BOUNDARY CONCENTRATIONS ON SEGMENTS

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Abstract. We consider the following singularly perturbed Neumann problem

$$\varepsilon^2\Delta u - u + u^p = 0\ , \quad u > 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega,$$

where $p > 2$ and Ω is a smooth and bounded domain in $\mathbb{R}^2$. We construct a new class of solutions which consist of large number of spikes concentrating on a **segment** of the boundary which contains a strict local minimum point of the mean curvature function and has the same mean curvature at the two end points. We find a continuum limit of ODE systems governing the interactions of spikes and show that the derivative of mean curvature function acts as *friction force*. Our construction is partly motivated by the construction of CMC surfaces on broken geodesics by Butscher and Mazzeo [10].

1. Introduction and statement of main results

1.1. **Introduction and Main Results.** In this paper, we establish **new** concentration phenomena for the following singularly perturbed elliptic problem

$$\begin{cases} 0\varepsilon^2\Delta u - u + u^p = 0 \quad \text{in} \quad \Omega, \\ \quad u > 0 \quad \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ with its unit outer normal ν, and the exponent p is greater than 2, and $\varepsilon > 0$ is a small parameter. We prove the existence of solutions concentrating on segment of $\partial\Omega$.

This equation is known as the stationary equation of the nonlinear Schrödinger equation:

$$i\tilde{h}\frac{\partial\psi}{\partial t} = -\frac{\tilde{h}^2}{2m}\Delta\psi + V\psi - \tilde{\gamma}|\psi|^{p-2}\psi \tag{1.2}$$

where $\tilde{h}$ is the Plank constant, V is the potential and $\tilde{\gamma}, m$ are positive constants. Then standing waves of (1.2) can be found by setting $\psi = e^{iEt/\tilde{h}}v(x)$ where E is a constant and the real function v satisfies the elliptic equation:

$$-\tilde{h}^2\Delta v + \tilde{V}v = |v|^{p-2}v \tag{1.3}$$

for some modified potential $\tilde{V}$. If we consider $\tilde{h} \to 0$, the above equation becomes a singularly perturbed one.

It can also be viewed as a stationary equation of Keller-Segel system in chemotaxis ([26]) and the Gierer-Meinhardt biological pattern formation system ([20]).

Although problem (1.1) is simple-looking, it has a rich and interesting structure of solutions. For the last twenty years, it has received considerable attention. In particular, the various concentration phenomena exhibited by the solutions of (1.1)

seem both mathematically intriguing and scientifically useful. We refer to three survey articles [36], [37] and [42] for more backgrounds and references.

In the subcritical case, problem (1.1) admits spike layer solutions, concentrating at one or multiple points of $\bar{\Omega}$. It was first established in [38],[39] by Ni and Takagi the existence of least energy (mountain pass) solutions to (1.1), that is, a solution u_ϵ with minimal energy. They showed in [38, 39] that, *for each $\epsilon > 0$ sufficiently small, u_ϵ has a spike at the most curved part of the boundary, i.e., the region where the mean curvature attains maximum value.*

Since the publication of [39], further studies on spike-layer solutions (for the Dirichlet problem and mixed boundary problem as well) have been made. For spike solutions, solutions with multiple boundary spikes as well as multiple interior spikes and mixed interior and boundary spikes have been established . (See [4], [6], [7], [14]-[17], [21]-[24], [27]-[28], [40], [41], [43]-[44] and the references therein.) Thanks to these works, the phenomenon of concentration at points is now well-understood. Necessary and sufficient conditions for the location of boundary and interior spikes are available.

In particular, concerning the interior spike layer solutions, Lin, Ni and Wei [29] showed that there are at least $\frac{C_N}{(\varepsilon|\log\varepsilon|)^N}$ number of interior spikes and recently the first author and the third author and Zeng [5] extended their result and obtained the optimal bound of number of interior spikes $\frac{C_N}{\varepsilon^N}$ for general smooth domain in $\mathbb{R}^N$.

A general principle is that for interior spike solutions, the distance function from the boundary $\partial\Omega$ plays an important role, while for the boundary spike solutions, the mean curvature function of the boundary plays an important role.

Besides the spike-layer solutions, it has been *conjectured* for a long time that *problem (1.1) should possess solutions which have $m-$dimensional concentration sets for every* $0 \le m \le N-1$. (See e.g. [36].) The case of $m = N$ is excluded since (1.1) is not expected to exhibit phase transitions.

Under symmetry conditions, some results have been obtained in [1], [2], [8], [9], [11] for problem (1.1) as well as the Dirichlet problem and the nonlinear Schrödinger equation.

In the general case, progress although still limited, has also been made in [18], [19], [30], [32], [33], [34], [35], [45], [46]. For solutions concentrating on interior higher dimensional sets, results were first obtained in [45], [46] where the third author and Yang constructed solutions concentrating on line segment in the interior of the domain Ω. For boundary concentration solutions, in a series of papers of Malchiodi and Montenegro [32]-[34], they proved the the existence of solutions concentrating on the *whole boundary* or arbitrary components of $\partial\Omega$ when $\Omega \subset \mathbb{R}^N$, and solutions concentrating on closed geodesics of $\partial\Omega$ when $\Omega \subset \mathbb{R}^3$ and later Mahmoudi and Malchiodi [18] extended the results and obtained the existence of solutions concentrating on the k submanifold of $\partial\Omega \subset \mathbb{R}^N$ provided that the sequence ε satisfies some gap condition. The latter condition is called *resonance*.

In [3], the first and third authors and Musso removed the resonance condition in [45] and proved the existence of solutions concentrating on the interior straight line by putting a large number of spikes distributing along the line. It is natural

to ask that whether one can remove the resonance condition for the boundary concentration solutions using similar ideas. Also in all the above mentioned papers, for higher dimensional boundary concentration solutions, the concentration sets are either the whole boundary or submanifold of the boundary. A natural question is:

Does problem (1.1) have solutions which concentrate on a broken segment of the boundary for all $\varepsilon \to 0$?

In this paper, we give an affirmative answer to the above question. We construct solutions concentrating on a broken segment γ of the boundary $\partial\Omega \subset \mathbb{R}^2$ for all $\varepsilon \to 0$ if γ satisfies the following condition:

$(\mathbf{H_1})$. Let $\gamma = \gamma([0,b])$ be the segment parametrized by arc length, and $H(q)$ be the curvature of $\partial\Omega$ at q. Denote by

$$H'(\gamma(s)) = \frac{d}{ds}H(\gamma(s)),\ H''(\gamma(s)) = \frac{d^2}{ds^2}H(\gamma(s)).$$

Assume that $H''(\gamma(s)) \geq c_0 > 0$ for all $s \in [0,b]$, and $\int_0^b H'(\gamma(s))ds = 0$.

