m, \rho \epsilon_m^{-1}\right) \quad \text{for any } \rho \in (0, \rho_5] \tag{5.2}$$

where $\mathcal{P}$ is the function defined in (2.8) with $f = (N-2)\hat{f}_m^{-\delta_m}$,

$$\mathcal{P}_{1m}(U, \rho)$$
$$= \int_{B_+^N(0,\rho)} \left[\left\{\frac{N-2}{4(N-1)}\right\} \epsilon_m^2 R[\tilde{g}_m](\epsilon_m \cdot)U + (\Delta - \Delta_{\hat{g}_m})U\right] \cdot \left[x_a \partial_a U + \left(\frac{N-2}{2}\right)U\right] dx$$
$$+ \left(\frac{N-2}{2}\right) \epsilon_m \int_{B^n(0,\rho)} H[\tilde{g}_m](\epsilon_m \cdot) \left[x_i \partial_i U + \left(\frac{N-2}{2}\right)U\right] U d\bar{x} \tag{5.3}$$

and

$$\mathcal{P}_{2m}(U, \rho) = -\int_{B^n(0,\rho)} x_i \partial_i \hat{f}_m \hat{f}_m^{-(\delta_m+1)} U^{p_m+1} d\bar{x} + \left(\frac{N-2}{2}\right) \int_{B^n(0,\rho)} \hat{f}_m^{-\delta_m} U^{p_m+1} d\bar{x}.$$

The left-hand side of (5.2) involves with the boundary integrals only. By (3.3), (3.4) and (4.11), there exists a constant $C > 0$ independent of $m \in \mathbb{N}$ and $\rho \in (0, \rho_5]$ such that

$$\mathcal{P}\left(\widetilde{V}_m, \rho \epsilon_m^{-1}\right) = O(\epsilon_m^{N-2}). \tag{5.4}$$

The right-hand side of (5.2) involves with the interior integrals. We can take ρ so small that

$$\mathcal{P}_{2m}\left(\widetilde{V}_m, \rho\epsilon_m^{-1}\right) \geq 0. \tag{5.5}$$

Also, choosing $\kappa \geq 2$ in (2.1), we may assume that the second integral in the right-hand side of (5.3) is bounded by

$$\epsilon_m \int_{B^n(0,\rho\epsilon_m^{-1})} |H[\tilde{g}_m](\epsilon_m\bar{x})| \left|x_i\partial_i\widetilde{V}_m + \left(\frac{N-2}{2}\right)\widetilde{V}_m\right| \left|\widetilde{V}_m\right| d\bar{x}$$

$$\leq C\epsilon_m^{\kappa+1} \int_{B^n(0,\rho\epsilon_m^{-1})} \frac{|\bar{x}|^\kappa}{1+|\bar{x}|^{2(N-2)}} d\bar{x} = O(\epsilon_m^3) + O(\epsilon_m^{N-2});$$

see the derivation of (7.2) below. Hence, by fixing ρ small enough and invoking (4.12), we get

$$\mathcal{P}_{1m}\left(\widetilde{V}_m, \rho\epsilon_m^{-1}\right)$$

$$= F_m(W_{1,0}, W_{1,0}) + [F_m(W_{1,0}, \Psi_m) + F_m(\Psi_m, W_{1,0})] + \begin{cases} O(\epsilon_m^2) & \text{for } N = 4, \\ O(\epsilon_m^3|\log\epsilon_m|) & \text{for } N = 5, \\ O(\epsilon_m^3) & \text{for } N \geq 6 \end{cases} \tag{5.6}$$

where

$$F_m(V_1, V_2) = \int_{B_+^N(0,\rho\epsilon_m^{-1})} \left[\left\{\frac{N-2}{4(N-1)}\right\}\epsilon_m^2 R[\tilde{g}_m](\epsilon_m x)V_1 + (\Delta - \Delta_{\hat{g}_m})V_1\right]$$

$$\times \left[x \cdot \nabla V_2 + \left(\frac{N-2}{2}\right)V_2\right] dx \tag{5.7}$$

and Ψ_m is the solution of (4.1) with $\epsilon = \epsilon_m$ and $\pi = \pi_m$. To estimate (5.6), we divide the cases according to the dimension N. We examine the case $N = 5$ first, $N = 6$ second, and $N = 4$ at last.

CASE $N = 5$: By putting $n = 4$ and $\gamma = \frac{1}{2}$ in (5.9) of [33], one can compute that

$$F_m(W_{1,0}, W_{1,0}) = C_1\epsilon_m^2\|\pi_m\|^2 + O(\epsilon_m^3|\log\epsilon_m|) \tag{5.8}$$

where

$$C_1 = -\frac{1}{8}\int_{\mathbb{R}_+^5} x_5^2|\nabla_{\bar{x}}W_{1,0}|^2 dx = -\frac{9}{8}|\mathbb{S}^3|\int_0^\infty \frac{x_5^2 dx_5}{(x_5+1)^4}\int_0^\infty \frac{t^5 dt}{(t^2+1)^5}$$

$$= -\frac{9}{8}|\mathbb{S}^3| \cdot \frac{1}{3} \cdot \frac{1}{24} = -\frac{1}{64}|\mathbb{S}^3|.$$

Besides, it was shown in (5.10) of [33] that

$$F_m(W_{1,0}, \Psi_m) + F_m(\Psi_m, W_{1,0}) \geq O(\epsilon_m^3|\log\epsilon_m|).$$

However, it is not enough to deduce the proposition because $C_1 < 0$. We will improve the estimate in the next result.

Lemma 5.2. *It holds that*

$$F_m(W_{1,0}, \Psi_m) + F_m(\Psi_m, W_{1,0}) \tag{5.9}$$

$$\geq |\mathbb{S}^3|\left(-\frac{1}{128} + \frac{a_1}{480} - \frac{11a_1^2}{60480} + \frac{a_2}{160} - \frac{a_1a_2}{1680} - \frac{a_2^2}{1680}\right)\epsilon_m^2\|\pi_m\|^2 + O(\epsilon_m^3|\log\epsilon_m| \cdot \|\pi_m\|).$$

Proof of Lemma 5.2. We see from Derivation of (5.10) of [33] that

$$F_m(W_{1,0}, \Psi_m) + F_m(\Psi_m, W_{1,0})$$

$$= -2\epsilon_m(\pi_m)_{ij} \left[\int_{\mathbb{R}^5_+} x_5 \partial_{ij} W_{1,0} \left(x \cdot \nabla \Psi_m + \frac{3}{2} \Psi_m \right) dx + \int_{\mathbb{R}^5_+} x_5 \partial_{ij} \Psi_m Z^0_{1,0} dx \right]$$

$$+ O(\epsilon_m^3 |\log \epsilon_m| \cdot \|\pi_m\|) \tag{5.10}$$

$$= -2\epsilon_m(\pi_m)_{ij} \int_{\mathbb{R}^5_+} x_5 \partial_i W_{1,0} \partial_j \Psi_m dx + O(\epsilon_m^3 |\log \epsilon_m| \cdot \|\pi_m\|)$$

$$= 2\epsilon_m(\pi_m)_{ij} \left(\int_{\mathbb{R}^5_+} x_5 \partial_{ij} W_{1,0} \Phi_m dx + \int_{\mathbb{R}^5_+} x_5 \partial_{ij} W_{1,0} \Xi_m dx \right) + O(\epsilon_m^3 |\log \epsilon_m| \cdot \|\pi_m\|)$$

where Φ_m and Ξ_m are defined by (4.5) and (4.7) with $\epsilon = \epsilon_m$ and $\pi = \pi_m$, and so $\Psi_m = \Phi_m + \Xi_m$.

On the other hand, by testing Ξ_m in (4.1), we obtain

$$2\epsilon_m(\pi_m)_{ij} \int_{\mathbb{R}^5_+} x_5 \partial_{ij} W_{1,0} \Xi_m dx$$

$$= \int_{\mathbb{R}^5_+} \nabla \Psi_m \cdot \nabla \Xi_m dx - 5 \int_{\mathbb{R}^4} w_{1,0}^{\frac{2}{3}} \Psi_m \Xi_m d\bar{x}$$

$$= \int_{\mathbb{R}^5_+} \nabla \Phi_m \cdot \nabla \Xi_m dx - 5 \int_{\mathbb{R}^4} w_{1,0}^{\frac{2}{3}} \Phi_m \Xi_m d\bar{x} + \int_{\mathbb{R}^5_+} |\nabla \Xi_m|^2 dx - 5 \int_{\mathbb{R}^4} w_{1,0}^{\frac{2}{3}} \Xi_m^2 d\bar{x}.$$

Testing Φ_m in (4.7), we find

$$\int_{\mathbb{R}^5_+} \nabla \Xi_m \cdot \nabla \Phi_m dx = 5 \int_{\mathbb{R}^4} w_{1,0}^{\frac{2}{3}} \Xi_m \Phi_m d\bar{x} + \int_{\mathbb{R}^4} q_m \Phi_m d\bar{x}$$

where q_m is the function defined by (4.8) with $\epsilon = \epsilon_m$ and $\pi = \pi_m$. Thus it follows from (4.9) that

$$2\epsilon_m(\pi_m)_{ij} \int_{\mathbb{R}^5_+} x_5 \partial_{ij} W_{1,0} \Xi_m dx \geq \int_{\mathbb{R}^4} q_m \Phi_m d\bar{x}. \tag{5.11}$$