Remark 1. From assumption (H_1), one can see that γ must contain a non degenerate local minimum point of the curvature H, and $\int_0^b H'(\gamma(s))ds = 0$ is equivalent to $H(\gamma(0)) = H(\gamma(b))$, i.e. the curvature at the two end points of γ must be the same.

Our main result in this paper states as follows:

Theorem 1.1. *Assume that γ satisfies $(\mathbf{H_1})$, then there exists $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, there exists boundary spike solutions to (1.1) concentrating on γ.*

Remark 2. In the paper [23], Gui, Winter and the third author proved the existence of multiple spike solutions concentrating at the local minimum point of the curvature $H(p)$. In this paper, we proved the existence of spike solutions concentrating on the segment which contains a local minimum of $H(p)$. Theorem 1.1 extends their result to a segment containing a local minimum point of H.

1.2. **Description of the construction.** The solutions we construct consist of large number $(O(\frac{1}{\varepsilon \ln \varepsilon}))$ of spikes distributed along the segment γ whose inter distance is sufficiently small $(O(\varepsilon \ln \varepsilon))$.

At first glance one may discard such kind of solutions as there seems to be no balancing force at the end points of the segment. In the following we will show that the derivative of the mean curvature function acts as *friction force*. This new phenomena was first discovered in the construction of CMC surfaces by Butscher and Mazzeo [10] in which they constructed CMC surfaces condensing to a finite geodesic segment. We will comment more on this later.

In this subsection, we will briefly describe the solutions to be constructed later and will give the main idea in the procedure of the construction.

More precisely let w be the unique solution of the following equation:

$$\begin{cases} \Delta w - w + w^p = 0 \text{ in } \mathbb{R}^2, \\ w > 0,\ w(0) = \max_{y\in\mathbb{R}^2} w(y), \\ w \to 0 \text{ as } |y| \to \infty. \end{cases} \tag{1.4}$$

It is well-known (see [25]) that w is radial, i.e., $w = w(r)$ and $w'(r) < 0$ and has the following asymptotic behaviour:

$$w(y) = c_{N,p}|y|^{-\frac{N-1}{2}} e^{-|y|}(1 + o(1)) \tag{1.5}$$

and

$$w'(y) = -(1 + o(1))w(y) \text{ as } |y| \to \infty. \tag{1.6}$$

For $q \in \partial\Omega$, we set

$$\Omega_\varepsilon = \{z : \varepsilon z \in \Omega\},\ \Omega_{\varepsilon,q} = \{z : \varepsilon z + q \in \Omega\},$$

and

$$\mathcal{P}w_q(z) = \mathcal{P}_{\Omega_{\varepsilon,q}} w(z - \frac{q}{\varepsilon}), w_q(z) = w(z - \frac{q}{\varepsilon}),\ z \in \Omega_\varepsilon,$$

where $\mathcal{P}_{\Omega_{\varepsilon,q}} w(z - \frac{q}{\varepsilon})$ is defined to be the unique solution of

$$\Delta u - u + w(\cdot - \frac{q}{\varepsilon})^p = 0 \text{ in } \Omega_{\varepsilon,q},\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_{\varepsilon,q} \tag{1.7}$$

We will put large number of boundary spikes along γ. Let the location of spikes be $(\gamma(s_1), \cdots, \gamma(s_k))$. We define

$$U = \sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,\gamma(s_i)}} w(z - \frac{\gamma(s_i)}{\varepsilon})$$

to be an approximate solution.

A natural and central question is how to choose s_i such that U is indeed a good approximation. By formal calculation, one has the following energy expansion for the energy functional corresponding to (1.1):

$$J(U) = \frac{k}{2} I(w) - \varepsilon\gamma_0 \sum_{i=1}^{k} H(\gamma(s_i)) - \frac{\gamma_1}{2} w(\frac{\gamma(s_i) - \gamma(s_j)}{\varepsilon}) + o(\varepsilon)$$

where γ_0, γ_1 are positive constants. One needs to find a critical point $(s_1, \cdots, s_k)$ of J in order to get a solution of (1.1), i.e. $\frac{\partial}{\partial s_i} J = 0$ for all i. The main point in this paper is to exploit the contribution of $H'(\gamma(s))$ in $\frac{\partial J}{\partial s_i}$. The novelty of this paper is the new method of constructing balance approximate spike solutions, i.e. the configuration space $\{(s_1, \cdots, s_k)\}$, such that $\frac{\partial J}{\partial s_i}$ is almost 0.

In this paper, we will establish a method to find such balance approximate solutions. It turns out that the number of spikes and their positions are determined by some nonlinear equations which involves the interaction of spikes and also the effect of the boundary curvature. To explain this, we need to introduce the interaction function $\Psi(s)$ (to describe the interactions of different spikes) which is defined for all $s \in \mathbb{R}$ by

$$\Psi(s) = -\int_{\mathbb{R}^2_+} w(y - (s, 0)) p w^{p-1} \frac{\partial w}{\partial y_1} dy.$$

It turns out that $\gamma(s_i)$ are determined by the following non-linear system:

$$\begin{cases} \Psi(\frac{|s_2-s_1|}{\varepsilon}) + \varepsilon^2 H'(\gamma(s_1)) = O(\varepsilon^3), \\ \Psi(\frac{|s_3-s_2|}{\varepsilon}) - \Psi(\frac{|s_2-s_1|}{\varepsilon}) + \varepsilon^2 H'(\gamma(s_2)) = O(\varepsilon^3), \\ \cdots \\ \Psi(\frac{|s_k-s_{k-1}|}{\varepsilon}) - \Psi(\frac{|s_{k-1}-s_{k-2}|}{\varepsilon}) + \varepsilon^2 H'(\gamma(s_{k-1})) = O(\varepsilon^3), \\ -\Psi(\frac{|s_k-s_{k-1}|}{\varepsilon}) + \varepsilon^2 H'(\gamma(s_k)) = O(\varepsilon^3), \end{cases} \tag{1.8}$$

and the number of spikes depending on ε is given by $k = k_\varepsilon = [\frac{b}{|\varepsilon \ln \varepsilon|}] + 1$.

One can see from the above equations that it is possible to balance the two end points of the segment using the derivative of the curvature function. But in general, the above nonlinear system is difficult to solve. Our new idea is to consider this non-linear system as a discretization of its continuum limiting ODE system (as the step size $h = \varepsilon|\ln \varepsilon|$ tends to 0):

$$\begin{cases} \frac{dx}{dt} = -\frac{1}{\ln \varepsilon}\Psi^{-1}(\frac{\varepsilon}{\ln \varepsilon}\rho(t)), \\ \frac{d\rho}{dt} = H'(\gamma(x(t))),\ 0 < t < b_\varepsilon, \\ \rho(0) = 0,\ \rho(b_\varepsilon) = \rho_b, \\ x'(b_\varepsilon) = -\frac{1}{\ln \varepsilon}\Psi^{-1}(\varepsilon^2 H'(\gamma(x(b_\varepsilon)))) \end{cases} \tag{1.9}$$

where Ψ^{-1} is the inverse function of Ψ, and $b_\varepsilon = (k_\varepsilon - 1)h = b + O(h)$ and $\rho_b < 0$ is a small constant depending on ε. The above overdetermined ODE is solvable under the assumption of the segment γ in $(\mathbf{H_1})$.

To describe the configuration space of $\gamma(s_i)$, we solve the ODE system (1.9) first and denote the solution as $x(t)$. Then we define the positions of spikes by midpoint approximation:

$$s_i^0 = x(\frac{t_i + t_{i+1}}{2}) \text{ for } i = 1, \cdots, k-1 \tag{1.10}$$

and

$$s_k^0 = s_{k-1}^0 + \varepsilon\Psi^{-1}(\varepsilon^2 H'(\frac{\varepsilon}{\ln \varepsilon}\rho_b)) \tag{1.11}$$

where

$$t_i = (i-1)|\varepsilon \ln \varepsilon|,\ i \geq 1 \tag{1.12}$$

The method to determine the approximate positions, i.e. s_i^0 is the main contribution of this paper which is contained in Section 6. The position defined in this way is indeed an almost balance one. We will find real solutions by perturbing these spike points.

Letting $y_i \in \mathbb{R}$, we define

$$s_i = s_i^0 + y_i,\ \text{for } i = 1, \cdots, k, \tag{1.13}$$

and y_i satisfies

$$\begin{cases} |y_1| \leq C|\varepsilon \ln(-\ln \varepsilon)|, \\ |(s_{i+1} - s_i) - (s_i - s_{i-1})| \leq \frac{C\varepsilon^3}{\min\{\Psi(\frac{s_i^0 - s_{i-1}^0}{\varepsilon}), \Psi(\frac{s_i^0 - s_{i+1}^0}{\varepsilon})\}} \end{cases} \tag{1.14}$$

for $i = 2, \cdots, k-1$ for some constant $C > 0$ large.

With these notations, we can define the configuration space of $(s_1, \cdots, s_k)$ by

$$\Lambda_k = \{(s_1, \cdots, s_k) \in \mathbb{R}^k | s_i \text{ is defined by (1.13) and satisfies (1.14)}\} \tag{1.15}$$

The reason to define the configuration space in this way will be made clear in Section 3.

Moreover, from the analysis of the ODE (1.9) in Section 6, one can get that

$$|s_i - s_{i-1}| \geq (1+o(1))|\varepsilon \ln \varepsilon|, \ w(\frac{s_i - s_{i-1}}{\varepsilon}) \leq \frac{c\varepsilon}{|\ln \varepsilon|} \tag{1.16}$$

for $i = 2, \cdots, k$ and

$$|s_i - s_{i-1}| = 2(1+o(1))|\varepsilon \ln \varepsilon|| \tag{1.17}$$

for $i = 2, k$.

We will prove Theorem 1.1 by showing the following result:

Theorem 1.2. *Assume γ be a segment of $\partial\Omega$ and satisfies $(\mathbf{H_1})$. Then there exists ε_0 such that for $\varepsilon < \varepsilon_0$, there exists positive number $k = k_{\varepsilon,\gamma} = [\frac{b}{|\varepsilon \ln \varepsilon|}] + 1$ and k points $(\gamma(s_1), \cdots, \gamma(s_k))$ on γ, where $(s_1, \cdots, s_k) \in \Lambda_k$ such that there exists a solution u_ε to problem (1.1) and u_ε has the following form:*

$$u_\varepsilon(x) = \sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,\gamma(s_i)}} w(\frac{x - \gamma(s_i)}{\varepsilon}) + o(1) \tag{1.18}$$

where $o(1) \to 0$ as $\varepsilon \to 0$ uniformly.

Remark 3. The motivation of our construction comes from the study of the constant mean curvature surface. In [10], Butscher and Mazzeo constructed CMC surface condensing to a geodesic segments by connecting large number $(O(\frac{1}{r}))$ of spheres of radius r distributing along the geodesic segment. Such surfaces can not exist in Euclidean space, but they are able to show that the gradient of the ambient scalar curvature acts as 'friction term' which permits the existence of balance surface. So the gradient of scalar curvature plays the same role as the gradient of the mean curvature in our case. In their paper, they require the symmetry condition on the geodesic segment. In our main theorem 1.1, if we further require that Ω is symmetric, it is easy to see that (H_1) can always be satisfied near the non-degenerate minimum point of the curvature $H(\gamma(s))$. Since we don't require any symmetry of the segment in Theorem 1.1, we believe that our idea can be used to construct CMC surface condensing to geodesic segments without the symmetry condition. This is the main contribution of our paper. We will discuss this in a forthcoming paper. (A. Butscher announced this result in a preprint [13] but the full details have not appeared.)

1.3. **Sketch of the proof of Theorem 1.2.** We will use the Lyapunov-Schmidt reduction method and perturbation argument to construct the solutions to (1.1). The perturbation argument used to produce a real solution is not so different from the ones appearing elsewhere in the literature, as we mentioned before, the main contribution of this paper is the new idea of constructing balanced approximate solutions. In the following, we give the sketch of the proof.

We introduce some notations first. Since after scaling $x = \varepsilon z$, the original problem becomes

$$\begin{cases} \Delta u - u + u^p = 0 \text{ in } \Omega_\varepsilon \\ u > 0 \text{ in } \Omega_\varepsilon \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_\varepsilon \end{cases} \tag{1.19}$$

Fixing $\mathbf{s} = (s_1, \cdots, s_k) \in \Lambda_k$, we denote by

$$\mathbf{P} = (P_1, \cdots, P_k) = (\frac{\gamma(s_1)}{\varepsilon}, \cdots, \frac{\gamma(s_k)}{\varepsilon})$$

and define the sum of k spikes as

$$U = \sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z - P_i).$$

Define the operator

$$S(u) = \Delta u - u + u^p.$$

We also define the following functions as the approximate kernels

$$Z_i = \frac{\partial \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z - P_i)}{\partial \tau_{P_i}}, \ i = 1, \cdots, k.$$

Using U as the approximate solution, and performing the Lyapunov-Schmidt reduction, we can show that there exists ε_0 such that for $\varepsilon < \varepsilon_0$, we can find a ψ of the following projected problem:

$$S(U + \psi) = \sum_{i=1}^{k} c_i Z_i, \ \int_{\Omega_\varepsilon} \psi Z_i = 0, \ i = 1, \cdots, k,$$

where c_i are constants depending on the form of ψ, Z_i.

Next, we need to solve the reduced problem

$$c_i = 0, \ i = 1, \cdots, k$$

by adjusting the points in Λ_k.