Combining (5.10) and (5.11), we obtain

$$F_m(W_{1,0}, \Psi_m) + F_m(\Psi_m, W_{1,0})$$

$$\geq (\pi_m)_{ij} \left(2\epsilon_m \int_{\mathbb{R}^5_+} x_5 \partial_{ij} W_{1,0} \Phi_m dx + \int_{\mathbb{R}^4} q_m \Phi_m d\bar{x} \right) + O(\epsilon_m^3 |\log \epsilon_m| \cdot \|\pi_m\|). \tag{5.12}$$

By applying (4.5), we evaluate

$$2\epsilon_m(\pi_m)_{ij} \int_{\mathbb{R}^5_+} x_5 \partial_{ij} W_{1,0} \Phi_m dx$$

$$= 30\epsilon_m(\pi_m)_{ij} \int_{\mathbb{R}^5_+} \frac{x_i x_j x_5}{(|\bar{x}|^2 + (x_5 + 1)^2)^{\frac{7}{2}}} \Phi_m dx$$

$$= 30\epsilon_m^2(\pi_m)_{ij}(\pi_m)_{kl} \int_{\mathbb{R}^5_+} x_i x_j x_k x_l \left[\frac{3}{2} \cdot \frac{x_5(x_5 - 1)}{(|\bar{x}|^2 + (x_5 + 1)^2)^6} \right.$$

$$\left. + a_1 \frac{x_5(x_5 + 1)}{(|\bar{x}|^2 + (x_5 + 1)^2)^8} + a_2 \frac{x_5}{(|\bar{x}|^2 + (x_5 + 1)^2)^7} \right] dx \tag{5.13}$$

$$= |\mathbb{S}^3| \left[\frac{15}{4} \int_0^\infty \frac{x_5(x_5 - 1) dx_5}{(x_5 + 1)^4} \int_0^\infty \frac{t^7 dt}{(t^2 + 1)^6} + \frac{5a_1}{2} \int_0^\infty \frac{x_5 dx_5}{(x_5 + 1)^7} \int_0^\infty \frac{t^7 dt}{(t^2 + 1)^8} \right.$$

$$+ \frac{5a_2}{2} \int_0^\infty \frac{x_5 dx_5}{(x_5+1)^6} \int_0^\infty \frac{t^7 dt}{(t^2+1)^7} \Bigg] \epsilon_m^2 \|\pi_m\|^2$$

$$= |\mathbb{S}^3| \left(\frac{1}{64} + \frac{a_1}{3360} + \frac{a_2}{960} \right) \epsilon_m^2 \|\pi_m\|^2$$

and

$$\int_{\mathbb{R}^4} q_m \Phi_m d\bar{x}$$

$$= \epsilon_m^2 (\pi_m)_{ij} (\pi_m)_{kl} \int_{\mathbb{R}^4} \frac{x_i x_j x_k x_l}{(|\bar{x}|^2+1)^5} \left[\frac{3}{2} + a_1 \left\{ \frac{1}{(|\bar{x}|^2+1)^2} - \frac{4}{(|\bar{x}|^2+1)^3} \right\} - \frac{2a_2}{(|\bar{x}|^2+1)^2} \right]$$

$$\times \left[-\frac{3}{2} + \frac{a_1}{(|\bar{x}|^2+1)^2} + \frac{a_2}{|\bar{x}|^2+1} \right] d\bar{x}$$

$$= \frac{1}{12} |\mathbb{S}^3| \int_0^\infty \frac{r^7}{(r^2+1)^5} \left[\frac{3}{2} + a_1 \left\{ \frac{1}{(r^2+1)^2} - \frac{4}{(r^2+1)^3} \right\} - \frac{2a_2}{(r^2+1)^2} \right] \tag{5.14}$$

$$\times \left[-\frac{3}{2} + \frac{a_1}{(r^2+1)^2} + \frac{a_2}{r^2+1} \right] d\bar{x} \cdot \epsilon_m^2 \|\pi_m\|^2$$

$$= |\mathbb{S}^3| \left(-\frac{3}{128} + \frac{a_1}{560} - \frac{11a_1^2}{60480} + \frac{a_2}{192} - \frac{a_1 a_2}{1680} - \frac{a_2^2}{1680} \right) \epsilon_m^2 \|\pi_m\|^2.$$

Putting (5.12)-(5.14), we deduce (5.9). $\qquad\square$

Corollary 5.3. *It holds that*

$$F_m(W_{1,0}, W_{1,0}) + [F_m(W_{1,0}, \Psi_m) + F_m(\Psi_m, W_{1,0})]$$

$$\geq \frac{3}{2560} |\mathbb{S}^3| \epsilon_m^2 \|\pi_m\|^2 + O(\epsilon_m^3 |\log \epsilon_m|) + O(\epsilon_m^3 |\log \epsilon_m| \cdot \|\pi_m\|). \tag{5.15}$$

Proof. Thus we conclude from (5.8) and (5.9) that

$$F_m(W_{1,0}, W_{1,0}) + [F_m(W_{1,0}, \Psi_m) + F_m(\Psi_m, W_{1,0})]$$

$$\geq |\mathbb{S}^3| P(a_1, a_2) \epsilon_m^2 \|\pi_m\|^2 + O(\epsilon_m^3 |\log \epsilon_m|) + O(\epsilon_m^3 |\log \epsilon_m| \cdot \|\pi_m\|)$$

where

$$P(a_1, a_2) = -\frac{3}{128} + \frac{a_1}{480} - \frac{11a_1^2}{60480} + \frac{a_2}{160} - \frac{a_1 a_2}{1680} - \frac{a_2^2}{1680}.$$

It holds that

$$\max_{a_1, a_2 \in \mathbb{R}} P(a_1, a_2) = P\left(-\frac{63}{4}, \frac{105}{8} \right) = \frac{3}{2560}.$$

Hence the assertion follows. $\qquad\square$

Completion of the proof of Proposition 5.1 for $N = 5$. Because $\tilde{g}_m \to \tilde{g}_0$ in $C^4(M, \mathbb{R}^{N \times N})$ as $m \to \infty$, the values of $\|\pi_m\|$ are uniformly bounded in $m \in \mathbb{N}$. From (5.2), (5.4), (5.5) and (5.15), we discover

$$O(\epsilon_m^3) \geq \frac{3}{2560} |\mathbb{S}^3| \epsilon_m^2 \|\pi_m\|^2 + O(\epsilon_m^3 |\log \epsilon_m|).$$

Accordingly,

$$O(\epsilon_m) \geq \frac{3}{2560} |\mathbb{S}^3| \|\pi_m\|^2 + O(\epsilon_m |\log \epsilon_m|).$$

Taking $m \to \infty$ on the both sides, we get (5.1) for $N = 5$. $\qquad\square$

CASE $N = 6$: The strategy is the same as the case $N = 5$. By inserting $n = 5$ and $\gamma = \frac{1}{2}$ in (5.9) of [33], one can compute that

$$F_m(W_{1,0}, W_{1,0}) = O(\epsilon_m^3).$$

Also, computing as in Lemma 5.2, we obtain

Lemma 5.4. *It holds that*

$$F_m(W_{1,0}, \Psi_m) + F_m(\Psi_m, W_{1,0})$$
$$\geq |\mathbb{S}^4| \left(-\frac{\pi}{320} + \frac{a_1\pi}{3584} - \frac{3a_1^2\pi}{163840} + \frac{a_2\pi}{1280} - \frac{a_1a_2\pi}{16384} - \frac{a_2^2\pi}{16384} \right) \epsilon_m^2 \|\pi_m\|^2 + O(\epsilon_m^3).$$

Choosing the parameters $a_1 = -\frac{128}{7}$ and $a_2 = \frac{544}{35}$, we get

Corollary 5.5. *It holds that*

$$F_m(W_{1,0}, W_{1,0}) + [F_m(W_{1,0}, \Psi_m) + F_m(\Psi_m, W_{1,0})] \geq \frac{31\pi}{78400} |\mathbb{S}^3| \, \epsilon_m^2 \|\pi_m\|^2 + O(\epsilon_m^3).$$

From this, the desired result (5.1) for $N = 6$ follows.

CASE $N = 4$: Because of the integrability issue on $W_{1,0}$, the computation becomes a little bit trickier than before. Especially, it turns out that the terms involving a_1 and a_2 contribute nothing. This is because the integrals involving them are $O(\epsilon_m^2)$, while the main order of $\mathcal{P}_{1m}(\widetilde{V}_m, \rho\epsilon_m^{-1})$ is $\epsilon_m^2 |\log \epsilon_m|$. Hence we set $a_1 = a_2 = 0$.