There are two main difficulties in solving the reduced problem. First we need to control the error projection produced by ψ. In order to control this projection, we need to work in a weighted norm, which estimates ψ locally (see Section 3), and also we need a further decomposition of ψ which is given in Section 4 from where one can see why we define the configuration space of s_i in (1.15). The reason that one needs to obtain a further decomposition of ψ is that the inter distance of spikes at main order is not the same. In fact, near the two end points, the inter distance of two neighbored spikes is $2(1 + o(1))\varepsilon|\ln \varepsilon|$, while in the more central part, the inter distance of two neighbored spikes is $(1+o(1))\varepsilon|\ln \varepsilon|$. Thus the global estimate for ψ is not enough for our estimates. We need a further decomposition near each spike. Second, we need to solve a non-linear system of the form (1.8), for which we use the discretezation of the ODE equation (1.9).

Finally, the paper is organized as follows. Some preliminary facts and useful estimates are explained in Section 2. Section 3 contains the standard Lyapunov-Schmidt reduction process: we study the linearized projected problem in 3.1 and then solve a non-linear projected problem in 3.2. In Section 4, we obtain a further asymptotic behavior of ψ which gives an expansion in ε. In Section 5, we derive the reduced nonlinear system of algebraic equations for the location. Section 6 is devoted to solving the reduced problem.

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2. Technical Analysis

In this section, we introduce a projection and derive some useful estimates. Most of the results in this section are quite standard now and has been extensively used in the literature (see [21], [22], [23], [38], [41], [43]).

Throughout this paper, we shall use the letter C to denote a generic positive constant which may vary from term to term. By the following rescaling

$$x = \varepsilon z,\ \ z \in \Omega_\varepsilon := \{\varepsilon z \in \Omega\}, \tag{2.1}$$

equation (1.1) becomes

$$\begin{cases} \Delta u - u + u^p = 0, \text{ in } \Omega_\varepsilon \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_\varepsilon. \end{cases} \tag{2.2}$$

We denote by $\mathbb{R}^2_+ = \{(y_1, y_2)|y_2 > 0\}$. Recall that w is the unique solution of (1.4).

Let $q \in \partial\Omega$. We can define a diffeomorphism straightening the boundary. We may assume that the inward normal to $\partial\Omega$ at q is pointing in the direction of the positive x_2 axis. Denote $B'(R) = \{|x_1| \le R\}$, and $\Omega_1 = \Omega \cap B(q, R) = \{(x_1, x_2) \in B(q, R)|x_2 - q_2 > \rho(x_1 - q_1)\}$ where $B(q, R) = \{x \in \mathbb{R}^2||x - q| < R\}$. Then since $\partial\Omega$ is smooth, we can find a constant R such that $\partial\Omega$ can be represented by the graph of a smooth function $\rho_q : B'(R) \to \mathbb{R}$ where $\rho_q(0) = 0$, and $\rho'_q(0) = 0$. From now on, we omit the use of q in ρ_q and write ρ instead if this can be done without confusion. So near q, $\partial\Omega$ can be represented by $(x_1, \rho(x_1))$. The curvature of $\partial\Omega$ at q is $H(q) = \rho''(0)$. After scaling, we know that near $Q = \frac{q}{\varepsilon}$, $\partial\Omega_\varepsilon$ can be represented by $(z_1, \varepsilon^{-1}\rho(\varepsilon z_1))$, where $(z_1, z_2) = \varepsilon^{-1}(x_1, x_2)$. By Taylor's expansion, we have the following:

$$\varepsilon^{-1}\rho(\varepsilon z_1) = \frac{1}{2}\rho''(0)\varepsilon z_1^2 + \frac{1}{6}\rho^{(3)}(0)\varepsilon^2 z_1^3 + O(\varepsilon^3 z_1^4). \tag{2.3}$$

Recall that for a smooth bounded domain $\mathcal{U}$, the projection $\mathcal{P}_\mathcal{U}$ of $H^2(\mathcal{U})$ onto $\{u \in H^2(\mathcal{U})|\frac{\partial u}{\partial \nu} = 0 \text{ at } \partial\mathcal{U}\}$ is defined as follows: For $v \in H^2(\mathcal{U})$, let $\mathcal{P}_\mathcal{U} v$ be the unique solution of the boundary value problem:

$$\begin{cases} \Delta u - u + v^p = 0, \text{ in } \mathcal{U}, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\mathcal{U}. \end{cases} \tag{2.4}$$

Let $h_P(z) = w(z - P) - \mathcal{P}_{\Omega_{\varepsilon,P}} w(z - P)$. Then h_P satisfies

$$\begin{cases} \Delta h_P(z) - h_P(z) = 0, \text{ in } \Omega_\varepsilon, \\ \frac{\partial h_P}{\partial \nu} = \frac{\partial}{\partial \nu} w(z - P) \text{ on } \partial\Omega_\varepsilon. \end{cases} \tag{2.5}$$

For $z \in \Omega_{1,\varepsilon}$, for $P = (\mathbf{P}_1, \mathbf{P}_2)$, set now

$$\begin{cases} y_1 = z_1 - \mathbf{P}_1, \\ y_2 = z_2 - \mathbf{P}_2 - \varepsilon^{-1}\rho(\varepsilon(z_1 - \mathbf{P}_1)). \end{cases} \tag{2.6}$$

Under this transformation, the Laplace operator and the boundary derivative operator become

$$\Delta_z = \Delta_y + \rho(\varepsilon z_1)^2 \partial_{y_2 y_2} - 2\rho'(\varepsilon z_1)\partial_{y_1 y_2} - \varepsilon\rho''(\varepsilon z_1)\partial_{y_2},$$
$$(1 + \rho'(\varepsilon z_1)^2)^{\frac{1}{2}} \frac{\partial}{\partial \nu} = \rho'(\varepsilon z_1)\partial_{y_1} - (1 + \rho'^2(\varepsilon z_1))\partial_{y_2}.$$

First we need to get the expansion of $h_P(z)$ in terms of ε, from which one can see the effect of the boundary curvature. In this paper, we need to expand it up to $O(\varepsilon^2)$. To be more specific, let $v^{(1)}$ be the unique solution of

$$\begin{cases} \Delta v - v = 0, \text{ in } \mathbb{R}^2_+ \\ \frac{\partial v}{\partial y_2} = \frac{w'}{|y|}\frac{\rho''(0)}{2}y_1^2 \text{ on } \partial\mathbb{R}^2_+, \end{cases} \tag{2.7}$$

where w' is the radial derivative of w, i.e. $w' = w_r(r)$, and $r = |z - P|$.

Let $v^{(2)}$ be the unique solution of

$$\begin{cases} \Delta v - v - 2\rho''(0)y_1\frac{\partial^2 v_1}{\partial y_1 \partial y_2} = 0 \text{ in } \mathbb{R}^2_+, \\ \frac{\partial v}{\partial y_2} = -\rho''(0)y_1\frac{\partial v_1}{\partial y_1} \text{ on } \partial\mathbb{R}^2_+. \end{cases}$$

Let $v^{(3)}$ be the unique solution of

$$\begin{cases} \Delta v - v = 0, \text{ in } \mathbb{R}^2_+, \\ \frac{\partial v}{\partial y_2} = \frac{w'}{|y|}\frac{1}{3}\rho^{(3)}(0)y_1^3, \text{ on } \partial\mathbb{R}^2_+. \end{cases} \tag{2.8}$$

Note that $v^{(1)}, v^{(2)}$ are even functions in y_1 and $v^{(3)}$ is odd function in y_1. Moreover, it is easy to see that $|v_i(y)| \le Ce^{-\mu|y|}$ for any $0 < \mu < 1$. Let $\chi(x)$ be a smooth cut-off function, such that $\chi(x) = 1$, $x \in B(0, R_0\varepsilon|\ln\varepsilon|)$, and $\chi(x) = 0$ for $x \in B(0, 2R_0\varepsilon|\ln\varepsilon|)^c$ for R_0 large enough, and $\chi_\varepsilon(z) = \chi(\varepsilon z)$ for $z \in \Omega_\varepsilon$. In this case, one has $w(R_0|\ln\varepsilon|) = O(\varepsilon^{R_0})$. Set

$$h_P(z) = -(\varepsilon v_1(y) + \varepsilon^2(v_2(y) + v_3(y)))\chi_\varepsilon(z - P) + \varepsilon^3\xi_P(z),\ z \in \Omega_\varepsilon. \tag{2.9}$$

Then we have the following estimate:

Proposition 2.1.

$$\|\xi(z)\|_{H^1(\Omega_\varepsilon)} \le C. \tag{2.10}$$

Proposition 2.1 was proved in [43] by Taylor expansion and a rigorous estimate for the reminder using estimates for elliptic equations. Moreover, one can checked that $|\xi(z)| \le Ce^{-\mu|z-P|}$ for some $0 < \mu < 1$.

In our proof, only the evenness property in y_1 of the functions $v^{(1)}$ and $v^{(2)}$ are used. But for the function $v^{(3)}$, both the oddness property and equation it satisfied will be used. In fact, it is from this term that the derivative of the curvature function appears.

Similarly we know from [43] that

Proposition 2.2.

$$[\frac{\partial w}{\partial \tau_P} - \frac{\partial \mathcal{P}_{\Omega_{\varepsilon,P}} w}{\partial \tau_P}](z - P) = \varepsilon\eta(y)\chi_\varepsilon(z - P) + \varepsilon^2\eta_1(z),\ z \in \Omega_\varepsilon, \tag{2.11}$$

where η is the unique solution of the following equation:

$$\begin{cases} \Delta\eta - \eta = 0 \text{ in } \mathbb{R}^2_+, \\ \frac{\partial\eta}{\partial y_2} = -\frac{1}{2}(\frac{w''}{|y|^2} - \frac{w'}{|y|^3})\rho''(0)y_1^3 - \frac{w'}{|y|}\rho''(0)y_1 \text{ on } \partial R^2_+. \end{cases} \tag{2.12}$$

Moreover,

$$\|\eta_1\|_{H^1(\Omega_\varepsilon)} \le C. \tag{2.13}$$

One can observe that $\eta(y)$ is an odd function in y_1. It can be seen that $|\eta_i(y)| \leq Ce^{-\mu|y|}$ for some $0 < \mu < 1$.

Finally, let

$$L_0 = \Delta - 1 + pw^{p-1}(z). \tag{2.14}$$

We have the following non degeneracy property:

Lemma 2.1.

$$Ker(L_0) \cap H^2_N(\mathbb{R}^2_+) = span\{\frac{\partial w}{\partial y_1}\}, \tag{2.15}$$

where $H^2_N(\mathbb{R}^2_+) = \{u \in H^2(\mathbb{R}^2_+), \frac{\partial u}{\partial y_2} = 0 \text{ on } \partial\mathbb{R}^2_+\}$.

Proof. See Lemma 4.2 in [38].

□

Next we state a useful lemma we will frequently use:

Lemma 2.2. *If* $|q_1 - q_2| << |q_1|$, *we have the following estimate:*

$$\int_{\mathbb{R}^2_+} pw(y)^{p-1}(w(y - q_1e_1) + w(y + q_2e_1))\frac{\partial w}{\partial y_1}dy = O(|q_1 - q_2|w(|q_1|) \tag{2.16}$$

as $|q_1| \to \infty$ *where* e_1 *is the unite vector* $(1, 0)$.

Proof. By the oddness of $\frac{\partial w}{\partial y_1}$ in y_1, one has

$$\begin{aligned}
&\int_{\mathbb{R}^2_+} pw(y)^{p-1}(w(y - q_1e_1) + w(y + q_2e_1))\frac{\partial w}{\partial y_1}dy \\
&= \int_{\mathbb{R}^2_+} pw(y)^{p-1}(w(y - q_1e_1) - w(y - q_2e_1))\frac{\partial w}{\partial y_1}dy \\
&= \int_{\mathbb{R}^2} pw(y)^{p-1}|\frac{\partial w}{\partial y_1}|O(w'(y - q_1e_1)|q_1 - q_2|)dy \\
&= O(|q_1 - q_2|)w(|q_1|).
\end{aligned}$$

□

Remark 4. In the following sections, we will denote by $y^i = (y^i_1, y^i_2)$ to be the transformation defined by (2.6) centered at the point P_i and $v^{(j)}_i$ be the corresponding solutions in the expansion of h_{P_i}.

3. Liapunov-Schmidt Reduction

In this section, we reduce problem (2.2) to finite dimension by the Liapunov-Schmidt reduction method. The argument by now is quite standard. We leave most of the proofs to the appendix. We first introduce some notations. Let $H^2_N(\Omega_\varepsilon)$ be the Hilbert space defined by

$$H^2_N(\Omega_\varepsilon) = \{u \in H^2(\Omega_\varepsilon)|\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_\varepsilon\}. \tag{3.1}$$

Define

$$S(u) = \Delta u - u + u^p \tag{3.2}$$

for $u \in H^2_N(\Omega_\varepsilon)$. Then solving equation (2.2) is equivalent to

$$S(u) = 0, u \in H^2_N(\Omega_\varepsilon). \tag{3.3}$$

To this end, we first study the linearized operator

$$L_\varepsilon(\psi) := \Delta\psi - \psi + p(\sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i))^{p-1}\psi,$$

and define the approximate kernels to be

$$Z_i = \frac{\partial \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i)}{\partial \tau_{P_i}},$$

for $i = 1, \cdots, k$.