Lemma 5.6. *It holds that*

$$F_m(W_{1,0}, W_{1,0}) = -\frac{\pi}{24} |\mathbb{S}^2| \|\pi_m\|^2 \epsilon_m^2 \log(\rho\epsilon_m^{-1}) + O(\epsilon_m^2). \tag{5.16}$$

Proof. Lemma 2.2 and the Gauss-Codazzi equation implies that

$$R[\tilde{g}_m](\epsilon_m x) = -\|\pi_m\|^2 + O(\epsilon_m|x|) \quad \text{in } B_+^5(0, \rho\epsilon_m^{-1}).$$

From this, Lemma 2.1 (more precisely, Lemmas 3.1 and 3.2 of [22]) and (5.7), we find that

$$F_m(W_{1,0}, W_{1,0}) = \widetilde{F}_{0m} + \widetilde{F}_{1m} + \widetilde{F}_{2m} + O(\epsilon_m^2) \tag{5.17}$$

where

$$\widetilde{F}_{0m} = \frac{1}{6} \epsilon_m^2 \int_{B_+^4(0,\rho\epsilon_m^{-1})} R[\tilde{g}_m](\epsilon_m x) W_{1,0} Z_{1,0}^0 dx$$
$$= -\frac{1}{6} \epsilon_m^2 \|\pi_m\|^2 \int_{B_+^4(0,\rho\epsilon_m^{-1})} W_{1,0} Z_{1,0}^0 dx + O(\epsilon_m^2),$$

$$\widetilde{F}_{1m} = \int_{B_+^4(0,\rho\epsilon_m^{-1})} (\delta^{ij} - \hat{g}_m^{ij}) \partial_{ij} W_{1,0} Z_{1,0} dx$$
$$= -\frac{1}{3} \epsilon_m^2 \left[3\|\pi_m\|^2 + R_{NN}[\tilde{g}_m](y_m) \right] \int_{B_+^4(0,\rho\epsilon_m^{-1})} x_4^2 \Delta_{\bar{x}} W_{1,0} Z_{1,0}^0 dx + O(\epsilon_m^2)$$
$$= -\frac{2}{3} \epsilon_m^2 \|\pi_m\|^2 \int_{B_+^4(0,\rho\epsilon_m^{-1})} x_4^2 \Delta_{\bar{x}} W_{1,0} Z_{1,0}^0 dx + O(\epsilon_m^2)$$

and

$$\widetilde{F}_{2m} = -\int_{B_+^4(0,\rho\epsilon_m^{-1})} \left(\frac{\partial_a \sqrt{|\hat{g}_m|}}{\sqrt{|\hat{g}_m|}} \right) \hat{g}_m^{ab} \partial_b W_{1,0} Z_{1,0} dx$$
$$= \epsilon_m^2 \left[\|\pi_m\|^2 + R_{NN}[\tilde{g}_m](y_m) \right] \int_{B_+^4(0,\rho\epsilon_m^{-1})} \hat{g}_m^{ab} \partial_b W_{1,0} Z_{1,0} dx + O(\epsilon_m^2) = O(\epsilon_m^2). \tag{5.18}$$

On the other hand, since

$$\int_{B_+^4(0,\rho\epsilon_m^{-1})} W_{1,0} Z_{1,0}^0 dx$$
$$= \int_0^{\rho\epsilon_m^{-1}} \int_{\mathbb{R}^3} \frac{1 - |\bar{x}|^2 - x_4^2}{(|\bar{x}|^2 + (x_4 + 1)^2)^3} d\bar{x} dx_4 + O(1)$$

$$= -\left|\mathbb{S}^2\right| \left[\int_0^{\rho\epsilon_m^{-1}} \frac{dx_4}{x_4+1} \int_0^\infty \frac{t^4 dt}{(t^2+1)^3} + \int_0^{\rho\epsilon_m^{-1}} \frac{x_4^2 dx_4}{(x_4+1)^3} \int_0^\infty \frac{t^2 dt}{(t^2+1)^3} \right] + O(1)$$

$$= -\frac{\pi}{4}\left|\mathbb{S}^2\right|\log(\rho\epsilon_m^{-1}) + O(1),$$

we have

$$\widetilde{F}_{0m} = \frac{\pi}{24}\left|\mathbb{S}^2\right|\|\pi_m\|^2\epsilon_m^2\log(\rho\epsilon_m^{-1}) + O(\epsilon_m^2). \tag{5.19}$$

Moreover,

$$\int_{B_+^4(0,\rho\epsilon_m^{-1})} x_4^2 \Delta_{\bar{x}} W_{1,0} Z_{1,0}^0 dx$$

$$= 2 \int_0^{\rho\epsilon_m^{-1}} \int_{\mathbb{R}^3} \frac{x_4^2\left[|\bar{x}|^2 - 3(x_4+1)^2\right](1-|\bar{x}|^2 - x_4^2)}{(|\bar{x}|^2 + (x_4+1)^2)^5} dx$$

$$= -2\left|\mathbb{S}^2\right| \int_0^{\rho\epsilon_m^{-1}} \int_0^\infty \frac{r^2 x_4^2\left[r^2 - 3(x_4+1)^2\right](r^2 + x_4^2)}{(r^2 + (x_4+1)^2)^5} dr dx_4 + O(1)$$

$$= \frac{\pi}{8}\left|\mathbb{S}^2\right|\log(\rho\epsilon_m^{-1}) + O(1),$$

from which we deduce that

$$\widetilde{F}_{1m} = -\frac{\pi}{12}\left|\mathbb{S}^2\right|\|\pi_m\|^2\epsilon_m^2\log(\rho\epsilon_m^{-1}) + O(\epsilon_m^2). \tag{5.20}$$

Combining (5.17)-(5.20), we obtain (5.16). $\square$

Unlike the cases $N = 5$ and 6, we do not exploit the mountain pass structure of the boundary Yamabe problem in $\mathbb{R}_+^N$. Instead, we use the integrability (or the decay property) of the functions involving the problem.

We define

$$\Phi_\delta(x) = \epsilon\pi_{ij}x_i x_j \left[\frac{x_4 - 1}{(|\bar{x}|^2 + (x_4+1)^2)^2} + \frac{\delta}{(|\bar{x}|^2 + (x_4+1)^2)^{\frac{3}{2}}} \right] \tag{5.21}$$

for δ small, which resembles the modified correction term $\psi_{\epsilon,\delta}$ defined in Page 400 of [43]. If $\delta = 0$, the function Φ_δ is reduced to Φ in (4.5) with $a_1 = a_2 = 0$. Let also $\Xi_\delta = \Psi - \Phi_\delta$ where Ψ is the solution of (4.1). Then it satisfies

$$\begin{cases} -\Delta\Xi_\delta = \dfrac{9\delta\epsilon\pi_{ij}x_i x_j}{(|\bar{x}|^2 + (x_4+1)^2)^{\frac{5}{2}}} & \text{in } \mathbb{R}_+^4, \\ -\displaystyle\lim_{x_4 \to 0} \dfrac{\partial\Xi_\delta}{\partial x_4} = 4w_{1,0}\Xi_\delta + q_\delta & \text{on } \mathbb{R}^3 \end{cases} \tag{5.22}$$

where

$$q_\delta(\bar{x}) = \frac{\epsilon\pi_{ij}x_i x_j}{(|\bar{x}|^2 + 1)^2} + \frac{\delta\epsilon\pi_{ij}x_i x_j}{(|\bar{x}|^2 + 1)^{\frac{5}{2}}} \quad \text{on } \mathbb{R}^3.$$

Lemma 5.7. *It holds that*

$$F_m(W_{1,0}, \Psi_m) + F_m(\Psi_m, W_{1,0})$$

$$\geq \left(\frac{\pi}{24} + \frac{64}{105}\delta + O(\delta^2) \right)\left|\mathbb{S}^2\right|\epsilon_m^2\log(\rho\epsilon_m^{-1})\|\pi_m\|^2 + O(\epsilon_m^2) \tag{5.23}$$

for δ small.