3.1. **Linear projected problem.** We first develop a solvability theory for the linear projected problem:

$$\begin{cases} L_\varepsilon(\psi) = h + \sum_{i=1}^{k} c_i Z_i, \\ \int_{\Omega_\varepsilon} \psi Z_i dz = 0, i = 1, \cdots, k, \\ \psi \in H_N^2(\Omega_\varepsilon) \end{cases} \tag{3.4}$$

Given $0 < \mu < 1$, consider the norm

$$\|h\|_* = \sup_{z\in\Omega_\varepsilon} |(\sum_j e^{-\mu|z-P_i|})^{-1} h(z)| \tag{3.5}$$

where $P_i \in \Lambda_k$ with Λ_k defined in (1.15).

The proof of the following Proposition on linearized operator, which we postpone to the appendix, is by now standard.

Proposition 3.1. *There exist positive numbers $\mu \in (0,1)$, ε_0 and C, such that for all $\varepsilon \le \varepsilon_0$, and for any points P_j, $j = 1, \ldots, k$ given by (1.15), there is a unique solution (ψ, c_i) to problem (3.4). Furthermore*

$$\|\psi\|_* \le C\|h\|_*. \tag{3.6}$$

In the following, if ψ is the unique solution given by Proposition 3.1, we set

$$\psi = \mathcal{A}(h). \tag{3.7}$$

Estimate (3.6) implies

$$\|\mathcal{A}(h)\|_* \le C\|h\|_*. \tag{3.8}$$

3.2. **Nonlinear projected problem.** We are now in the position to solve the nonlinear equation:

$$\begin{cases} L_\varepsilon(\psi) + E + N(\psi) = \sum_{i=1}^{k} c_i Z_i, \\ \int_{\Omega_\varepsilon} \psi Z_i = 0 \text{ for } i = 1, \cdots, k, \\ \psi \in H_N^2(\Omega_\varepsilon) \end{cases} \tag{3.9}$$

where E is the error of the approximate solution U:

$$\begin{aligned} E &= \Delta(\sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i)) - (\sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i)) \\ &\quad + (\sum_{i=1}^{k} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i))^p, \end{aligned} \tag{3.10}$$

and $N(\psi)$ is the nonlinear term:

$$\begin{aligned} N(\psi) &= ((\sum_{i=1}^{k}\mathcal{P}_{\Omega_{\varepsilon,P_i}}w(z-P_i))+\psi)^p-(\sum_{i=1}^{k}\mathcal{P}_{\Omega_{\varepsilon,P_i}}w(z-P_i))^p \\ &\quad -p(\sum_{i=1}^{k}\mathcal{P}_{\Omega_{\varepsilon,P_i}}w(z-P_i))^{p-1}\psi. \end{aligned} \tag{3.11}$$

By Proposition 3.1, we can rewrite (3.9) as

$$\psi=-\mathcal{A}\left(E+N(\psi)\right) \tag{3.12}$$

where $\mathcal{A}$ is the operator introduced in (3.7). In other words, ψ solves (3.9) if and only if ψ is a fixed point for the operator

$$T(\psi):=-\mathcal{A}\left(E+N(\psi)\right).$$

We are going to show that the operator T defined above for $\psi\in H_N^2(\Omega_\varepsilon)$ is a contraction on

$$\mathcal{B}=\{\psi\in H_N^2(\Omega_\varepsilon)\,:\,\|\psi\|_*\le C\varepsilon,\ \int_{\Omega_\varepsilon}\psi Z_i=0\}$$

for some $C>0$ large enough.

In fact we have the following lemma:

Lemma 3.1. *There exist $\mu\in(0,1)$, and positive numbers ε_0, C, such that for all $\varepsilon\le\varepsilon_0$, for any points P_j, $j=1,\dots,k$ given by (1.15), the following estimates hold:*

$$\|E\|_*\le C\varepsilon \tag{3.13}$$

and

$$\|N(\phi)\|_*\le C\|\phi\|_*^2 \tag{3.14}$$

Proof. We start with the proof for (3.13). Fix $j\in\{1,\dots,k\}$ and consider the region $|z-P_j|\le\frac{\min\{|P_j-P_{j-1}|,|P_j-P_{j+1}|\}}{2}$. In this region the error E, whose definition is in (3.11), can be estimated in the following way

$$\begin{aligned} |E(z)| &\le Cw^{p-1}(z-P_j)[\sum_{P_i\neq P_j}w(z-P_i)+\sum_i h_{P_i}(z)] \\ &\le C(\varepsilon+\varepsilon^{\frac{p-\mu}{2}})e^{-\mu|z-P_j|}\le C\varepsilon e^{-\mu|z-P_j|} \end{aligned} \tag{3.15}$$

if we choose μ small enough such that $p-\mu>2$.

Consider now the region $|z-P_j|>\frac{\min\{|P_j-P_{j-1}|,|P_j-P_{j+1}|\}}{2}$, for all j. From the definition of E, we get in the region under consideration

$$\begin{aligned} |E(z)| &\le C\left[\sum_i h_{P_i}(z)+(\sum_{i=1}^{k}P_{\Omega_{\varepsilon,P_i}}w(z-P_i))^p-\sum_i w(x-P_i)^p\right] \\ &\le C\sum_i e^{-\mu|z-P_i|}(\varepsilon+\varepsilon^{\frac{p-\mu}{2}}) \\ &\le C\varepsilon\sum_i e^{-\mu|z-P_i|}. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16) we get (3.13).

We now prove (3.14). Let $\psi \in \mathcal{B}$. Then

$$\begin{aligned} |N(\psi)| &\leq |((\sum_{i=1}^{k} P_{\Omega_{\varepsilon,P_i}} w(z-P_i)) + \psi)^p - (\sum_{i=1}^{k} P_{\Omega_\varepsilon} w(z-P_i))^p \\ &- p(\sum_{i=1}^{k} P_{\Omega_{\varepsilon,P_i}} w(z-P_i))^{p-1}\psi| \leq C\psi^2. \end{aligned}$$

Thus we have

$$|(\textstyle\sum_j e^{-\mu|x-P_j|})^{-1} N(\psi)| \quad \leq C\|\psi\|_*^2$$

This gives (3.14).

□

Using the above estimates, we have the validity of the following result:

Proposition 3.2. *There exist $\mu \in (0,1)$, and positive numbers ε_0, C, such that for all $\varepsilon \leq \varepsilon_0$, for any points P_j, $j = 1, \ldots, k$ given by (1.15), there is a unique solution (ψ, c_i) to problem (3.9). This solution depends continuously on the parameters of the construction (namely P_j, $j = 1, \ldots, k$) and furthermore*

$$\|\psi\|_* \leq C\varepsilon. \tag{3.17}$$

Proof. As we mentioned before, we are going to show that the operator T is a contraction mapping in $\mathcal{B}$.

By the estimates in Lemma 3.1, (3.13) and (3.14) and taking into account (3.8), we have for any $\psi \in \mathcal{B}$

$$\begin{aligned} \|T(\psi)\|_* \leq \quad & C\left[\|E + N(\psi)\|_*\right] \leq C(\varepsilon + \varepsilon^2) \\ & \leq C_1\varepsilon \end{aligned}$$

for a proper choice of C_1 in the definition of $\mathcal{B}$. Take now ψ_1 and ψ_2 in $\mathcal{B}$. Then it is straightforward to show that

$$\begin{aligned} \|T(\psi_1) - T(\psi_2)\|_* \quad & \leq C\|N(\psi_1) - N(\psi_2)\|_* \\ & \leq C\left[\|\psi_1\|_* + \|\psi_2\|_*\right] \|\psi_1 - \psi_2\|_* \\ & \leq o(1)\|\psi_1 - \psi_2\|_* \end{aligned}$$

This means that T is a contraction mapping from $\mathcal{B}$ into itself.

The existence of a fixed point ψ now follows from the contraction mapping principle, and ψ is a solution of (3.9) and satisfies (3.17).

A direct consequence of the fixed point characterization of ψ given above together with the fact that the error term E depends continuously (in the *-norm) on the parameters P_j, $j = 1, \ldots, k$ is that the map $(P_1, \cdots, P_k) \to \psi$ into the space $C(\bar{\Omega}_\varepsilon)$ is continuous (in the $*$-norm). This concludes the proof of the Proposition.

□

4. Further expansion of the error

In the previous section, we obtained a solution ψ to (3.9), which satisfies $\|\psi\|_* \leq C\varepsilon$. But this estimate is not enough in solving the reduced problem. For later purpose, we need to obtain the asymptotic behaviour of the function ψ as $\varepsilon \to 0$. This is needed to compute the neighboring interactions. The idea is that although the $\|\cdot\|_*$ norm of ψ is not small enough, but we can get a more accurate decomposition such that the projection with respect to Z_i is small enough for our purpose.

Before we state the result, we first consider the following equation:

$$\begin{cases} \Delta\phi - \phi + pw(y)^{p-1}\phi = h + d\frac{\partial w(y)}{\partial y_1} \text{ in } \mathbb{R}^2_+, \\ \frac{\partial \phi}{\partial y_2} = 0 \text{ on } \partial\mathbb{R}^2_+, \\ \int_{\mathbb{R}^2_+} \phi \frac{\partial w(y)}{\partial y_1} dy = 0 \end{cases} \tag{4.1}$$

where $d = -\frac{\int_{\mathbb{R}^2_+} h \frac{\partial w}{\partial y_1}}{\int_{\mathbb{R}^2_+} (\frac{\partial w}{\partial y_1})^2}$. We consider the above equation in the space $\{\|h\|_{**} < +\infty\}$, where $\|h\|_{**} = \sup_{y\in\mathbb{R}^2_+} |e^{\mu_1|y|}h|$ for some $0 < \mu_1 < 1$. It is quite standard to show the solvability of the above equation and ϕ will satisfy the following estimate:

$$\|\phi\|_{**} \leq C\|h\|_{**}. \tag{4.2}$$

Now we decompose ψ as follows:

Proposition 4.1.

$$\psi = \sum_{i=1}^{k} \chi_\varepsilon(z - P_i)\phi_i + \varepsilon^2\psi_1, \tag{4.3}$$

where

$$\|\psi_1\|_* \leq C. \tag{4.4}$$

and $\phi_i = \phi_i(y^i)$ is the unique solution of

$$\begin{cases} \Delta\phi_i - \phi_i + pw(y^i)^{p-1}\phi_i = H_i + d_i\frac{\partial w(y^i)}{\partial y_i} \text{ in } \mathbb{R}^2_+, \\ \frac{\partial \phi_i}{\partial y_2} = 0 \text{ on } \partial\mathbb{R}^2_+, \\ \int_{\mathbb{R}^2_+} \phi_i \frac{\partial w(y^i)}{\partial y_1} dy = 0 \end{cases} \tag{4.5}$$

where d_i is defined such that the right hand side of the above equation is orthogonal to $\frac{\partial w(y^i)}{\partial y_1}$ in L^2 norm, and

$$H_i = -pw(y^i)^{p-1}[w(y^i - \frac{s_{i-1} - s_i}{\varepsilon}e_1) + w(y^i - \frac{s_{i+1} - s_i}{\varepsilon}e_1) + \varepsilon v_i^{(1)}], \tag{4.6}$$

for $i = 2, \cdots, k-1$ and

$$H_1 = -pw(y^1)^{p-1}[w(y^1 - \frac{s_2 - s_1}{\varepsilon}e_1) + \varepsilon v_1^{(1)}], \tag{4.7}$$

and

$$H_k = -pw(y^k)^{p-1}[w(y^k - \frac{s_{k-1} - s_k}{\varepsilon}e_1) + \varepsilon v_k^{(1)}], \tag{4.8}$$

and we denote

$$v_i^{(1)} = v_{P_i}^{(1)}(y^i) \tag{4.9}$$

are the solutions obtained in Section 1.2 centered at the point P_i.

Proof. First by the definition of d_i, there holds

$$d_i = \int_{\mathbb{R}^2_+} H_i \frac{\partial w(y^i)}{\partial y_1} dy. \tag{4.10}$$

Then from Lemma 2.2, and the evenness of $v_i^{(1)}$ with respect to y_1^i, and the definition of the configuration space (1.15), we know that for $i = 2, \cdots, k-1$

$$|d_i| \leq C\varepsilon^{-1}||s_{i+1} - s_i| - |s_i - s_{i-1}| \min\{w(\frac{s_i - s_{i+1}}{\varepsilon}), w(\frac{s_i - s_{i-1}}{\varepsilon})\} \leq C\varepsilon^2, \tag{4.11}$$

and for $i = 1, k$,

$$|d_1| = O(w(\frac{s_1 - s_2}{\varepsilon})) = O(\varepsilon^2) \text{ and } |d_k| = O(w(\frac{s_k - s_{k-1}}{\varepsilon})) = O(\varepsilon^2). \tag{4.12}$$

Moreover, from (4.1), we have the following estimate:

$$\|\phi_i\|_{**} \leq C\varepsilon \text{ if } p > 2 + \mu_1. \tag{4.13}$$

Our strategy to estimate ψ_1 is to decompose ψ_1 into three parts and show that each of them is bounded in $\|\cdot\|_*$ as $\varepsilon \to 0$. We write ψ_1 as

$$\psi_1 = \psi_{11} + \psi_{12} + \psi_{13}, \tag{4.14}$$

where ψ_{11} satisfies

$$\begin{cases} \Delta\psi_{11} - \psi_{11} = 0, \text{ in } \Omega_\varepsilon \\ \frac{\partial \psi_{11}}{\partial \nu} = -\frac{1}{\varepsilon^2} \frac{\partial \sum_{i=1}^k \chi_\varepsilon(z - P_i)\phi_i}{\partial \nu} \text{ on } \partial\Omega_\varepsilon. \end{cases} \tag{4.15}$$