Proof. Let $\Phi_{m,\delta}$ be the function Φ_δ in (5.21) with $\epsilon = \epsilon_m$ and $\pi = \pi_m$. Set $\Xi_{m,\delta}$ and $q_{m,\delta}$ in an analogous manner. By (4.5) and (4.2) of [33], it holds that

$$|\Phi_{m,\delta}(x)| + |\Xi_{m,\delta}(x)| \leq \frac{C\epsilon_m|\pi_m|_\infty}{1+|x|} \quad \text{and} \quad |\nabla\Phi_{m,\delta}(x)| + |\nabla\Xi_{m,\delta}(x)| \leq \frac{C\epsilon_m|\pi_m|_\infty}{1+|x|^2} \tag{5.24}$$

where $|\pi_m|_\infty = \max_{i,j=1,2,3} |(\pi_m)_{ij}|$. Integrating by parts, and employing (5.24),

$$\int_{B^3(0,\rho\epsilon_m^{-1})} \frac{d\bar{x}}{1 + |\bar{x}|^3 + (\rho\epsilon_m^{-1})^3} \leq |\mathbb{S}^2| \int_0^{\rho\epsilon_m^{-1}} \frac{r^2 dr}{r^3 + (\rho\epsilon_m^{-1})^3} = |\mathbb{S}^2| \int_0^1 \frac{dt}{t^3 + 1} = O(1)$$

and

$$\int_0^{\rho\epsilon_m^{-1}} \int_{\partial B^3(0,\rho\epsilon_m^{-1})} \frac{dx}{1 + |x|^3} \leq C \int_0^{\rho\epsilon_m^{-1}} \frac{(\rho\epsilon_m^{-1})^2 dx_4}{1 + (\rho\epsilon_m^{-1})^3 + x_4^3} \leq C \int_0^1 \frac{dt}{t^3 + 1} = O(1),$$

we calculate that

$$F_m(W_{1,0}, \Psi_m) + F_m(\Psi_m, W_{1,0})$$

$$= -2\epsilon_m (\pi_m)_{ij} \int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} x_4 \left[\partial_{ij} W_{1,0} \left(x_k \partial_k \Psi_m + x_4 \partial_4 \Psi_m + \Psi_m \right) \right.$$

$$\left. + \partial_{ij} \Psi_m \left(x_k \partial_k W_{1,0} + x_4 \partial_4 W_{1,0} + W_{1,0} \right) \right] d\bar{x} dx_4 + O(\epsilon_m^2)$$

$$= 2\epsilon_m (\pi_m)_{ij} \int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} x_4 \left[4\, \partial_i W_{1,0} \partial_j \Psi_m + \partial_i W_{1,0}(x_k \partial_{jk}\Psi_m + x_4 \partial_{j4}\Psi_m) \right. \quad (5.25)$$

$$\left. + (x_k \partial_{ik} W_{1,0} + x_4 \partial_{i4} W_{1,0}) \partial_j \Psi_m \right] dx + O(\epsilon_m^2)$$

$$= -2\epsilon_m (\pi_m)_{ij} \int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} x_4 \partial_i W_{1,0} \partial_j \Psi_m dx + O(\epsilon_m^2)$$

$$= 2\epsilon_m (\pi_m)_{ij} \int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} (x_4 \partial_{ij} W_{1,0} \Phi_{m,\delta} + x_4 \partial_{ij} W_{1,0} \Xi_{m,\delta})\, dx + O(\epsilon_m^2).$$

On the other hand, by testing $\Xi_{m,\delta}$ in (4.1) and applying (5.24) once more, we obtain

$$2\epsilon_m (\pi_m)_{ij} \int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} x_4 \partial_{ij} W_{1,0} \Xi_{m,\delta} dx$$

$$= \int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} \nabla \Phi_{m,\delta} \cdot \nabla \Xi_{m,\delta} dx + \int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} |\nabla \Xi_{m,\delta}|^2 dx + O(\epsilon_m^2).$$

Also, testing $\Phi_{m,\delta}$ in (5.22) shows

$$\int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} \nabla \Xi_{m,\delta} \cdot \nabla \Phi_{m,\delta} dx$$

$$= \int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} \frac{9\delta\epsilon_m \pi_{ij} x_i x_j}{(|\bar{x}|^2 + (x_4 + 1)^2)^{\frac{5}{2}}} \Phi_{m,\delta} dx + \int_{B^3(0,\rho\epsilon_m^{-1})} q_{m,\delta} \Phi_{m,\delta} d\bar{x} + O(\epsilon_m^2).$$

Consequently,

$$2\epsilon_m (\pi_m)_{ij} \int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} x_4 \partial_{ij} W_{1,0} \Xi_m dx$$

$$\geq \int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} \frac{9\delta\epsilon_m \pi_{ij} x_i x_j}{(|\bar{x}|^2 + (x_4 + 1)^2)^{\frac{5}{2}}} \Phi_{m,\delta} dx + \int_{B^3(0,\rho\epsilon_m^{-1})} q_{m,\delta} \Phi_{m,\delta} d\bar{x} + O(\epsilon_m^2). \quad (5.26)$$

Combining (5.25) and (5.26), we obtain

$$F_m(W_{1,0}, \Psi_m) + F_m(\Psi_m, W_{1,0}) \geq (\pi_m)_{ij} \left(2\epsilon_m \int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} x_4 \partial_{ij} W_{1,0} \Phi_{m,\delta} dx \right.$$

$$\left. + \int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} \frac{9\delta\epsilon_m \pi_{ij} x_i x_j}{(|\bar{x}|^2 + (x_4 + 1)^2)^{\frac{5}{2}}} \Phi_{m,\delta} dx \right.$$

$$+ \int_{B^3(0,\rho\epsilon_m^{-1})} q_{m,\delta}\Phi_{m,\delta}d\bar{x} \Bigg) + O(\epsilon_m^2). \qquad (5.27)$$

By applying (5.21), we evaluate

$$2\epsilon_m(\pi_m)_{ij} \int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} x_4 \partial_{ij}W_{1,0}\Phi_{m,\delta}dx$$

$$= 16\epsilon_m^2(\pi_m)_{ij}(\pi_m)_{kl} \int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} x_i x_j x_k x_l$$

$$\left[\frac{x_4^2}{(|\bar{x}|^2 + (x_4+1)^2)^5} + \delta \frac{x_4}{(|\bar{x}|^2 + (x_4+1)^2)^{\frac{9}{2}}} \right] dx + O(\epsilon_m^2)$$

$$= \frac{32}{15}|\mathbb{S}^2| \left[\int_0^{\rho\epsilon_m^{-1}} \frac{x_4^2 dx_4}{(x_4+1)^3} \int_0^\infty \frac{t^6 dt}{(t^2+1)^5} \right.$$

$$\left. + \delta \int_0^{\rho\epsilon_m^{-1}} \frac{x_4 dx_4}{(x_4+1)^2} \int_0^\infty \frac{t^6 dt}{(t^2+1)^{\frac{9}{2}}} \right] \epsilon_m^2 \|\pi_m\|^2 + O(\epsilon_m^2)$$

$$= \left(\frac{\pi}{24} + \frac{32}{105}\delta \right) |\mathbb{S}^2| \epsilon_m^2 \log(\rho\epsilon_m^{-1})\|\pi_m\|^2 + O(\epsilon_m^2), \qquad (5.28)$$

$$9\delta\epsilon_m(\pi_m)_{ij} \int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} \frac{x_i x_j}{(|\bar{x}|^2 + (x_4+1)^2)^{\frac{5}{2}}} \Phi_{m,\delta}dx$$

$$= 9\delta\epsilon_m^2(\pi_m)_{ij}(\pi_m)_{kl} \int_0^{\rho\epsilon_m^{-1}} \int_{B^3(0,\rho\epsilon_m^{-1})} \frac{x_i x_j x_k x_l x_4}{(|\bar{x}|^2 + (x_4+1)^2)^{\frac{9}{2}}} dx$$

$$+ O(\delta^2 \log(\rho\epsilon_m^{-1})\|\pi_m\|^2) + O(\epsilon_m^2) \qquad (5.29)$$

$$= \frac{6}{35}\delta |\mathbb{S}^2| \epsilon_m^2 \log(\rho\epsilon_m^{-1})\|\pi_m\|^2 + O(\delta^2 \log(\rho\epsilon_m^{-1})\|\pi_m\|^2) + O(\epsilon_m^2)$$

and

$$\int_{B^3(0,\rho\epsilon_m^{-1})} q_{m,\delta}\Phi_{m,\delta}d\bar{x} = \epsilon_m^2\delta(\pi_m)_{ij}(\pi_m)_{kl} \int_{B^3(0,\rho\epsilon_m^{-1})} \frac{x_i x_j x_k x_l}{(|\bar{x}|^2 + 1)^{\frac{7}{2}}} d\bar{x} + O(\epsilon_m^2)$$

$$= \frac{2}{15}\delta |\mathbb{S}^2| \epsilon_m^2 \log(\rho\epsilon_m^{-1})\|\pi_m\|^2. \qquad (5.30)$$

Putting (5.27)-(5.30), we deduce (5.23). □

Corollary 5.8. *It holds that*

$$F_m(W_{1,0}, W_{1,0}) + [F_m(W_{1,0}, \Psi_m) + F_m(\Psi_m, W_{1,0})]$$

$$\geq \left(\frac{64}{105}\delta + O(\delta^2) \right) |\mathbb{S}^2| \epsilon_m^2 \log(\rho\epsilon_m^{-1})\|\pi_m\|^2 + O(\epsilon_m^2)$$

for δ small.

Proof. The result immediately follows from (5.16) and (5.23). □

Completion of the proof of Proposition 5.1 for $N = 4$. By taking $\delta > 0$ in Corollary 5.8 small enough, we infer from (5.2), (5.4) and (5.5) that

$$O(\epsilon_m^2) \geq \frac{32}{105}\delta |\mathbb{S}^2| \epsilon_m^2 \log(\rho\epsilon_m^{-1})\|\pi_m\|^2 + O(\epsilon_m^2).$$

Accordingly,

$$O\left(\frac{1}{|\log \epsilon_m|} \right) \geq \|\pi_m\|^2 + O\left(\frac{1}{|\log \epsilon_m|} \right).$$

This implies that (5.1) holds for $N = 4$. $\qquad\square$

5.2. **Non-negativity of a sum of the second-order derivatives of the trace-free second fundamental form.** To derive Proposition 5.1, we analyzed the ϵ_m^2-order of the asymptotic expansion of the term $\mathcal{P}_{1m}\left(\widetilde{V}_m, \rho\epsilon_m^{-1}\right)$. We will prove the next result by examining its $\epsilon_m^3|\log\epsilon_m|$-order.

Proposition 5.9. *Suppose that $N = 5$ and $y_m \to y_0 \in \partial M$ is an isolated simple blow-up point of the sequence $\{U_m\}_{m\in\mathbb{N}}$ of the solutions to (2.5). If $\{\tilde{g}_m\}_{m\in\mathbb{N}}$ is a sequence of the metrics constructed in Subsection 3.1, then*

$$\pi[\tilde{g}_0]_{ij,ij}(y_0) \geq 0. \tag{5.31}$$

Proof. Fix any $\rho \in (0, \rho_5]$. By appealing (5.1), one can improve the error in (5.6) so that

$$\mathcal{P}_{1m}\left(\widetilde{V}_m, \rho\epsilon_m^{-1}\right) = F_m(W_{1,0}, W_{1,0}) + [F_m(W_{1,0}, \Psi_m) + F_m(\Psi_m, W_{1,0})] + O(\epsilon_m^3)$$

where F_m is the map defined in (5.7). From this, (5.2), (5.4) and (5.5), we deduce

$$O(\epsilon_m^3) \geq F_m(W_{1,0}, W_{1,0}) + [F_m(W_{1,0}, \Psi_m) + F_m(\Psi_m, W_{1,0})] + O(\epsilon_m^3). \tag{5.32}$$

Moreover, arguing as in the proof of Lemma 5.6, we see

$$F_m(W_{1,0}, W_{1,0}) + \frac{1}{64}\left|\mathbb{S}^3\right|\epsilon_m^2\|\pi_m\|^2$$

$$= -\epsilon_m^3(\pi_m)_{ij,kl}\int_{B_+^5(0,\rho\epsilon_m^{-1})} x_5 x_k x_l \partial_{ij} W_{1,0} Z_{1,0}^0 dx$$

$$\quad - 2\epsilon_m^3(\pi_m)_{ij,ik}\int_{B_+^5(0,\rho\epsilon_m^{-1})} x_5 x_k \partial_j W_{1,0} Z_{1,0}^0 dx + O(\epsilon_m^3)$$

$$= -\frac{15}{8}\epsilon_m^3(\pi_m)_{ij,ij}\int_0^{\rho\epsilon_m^{-1}}\int_{\mathbb{R}^4} \frac{x_5|\bar{x}|^4(1 - |\bar{x}|^2 - x_5^2)}{(|\bar{x}|^2 + (x_5 + 1)^2)^6} dx$$

$$\quad + \frac{9}{4}\epsilon_m^3(\pi_m)_{ij,ij}\int_0^{\rho\epsilon_m^{-1}}\int_{\mathbb{R}^4} \frac{x_5|\bar{x}|^2(1 - |\bar{x}|^2 - x_5^2)}{(|\bar{x}|^2 + (x_5 + 1)^2)^5} dx + O(\epsilon_m^3)$$

$$= -\frac{9}{64}\left|\mathbb{S}^3\right|\epsilon_m^3\log(\rho\epsilon_m^{-1})(\pi_m)_{ij,ij} + O(\epsilon_m^3).$$

To deduce each equality, we took $\kappa \geq 4$ in Lemma 2.2 so that $R_{NN,N}[\tilde{g}_m](y_m) = 0$, and used Lemma 2.1, the symmetry of the integral and (5.1). After setting $a_1 = -\frac{63}{4}$ and $a_2 = \frac{105}{8}$ as in the proof of Corollary 5.3 and applying (5.1) once more, we arrive at

$$F_m(W_{1,0}, W_{1,0}) + [F_m(W_{1,0}, \Psi_m) + F_m(\Psi_m, W_{1,0})]$$

$$\geq \frac{3}{2560}\left|\mathbb{S}^3\right|\epsilon_m^2\|\pi_m\|^2 - \frac{9}{64}\left|\mathbb{S}^3\right|\epsilon_m^3\log(\rho\epsilon_m^{-1})(\pi_m)_{ij,ij} + O(\epsilon_m^3) + O(\epsilon_m^3|\log\epsilon_m|\cdot\|\pi_m\|)$$

$$\geq -\frac{9}{64}\left|\mathbb{S}^3\right|\epsilon_m^3\log(\rho\epsilon_m^{-1})(\pi_m)_{ij,ij} + O(\epsilon_m^3) + O\left(\epsilon_m^{\frac{7}{2}}|\log\epsilon_m|^{\frac{3}{2}}\right).$$

Inserting this estimate to (5.32), we obtain

$$O\left(\frac{1}{|\log\epsilon_m|}\right) \geq -\frac{9}{64}\left|\mathbb{S}^3\right|(\pi_m)_{ij,ij} + O\left(\frac{1}{|\log\epsilon_m|}\right) + O\left(\epsilon_m^{\frac{1}{2}}|\log\epsilon_m|^{\frac{3}{2}}\right).$$

Taking $m \to \infty$ on the both sides, we obtain (5.31). $\qquad\square$

6. Local sign restriction and set of blow-up points

Under the validity of Proposition 5.1, we derive the local sign restriction of the function $\mathcal{P}'$.

Proposition 6.1. *Assume that $N \geq 4$ and $y_m \to y_0 \in \partial M$ is an isolated simple blow-up point for the sequence $\{U_m\}_{m \in \mathbb{N}}$ to the solutions to (2.5). Then, given $m \in \mathbb{N}$ large and $\rho > 0$ small, there exist constants $\mathcal{C}_0 \geq 0$ and $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 > 0$ independent of m and ρ such that*

$$\epsilon_m^{N-2+o(1)} \mathcal{P}' \left(\widetilde{U}_m(0) \widetilde{U}_m, \rho \right) \geq \epsilon_m^2 \mathcal{C}_0 - \epsilon_m^{2+\eta} \rho^{2-\eta} \mathcal{C}_1 - \epsilon_m^{N-2} \rho^{-N+3} \mathcal{C}_2 - \frac{\epsilon_m^{N-1} \rho^{N-1} \mathcal{C}_3}{\epsilon_m^{2(N-1)+o(1)} + \rho^{2(N-1)+o(1)}}$$

for $N \geq 5$ and

$$\epsilon_m^{2+o(1)} \mathcal{P}' \left(\widetilde{U}_m(0) \widetilde{U}_m, \rho \right) \geq \epsilon_m^2 \log(1 + \rho \epsilon_m^{-1}) \mathcal{C}_0 - \epsilon_m^2 \mathcal{C}_1 - \frac{\epsilon_m^3 \rho^3 \mathcal{C}_2}{\epsilon_m^{6+o(1)} + \rho^{6+o(1)}}$$

for $N = 4$, in $\tilde{g}_m$-Fermi coordinates centered in y_m. Here, $\mathcal{P}'$ is the function defined in (2.7), $\eta > 0$ is an arbitrarily small number and $\epsilon_m^{o(1)} \to 1$ as $m \to \infty$.

Proof. If $N \geq 5$, the proof follows the same lines as that of Lemma 6.1 in [33]; cf. Theorem 7.2 of [2]. Slightly modifying the argument, one can also establish the inequality for $N = 4$. Here we allow the possibility that $\pi[\tilde{g}_0](y_0) = 0$ as opposed to [33]. Thus we cannot exclude that $\mathcal{C}_0 = 0$. $\square$

From the previous proposition, we conclude the following results. It can be derived as in Section 6 of [33].

Lemma 6.2. *Assume that $N \geq 4$, and $y_0 \in \partial M$ is an isolated blow-up point for the sequence $\{U_m\}_{m \in \mathbb{N}}$ to (2.5). Then it is an isolated simple blow-up point of $\{U_m\}_{m \in \mathbb{N}}$.*

Proposition 6.3. *Assume the hypotheses of Theorem 1.1. Let $\varepsilon_0, \varepsilon_1, R, C_0$ and C_1 be positive numbers in the statement of Proposition 3.2. Suppose that $U \in H^1(M)$ is a solution to (2.5) and $\{y_1, \cdots, y_{\mathcal{N}}\}$ is the set of its local maxima on ∂M. Then there exists a constant $C_2 > 0$ depending only on (M, g), N, ε_0, ε_1 and R such that if $\max_{\partial M} U \geq C_0$, then $d_h(y_{m_1}, y_{m_2}) \geq C_2$ for all $1 \leq m_1 \neq m_2 \leq \mathcal{N}(U)$. In particular, the set of blow-up points of $\{U_m\}_{m \in \mathbb{N}}$ is finite and it consists of isolated simple blow-up points.*

7. The compactness result

Let G_{y_0} be the normalized Green's function of the conformal Laplacian on $(M, \tilde{g}_0)$ with Neumann boundary condition with pole at $y_0 \in \partial M$, that is, the solution of

$$\begin{cases} L_{\tilde{g}_0} G_{y_0}(y) = 0 & \text{in } M, \\ B_{\tilde{g}_0} G_{y_0}(y) = \delta_{y_0} & \text{on } \partial M, \\ \lim_{d_{\tilde{g}_0}(y, y_0) \to 0} d_{\tilde{g}_0}(y, y_0)^{N-2} G_{y_0}(y) = 1 \end{cases} \tag{7.1}$$

where δ_{y_0} is the Dirac measure centered at y_0. It will serve as the function G when we apply the positive mass theorem (described in Lemma 2.4).

In the following two lemmas, we verify the necessary conditions to apply Lemma 2.4 for 4- and 5-manifolds. Note that the number d in (2.10) is 1.

Lemma 7.1. *Suppose that $N = 4$ or 5, $y_0 \in \partial M$ is an isolated simple blow-up point of the sequence $\{U_m\}_{m \in \mathbb{N}}$ of the solutions to (2.5). If we take $\kappa \geq 4$ in (2.1), we can expand the metric $g = \tilde{g}_0$ as in (2.10) and (2.11).*

Proof. By Lemma 2.1 and Proposition 5.1, it clearly holds that

$$A_{iN}(x) = A_{NN}(x) = 0 \quad \text{and} \quad A_{ij}(x) = O(|x|^2).$$

Therefore,

$$\exp A(x) = I + A(x) + O(|x|^4) \quad \text{and so} \quad g(x) = \exp A(x) + O(|x|^4)$$

where I is the $N \times N$-identity matrix. From this, we see that

$$\det g(x) = e^{\operatorname{trace}A(x)+O(|x|^4)} = 1 + \operatorname{trace}(A(x)) + O(|x|^4).$$

By virtue of our choice $\kappa \geq 4$, it follows that $\operatorname{trace}(A(x)) = O(|x|^4)$ as desired. $\square$

Lemma 7.2. *Suppose that $N = 4$ or 5, and $y_0 \in \partial M$ is an isolated simple blow-up point of the sequence $\{U_m\}_{m \in \mathbb{N}}$ of the solutions to (2.5). If we choose the integer κ in (2.1) large enough, we obtain that*

$$G_{y_0}(x) = \begin{cases} |x|^{-2} + O(|\log|x||) & \text{for } N = 4, \\ |x|^{-3} + O(|x|^{-1}|\log|x||) & \text{for } N = 5 \end{cases}$$

in $\tilde{g}_0$-Fermi coordinates centered at y_0. As a particular consequence, G_{y_0} is a smooth positive function on $M \setminus \{y_0\}$ which can be expressed as in (2.12)-(2.13).

Proof. We will employ Proposition B.2 of [6], in which Almaraz and Sun constructed the Green's function on manifolds with boundary using parametrices.

According to their result, if there exists a sufficiently large integer κ_0 such that

$$|H[\tilde{g}_0](y)| \leq C d_{\tilde{g}_0}(y, y_0)^{\kappa_0} \quad \text{for all } y \in \partial M, \tag{7.2}$$

then one can find a smooth positive solution G_{y_0} on $M \setminus \{y_0\}$ to (7.1) with $g = \tilde{g}_0$. Moreover, if $\tilde{g}_0 = \exp B$ for some 2-tensor B on M, then

$$\left| G_{y_0}(x) - |x|^{2-N} \right|$$
$$\leq C \sum_{i,j=1}^{n} \sum_{|\alpha|=1}^{d} |B_{ij,\alpha}(0)| \, |x|^{|\alpha|+2-N} + \begin{cases} C(1+|\log|x||) & \text{for } N = 3, 4, \\ C|x|^{d+3-N} & \text{for } N \geq 5 \end{cases} \tag{7.3}$$

in $\tilde{g}_0$-Fermi coordinates centered at y_0, where $d = \lfloor \frac{N-2}{2} \rfloor$ as before. Check also (1.3) for the notations involving multi-indices.

Differentiating (3.4) of [22] $|\beta|$-times, we obtain

$$\frac{\partial}{\partial x_N} \frac{\partial \sqrt{|\tilde{g}_0|}}{\partial x_\beta}(\bar{x}, 0) = -n \sum_{\beta'+\beta''=\beta} \frac{\beta!}{\beta'!\beta''!} \left(\frac{\partial \sqrt{|\tilde{h}_0|}}{\partial x_{\beta'}} \frac{\partial H[\tilde{g}_0]}{\partial x_{\beta''}} \right)(\bar{x}) \quad \text{for } \bar{x} \in \mathbb{R}^n, \tag{7.4}$$

in normal coordinates on ∂M centered at y_0. Here $\tilde{h}_0$ is the restriction of $\tilde{g}_0$ to ∂M. In light of (2.1) and (7.4), the coefficient of $x_\beta x_N$ in the Taylor expansion of $\sqrt{|\tilde{g}_0|}$ at $x = 0$ has to be

$$-\frac{n}{(|\beta|+1)!} \frac{\partial^\beta H[\tilde{g}_0]}{\partial x_\beta}(0) = 0 \quad \text{for all } |\beta| \leq \kappa - 1.$$

Thus, if we take $\kappa \geq \kappa_0$, all partial derivatives of H of order $\leq \kappa_0 - 1$ must vanish at 0, and so (7.2) holds.

On the other hand, we know that $A(x) = O(|x|^2)$ and $\tilde{g}_0 = \exp A + O(|x|^4)$ from the proof of Lemma 7.1. Therefore, for $|\alpha| = 1$,

$$B_{ij,\alpha}(x) = A_{ij,\alpha}(x) + O(|x|) = O(|x|), \quad \text{and so } B_{ij,\alpha}(0) = 0.$$

This implies that the right-hand side of (7.3) is bounded by

$$\begin{cases} C(|x|^{4-N} + 1 + |\log|x||) = C(1 + |\log|x||) = O(|\log|x||) & \text{for } N = 4, \\ C|x|^{4-N} = C|x|^{-1} = O(|x|^{-1}|\log|x||) & \text{for } N = 5. \end{cases}$$

The proof is finished. $\square$

We next examine the relationship between the flux integral $\mathcal{I}(y_0, \rho)$ given in (2.15) and the quantity $\mathcal{P}'(G_{y_0}, \rho)$ defined by (2.7).

Lemma 7.3. *Under the assumptions of Lemma 7.1, it holds that*

$$\mathcal{P}'(G_{y_0}, \rho) = \begin{cases} -\dfrac{1}{6}\mathcal{I}(y_0, \rho) + O(\rho) & \text{for } N = 4, \\ -\dfrac{9}{32}\mathcal{I}(y_0, \rho) - \dfrac{3}{512}\left|\mathbb{S}^3\right|\pi[\tilde{g}_0]_{ij,ij}(y_0) + O(\rho|\log\rho|) & \text{for } N = 5. \end{cases} \tag{7.5}$$

Proof. Lemma 3.2 of [5] leads us that

$$\mathcal{P}'(G_{y_0}, \rho) = -\left(\frac{N-2}{2}\right)\int_{\partial_I B_+^N(0,\rho)} \left(|x|^{2-N}\partial_a G_{y_0}(x) - \partial_a|x|^{2-N}G_{y_0}(x)\right)\frac{x_a}{|x|}\,dS_x \tag{7.6}$$
$$+ O(\rho^{6-N}|\log\rho|).$$

Therefore, we infer from (2.15) that

$$\mathcal{P}'(G_{y_0}, \rho) = -\frac{(N-2)^2}{8(N-1)}\left[\mathcal{I}(y_0, \rho)\right.$$

$$+ \int_{\partial_I B_+^N(0,\rho)} \left(\rho^{3-2N}x_a\partial_b A_{ab}(x) - 2N\rho^{1-2N}x_a x_b A_{ab}(x)\right)dS_x\bigg] + O(\rho^{6-N}|\log\rho|). \tag{7.7}$$

On the other hand, by setting $\kappa \geq 4$ in Lemma 2.2 and applying Proposition 5.1, we obtain

$$\text{Sym}_{klm}R_{ikjl,m}[\tilde{h}_0] = \text{Sym}_{kl}H_{,kl}[\tilde{g}_0] = R_{NN,k}[\tilde{g}_0] = R_{NN,N}[\tilde{g}_0] = 0 \quad \text{at } y_0 \tag{7.8}$$

where $\tilde{h}_0 = \tilde{g}_0|_{T\partial M}$. Thanks to (2.11), (2.2), (7.8), the Ricci identity and the symmetry of the integral, we find

$$\int_{\partial_I B_+^N(0,\rho)} \left(\rho^{3-2N}x_i\partial_j A_{ij}(x) - 2N\rho^{1-2N}x_i x_j A_{ij}(x)\right)dS_x$$

$$= \rho^{5-N}|\mathbb{S}^{n-1}|\left[\frac{2(N-3)}{(N-1)(N+1)(N+3)}\right]\pi[\tilde{g}_0]_{ij,ij}(y_0) + O(\rho^{6-N}) \tag{7.9}$$

for any $N \geq 4$.

Combining (7.7) and (7.9), we derive (7.5). $\qquad\square$

The above lemma shows that Proposition 3.6 of [5] is valid for $N = 4$, but is not in general for $N = 5$.

As a by-product of the previous lemma, we can evaluate the mass m_0.

Corollary 7.4. *Under the assumptions of Lemma 7.1, it holds that*

$$m_0 = \begin{cases} -6\lim_{\rho\to 0}\mathcal{P}'(G_{y_0}, \rho) & \text{for } N = 4, \\ -\dfrac{32}{9}\lim_{\rho\to 0}\mathcal{P}'(G_{y_0}, \rho) - \dfrac{1}{48}\left|\mathbb{S}^3\right|\pi[\tilde{g}_0]_{ij,ij}(y_0) & \text{for } N = 5. \end{cases}$$

Proof. Taking $\rho \to 0$ on the both sides of (7.5) and using (2.14), we get the result. $\qquad\square$

One can see from (2.7) or (7.6) that the value of $\mathcal{P}'(G_{y_0}, \rho)$ is completely determined by the Green's function G_{y_0}. Therefore, the above corollary tells us that the mass is involved with not only the Green's function but also the trace-free second fundamental form if $N = 5$. As mentioned in Remark 1.3, it is a unique property of manifolds with boundary.

We are now ready to complete the proof of our main result.

Proof of Theorem 1.1. Suppose that $y_0 \in \partial M$ is a blow-up point of of the sequence $\{U_m\}_{m \in \mathbb{N}}$ of the solutions to (2.5). By Proposition 6.3, it is isolated simple. By Proposition 3.5 and elliptic regularity theory, there also exists a constant $a > 0$ such that

$$U_m(y_m) U_m \to a G_{y_0} \quad \text{in } C^2(B_{\tilde{g}_0}(y_0, \rho) \setminus \{y_0\}) \quad \text{as } m \to \infty.$$

Thanks to Proposition 6.1, it follows that

$$\liminf_{\rho \to 0} \mathcal{P}'(a G_{y_0}, \rho) = a^2 \liminf_{\rho \to 0} \mathcal{P}'(G_{y_0}, \rho) \geq 0. \tag{7.10}$$

We split the proof into two cases according to the dimension of the manifold M.

CASE $N = 4$ AND 5: By virtue of Lemmas 7.1 and 7.2, all the conditions needed to apply Lemma 2.4 hold. Besides, (7.1) yields that $R\big[G_{y_0}^{\frac{4}{N-2}} g\big] = 0$ and $H\big[G_{y_0}^{\frac{4}{N-2}} g\big] = 0$ on their respective domains, which trivially implies (2.16). Employing Lemma 2.4, Corollary 7.4, Proposition 5.9 and (7.10), we deduce

$$0 < m_0 = \begin{cases} -6 \lim_{\rho \to 0} \mathcal{P}'(G_{y_0}, \rho) \leq 0 & \text{for } N = 4, \\ -\dfrac{32}{9} \lim_{\rho \to 0} \mathcal{P}'(G_{y_0}, \rho) - \dfrac{1}{48} \big|\mathbb{S}^3\big| \, \pi[\tilde{g}_0]_{ij,ij}(y_0) \leq 0 & \text{for } N = 5, \end{cases}$$

a contradiction. Consequently, there is no blow-up point of a solution to (2.5), which means that its solution set is $L^\infty(M)$-bounded. Elliptic regularity tells us that it is $C^2(M)$-compact. Theorem 1.1 must be true in this case.

CASE $N = 6$: We remind that the trace-free second fundamental form $\pi[g]$ is assumed to be never zero on ∂M. There is a positive smooth function ω_0 on M such that $\tilde{g}_0 = \omega_0 g$ on M. In Proposition 1.2 of [23], it was proved that $\pi[\tilde{g}_0] = \sqrt{\omega}\, \pi[g]$ on ∂M. This produce a contradiction, since Proposition 5.1 reads

$$0 = \|\pi[\tilde{g}_0](y_0)\| = \omega(y_0)^{-\frac{1}{2}} \|\pi[g](y_0)\| > 0.$$

The same reasoning as above shows that Theorem 1.1 is also valid in this case. $\square$

APPENDIX A. PROOF OF LEMMAS 4.3 AND 4.5

Throughout this section, we assume that $N \geq 5$. The case $N = 4$ can be handled similarly.

In order to prove Lemma 4.3, we first need two preliminary observations.

Lemma A.1. *Suppose that $N \geq 5$. The function*

$$\Phi_1(x) = \frac{1}{4(N-4)} \frac{x_N + 1}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N-4}{2}}} + a_1 \frac{x_N + 1}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N}{2}}} \quad \text{in } \mathbb{R}_+^N$$

for $a_1 \in \mathbb{R}$ satisfies

$$-\Delta \Phi_1 = \frac{x_N + 1}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N-2}{2}}} \quad \text{in } \mathbb{R}_+^N.$$

Proof. It holds that

$$\frac{x_N + 1}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N-2}{2}}} = -\left(\frac{1}{N-4}\right) \partial_N \left[\frac{1}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N-4}{2}}}\right] \quad \text{in } \mathbb{R}_+^N.$$

Thus, if we have a solution Φ_0 of the equation

$$-\Delta \Phi_0 = \frac{1}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N-4}{2}}} \quad \text{in } \mathbb{R}_+^N,$$

we will be able to choose

$$\Phi_1 = -\left(\frac{1}{N-4}\right) \partial_N \Phi_0. \tag{A.1}$$

On the other hand, we see that

$$-\Delta[\Phi_0(\bar{x}, x_N - 1)] = \frac{1}{(|\bar{x}|^2 + x_N^2)^{\frac{N-4}{2}}} = \frac{1}{|x|^{N-4}} \quad \text{in } \mathbb{R}^n \times (1, \infty).$$

If we assume that $\Phi_0(\bar{x}, x_N - 1)$ is radial symmetric, i.e., $\phi_0(|x|) = \Phi_0(\bar{x}, x_N - 1)$, then it is reduced to

$$-\phi_0'' - \frac{N-1}{r}\phi_0' = \frac{1}{r^{N-4}} \quad \text{in } (0, \infty).$$

Its general solution is expressed as

$$\phi_0(r) = \begin{cases} \dfrac{1}{4(N-6)}\dfrac{1}{r^{N-6}} + \dfrac{a_1}{r^{N-2}} + a_1' & \text{for } N = 5 \text{ or } N \geq 7, \\ -\dfrac{\log r}{4} + \dfrac{a_1}{r^4} + a_1' & \text{for } N = 6 \end{cases}$$

for $r \in (0, \infty)$ and $a_1, a_1' \in \mathbb{R}$. Consequently,

$$\Phi_0(x) = \begin{cases} \dfrac{1}{4(N-6)}\dfrac{1}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N-6}{2}}} + \dfrac{a_1}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N-2}{2}}} + a_2 \\ \hspace{8cm} \text{for } N = 5 \text{ or } N \geq 7, \\ -\dfrac{1}{8}\log(|\bar{x}|^2 + (x_N + 1)^2) + \dfrac{a_1}{(|\bar{x}|^2 + (x_N + 1)^2)^2} + a_1' \\ \hspace{8cm} \text{for } N = 6 \end{cases} \tag{A.2}$$

in $\mathbb{R}_+^N$.

By (A.1) and (A.2), the assertion in the statement holds. $\qquad\square$

Lemma A.2. *The function*

$$\Phi_2(x) = \frac{1}{2(N-4)}\frac{1}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N-4}{2}}} + a_2\frac{1}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N-2}{2}}} + a_2' \quad in \ \mathbb{R}_+^N$$

for $a_2, a_2' \in \mathbb{R}$ satisfies

$$-\Delta\Phi_2 = \frac{1}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N-2}{2}}} \quad in \ \mathbb{R}_+^N. \tag{A.3}$$

Proof. Equation (A.3) is equivalent to

$$-\Delta[\Phi_2(\bar{x}, x_N - 1)] = \frac{1}{(|\bar{x}|^2 + x_N^2)^{\frac{N-2}{2}}} = \frac{1}{|x|^{N-2}} \quad \text{in } \mathbb{R}_+^N.$$

If we assume that $\Phi_2(\bar{x}, x_N - 1)$ is radial symmetric, i.e., $\phi_2(|x|) = \Phi_2(\bar{x}, x_N - 1)$, then it is reduced to

$$-\phi_2'' - \frac{n}{r}\phi_2' = \frac{1}{r^{N-2}} \quad \text{in } (0, \infty).$$

The general solution is expressed as

$$\phi_2(r) = \frac{1}{2(N-4)r^{N-4}} + \frac{a_2}{r^{N-2}} + a_2'$$

for $r \in (0, \infty)$ and $a_2, a_2' \in \mathbb{R}$. As a result, the assertion in the statement holds. $\qquad\square$

Corollary A.3. *The function*

$$(\Phi_1 - \Phi_2)(x) = \frac{1}{4(N-4)}\frac{x_N - 1}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N-4}{2}}} + a_1\frac{x_N + 1}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N}{2}}}$$
$$+ a_2\frac{1}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N-2}{2}}} + a_2' \qquad in \ \mathbb{R}_+^N$$

for a_1, a_2, $a_2' \in \mathbb{R}$ satisfies

$$-\Delta(\Phi_1 - \Phi_2) = \frac{x_N}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N-2}{2}}} = x_N W_{1,0} \quad in \ \mathbb{R}_+^N.$$

Proof. It is a direct consequence of Lemmas A.1 and A.2. □

Completion of the proof of Lemma 4.3. Define $\widetilde{\Phi}_{ij} = \partial_{ij}(\Phi_1 - \Phi_2)$ so that $\Phi = 2\pi_{ij}\widetilde{\Phi}_{ij}$. By Corollary A.3,

$$\widetilde{\Phi}_{ij}(x) = -\frac{x_N - 1}{4}\left[\frac{\delta_{ij}}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N-2}{2}}} - (N-2)\frac{x_i x_j}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N}{2}}}\right]$$

$$+ a_1(x_N + 1)\left[\frac{\delta_{ij}}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N+2}{2}}} - (N+2)\frac{x_i x_j}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N+4}{2}}}\right] \quad in \ \mathbb{R}_+^N.$$

$$+ a_2\left[\frac{\delta_{ij}}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N}{2}}} - N\frac{x_i x_j}{(|\bar{x}|^2 + (x_N + 1)^2)^{\frac{N+2}{2}}}\right]$$

Since the trace of π is assumed to be 0, we have (4.5). This completes the proof. □

Completion of the proof of Lemma 4.5. It follows from (4.1) and (4.6) that U is harmonic in $\mathbb{R}_+^N$. Note also that

$$\lim_{x_N \to 0} \frac{\partial U}{\partial x_N} + N w_{1,0}^{\frac{2}{N-2}} U = -\lim_{x_N \to 0} \frac{\partial \Phi}{\partial x_N} - N w_{1,0}^{\frac{2}{N-2}} \Phi \quad \text{on } \mathbb{R}^n.$$

Plugging (4.5) into the right-hand side, we find the boundary condition that U satisfies. □

References

[1] S. Almaraz, An existence theorem of conformal scalar-flat metrics on manifolds with boundary, Pacific J. Math. **248** (2010), 1–22.

[2] ———, A compactness theorem for scalar-flat metrics on manifolds with boundary, Calc. Var. Partial Differential Equations **41** (2011), 341–386.

[3] ———, Blow-up phenomena for scalar-flat metrics on manifolds with boundary, J. Differential Equations **251** (2011), 1813–1840.

[4] S. Almaraz, E. Barbosa, L. L. de Lima, A positive mass theorem for asymptotically flat manifolds with a non-compact boundary, Comm. Anal. Geom. **24** (2016), 673–715.

[5] S. Almaraz, O. S. de Queiroz, S. Wang, A compactness theorem for scalar-flat metrics on 3-manifolds with boundary, J. Funct. Anal. in press.

[6] S. Almaraz, L. Sun, Convergence of the Yamabe flow on manifolds with minimal boundary, Ann. Sc. Norm. Super. Pisa Cl. Sci., in press.

[7] T. Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl. **55** (1976), 269–296.

[8] S. Brendle, Blow-up phenomena for the Yamabe equation, J. Amer. Math. Soc. **21** (2008), 951–979.

[9] S. Brendle, S. Chen, An existence theorem for the Yamabe problem on manifolds with boundary, J. Eur. Math. Soc. (JEMS) **16** (2014), 991–1016.

[10] S. Brendle, F. Marques, Blow-up phenomena for the Yamabe equation II, J. Differential Geom. **81** (2009), 225–250.

[11] E. Cárdenas, W. Sierra, Uniqueness of solutions of the Yamabe problem on manifolds with boundary, Nonlinear Anal. **187** (2019), 125–133.

[12] S. S. Chen, Conformal deformation to scalar flat metrics with constant mean curvature on the boundary in higher dimensions, preprint, arXiv:0912.1302.

[13] P. Cherrier, Problèmes de Neumann non linéaires sur les variétés Riemanniennes, J. Funct. Anal. **57** (1984) 154–206.

[14] J. Dávila, M. del Pino, Y. Sire, Nondegeneracy of the bubble in the critical case for nonlocal equations, Proc. Amer. Math. Soc. **141** (2013), 3865–3870.

[15] S. Deng, S. Kim, A. Pistoia, Linear perturbations of the fractional Yamabe problem on the minimal conformal infinity, to appear in Comm. Anal. Geom.

[16] M. M. Disconzi, M. A. Khuri, Compactness and non-compactness for the Yamabe problem on manifolds with boundary, J. Reine Angew. Math. **724** (2017), 145–201.

[17] Z. Djadli, A. Malchiodi, M. Ould Ahmedou, Prescribing scalar and boundary mean curvature on the three dimensional half sphere. J. Geom. Anal. **13** (2003), 255-289.

[18] _____, The prescribed boundary mean curvature problem on $\mathbb{B}^4$. J. Differential Equations **206** (2004), 373–398.

[19] O. Druet, Compactness for Yamabe metrics in low dimensions, Int. Math. Res. Not. **23** (2004), 1143–1191.

[20] J. F. Escobar, Sharp constant in a Sobolev trace inequality, Indiana Univ. Math. J. **37** (1988), 687–698.

[21] _____, Uniqueness theorems on conformal deformation of metrics, Sobolev inequalities, and an eigenvalue estimate, Comm. Pure Appl. Math. **43** (1990), 857–883.

[22] _____, Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary, Ann. of Math. **136** (1992), 1–50.

[23] _____, The Yamabe problem on manifolds with boundary, J. Differential Geom. **35** (1992), 21–84.

[24] _____, Conformal metrics with prescribed mean curvature on the boundary, Calc. Var. Partial Differential Equations **4** (1996), 559–592.

[25] V. Felli, A. Ould Ahmedou, Compactness results in conformal deformations of Riemannian metrics on manifolds with boundaries, Math. Z. **244** (2003), 175–210.

[26] _____, A geometric equation with critical nonlinearity on the boundary, Pacific J. Math. **218** (2005), 75–99.

[27] M. Ghimenti, A. M. Micheletti, A compactness result for scalar-flat metrics on manifolds with umbilic boundary, preprint, arXiv:1903.10990.

[28] M. Ghimenti, A. M. Micheletti, A. Pistoia Linear perturbation of the Yamabe problem on manifolds with boundary, J. Geom. Anal. **28** (2018), 1315–1340.

[29] _____, Blow-up phenomena for linearly perturbed Yamabe problem on manifolds with umbilic boundary, J. Differential Equations **267** (2019), 587–618.

[30] Z.-C. Han, Y. Y. Li, The Yamabe problem on manifolds with boundary: existence and compactness results, Duke Math. J. **99** (1999), 485–542.

[31] M. Khuri, F. Marques, R. Schoen, A compactness theorem for the Yamabe problem, J. Differential Geom. **81** (2009), 143–196.

[32] S. Kim, M. Musso, and J. Wei, Existence theorems of the fractional Yamabe problem, Anal. PDE **11** (2018), 75–113.

[33] _____, A compactness theorem of the fractional Yamabe problem, Part I: The non-umbilic conformal infinity, preprint, arXiv:1808.04951.

[34] M. Mayer, C. B. Ndiaye, Barycenter technique and the Riemann mapping problem of Cherrier-Escobar, J. Differential Geom. **107** (2017), 519–560.

[35] G. Li, A compactness theorem on Branson's Q-curvature equation, preprint, arXiv:1505.07692.

[36] Y. Y. Li, J. Xiong, Compactness of conformal metrics with constant Q-curvature. I, Adv. Math. **345** (2019), 116–160.

[37] Y. Y. Li, L. Zhang, Compactness of solutions to the Yamabe problem II, Calc. Var. and Partial Differential Equations **25** (2005), 185–237.

[38] _____, Compactness of solutions to the Yamabe problem III, J. Funct. Anal. **245** (2006), 438–474.

[39] Y. Li, M. Zhu, Uniqueness theorems through the method of moving spheres, Duke Math. J. **80** (1995), 383–417.

[40] _____, Yamabe type equations on three dimensional Riemannian manifolds, Comm. Contemp. Math. **1** (1999), 1–50.

[41] F. C. Marques, A priori estimates for the Yamabe problem in the non-locally conformally flat case, J. Differential Geom. **71** (2005), 315–346.

[42] _____, Existence results for the Yamabe problem on manifolds with boundary, Indiana Univ. Math. J. **54** (2005), 1599–1620.

[43] _____, Conformal deformations to scalar-flat metris with constant mean curvature on the boundary, Comm. Anal. Geom. **15** (2007), 381–405.

[44] R. Schoen, Course notes on 'Topics in differential geometry' at Stanford University, (1988), available at `https://www.math.washington.edu/~pollack/research/Schoen-1988-notes.html`

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