Define ψ_{12} by

$$\psi_{12} = \frac{1}{\varepsilon^2} \sum_{i=1}^k s_i Z_i, \tag{4.16}$$

and s_i is determined by

$$M(s_i) = -\int_{\Omega_\varepsilon} (\sum_{i=1}^k \chi_\varepsilon(z - P_i)\phi_i + \varepsilon^2\psi_{11}) Z_i. \tag{4.17}$$

Finally define ψ_{13} to be the solution of the following equation:

$$\begin{cases} L_\varepsilon(\psi_{13}) = \frac{1}{\varepsilon^2} L_\varepsilon(\psi - \sum_{i=1}^k \chi_\varepsilon(z - P_i)\phi_i - \varepsilon^2(\psi_{11} + \psi_{12})) \text{ in } \Omega_\varepsilon \\ \frac{\partial \psi_{13}}{\partial \nu} = 0 \text{ on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} \psi_{13} Z_i dz = 0. \end{cases} \tag{4.18}$$

Next we will estimate $\psi_{11}, \psi_{12}, \psi_{13}$ term by term. First we estimate $g_{1\varepsilon} = \frac{1}{\varepsilon^2}\frac{\partial \sum_{i=1}^k \chi_\varepsilon(z-P_i)\phi_i}{\partial \nu}$. By direct calculation

$$\begin{aligned} g_{1\varepsilon} &= \frac{1}{\varepsilon^2}\sum_{i=1}^k(\chi_\varepsilon(z-P_i)\frac{\partial \phi_i}{\partial \nu} + \phi_i \frac{\partial \chi_\varepsilon(z-P_i)}{\partial \nu}) \\ &= \frac{1}{\varepsilon^2}\sum_{i=1}^k \varepsilon e^{-\mu_1|y-\frac{s_i}{\varepsilon}|}\frac{\partial \chi_\varepsilon(z-P_i)}{\partial \nu} + O(\varepsilon^2) \\ &= O(\varepsilon^{-2}e^{-(\mu_1-\mu)R_0|\ln \varepsilon|})\sum_{i=1}^k e^{-\mu|z-P_i|} \\ &\leq C\sum_{i=1}^k e^{-\mu|z-P_i|} \end{aligned}$$

if we choose $\mu_1 > \mu$ and the cutoff function in such a way that $(\mu_1-\mu)R_0 \geq 1$. In the above estimate, we use the definition of ϕ_i and the Neumann boundary satisfied by it and the definition of the cut-off function χ. Thus we have that $\|g_{1\varepsilon}\|_* \leq C$, therefore, there exists constant $C > 0$, such that

$$\|\psi_{11}\|_* \leq C. \tag{4.19}$$

By the definition of ψ_{12}, ϕ_i and the estimate on ψ_{11}, one can obtain that

$$\begin{aligned} &\int_{\Omega_\varepsilon}(\sum_{j=1}^k \chi_\varepsilon(z-P_j)\phi_j + \varepsilon^2\psi_{11})Z_i dz \\ &= \int_{\Omega_\varepsilon}\chi_\varepsilon(z-P_i)\phi_i Z_i dz + \sum_{j=i-1,i+1}\chi_\varepsilon(z-p_j)\phi_j Z_i dz \\ &+O(\varepsilon^{1+(1+\mu)(1+o(1))}) + O(\varepsilon^2) \end{aligned}$$

In order to estimate the above term, we first consider a general function which is the solution of the following equation:

$$\begin{cases} \Delta\phi - \phi + pw(y)^{p-1}\phi \\ = -pw(y)^{p-1}(w(y-q_1e_1) + w(y+q_2e_1) + \varepsilon v^{(1)}) + d\frac{\partial w(y)}{\partial y_1} \text{ in } \mathbb{R}^2_+, \\ \frac{\partial \phi}{\partial y_2} = 0 \text{ on } \partial\mathbb{R}^2_+, \\ \int_{\mathbb{R}^2_+}\phi\frac{\partial w(y)}{\partial y_1}dy = 0 \end{cases} \tag{4.20}$$

We can decompose it as

$$\phi = \phi^1 + \phi^2, \tag{4.21}$$

where

$$\begin{cases} \Delta\phi^1 - \phi^1 + pw(y)^{p-1}\phi^1 \\ = -pw(y)^{p-1}(w(y-q_1e_1) + w(y+q_1e_1) + \varepsilon v^{(1)}) + d_1\frac{\partial w(y)}{\partial y_1} \text{ in } \mathbb{R}^2_+, \\ \frac{\partial \phi^1}{\partial y_2} = 0 \text{ on } \partial\mathbb{R}^2_+, \\ \int_{\mathbb{R}^2_+}\phi^1\frac{\partial w(y)}{\partial y_1}dy = 0 \end{cases} \tag{4.22}$$

and

$$\begin{cases} \Delta\phi^2 - \phi^2 + pw(y)^{p-1}\phi^2 \\ = -pw(y)^{p-1}(w(y+q_2e_1) - w(y+q_1e_1)) + d_2\frac{\partial w(y)}{\partial y_1} \text{ in } \mathbb{R}^2_+, \\ \frac{\partial\phi^2}{\partial y_2} = 0 \text{ on } \partial\mathbb{R}^2_+, \\ \int_{\mathbb{R}^2_+} \phi^2\frac{\partial w(y)}{\partial y_1}dy = 0 \end{cases} \tag{4.23}$$

where d_i are defined such that the right hand sides of the above equations are orthogonal to $\frac{\partial w}{\partial y_1}$ in L^2 norm It is easy to see that ϕ^1 is even in y_1 and by Lemma 2.2, ϕ^2 satisfies

$$\|\phi^2\|_{**} \le Cw(q_1)|q_1 - q_2|,$$

if $|q_1 - q_2| << |q_1|$ and $|q_1| \to \infty$.

Using the above estimates, we can decompose ϕ_i as

$$\phi_i = \phi_{i,1} + \phi_{i,2} \tag{4.24}$$

and $\phi_{i,1}$ is even in y_1^i and

$$\|\phi_{i,2}\|_{**} \le C||\frac{s_i - s_{i-1}}{\varepsilon}| - |\frac{s_i - s_{i+1}}{\varepsilon}|| \min\{w(\frac{s_i - s_{i-1}}{\varepsilon}), w(\frac{s_i - s_{i+1}}{\varepsilon})\} \le C\varepsilon^2. \tag{4.25}$$

Then by the above estimate and the decomposition in Proposition 2.2, we have

$$\int_{\Omega_\varepsilon} \chi_\varepsilon(z - P_i)\phi_i Z_i dz = O(\varepsilon^2), \tag{4.26}$$

and similar to the decomposition of ϕ_i, one can also decompose $\phi_{i-1} + \phi_{i+1}$ as an even function of y_1^i and an $O(\varepsilon^2)$ function, so we get that

$$\begin{aligned} &\sum_{j=i-1,i+1} \int_{\Omega_\varepsilon} \chi_\varepsilon(z - P_j)\phi_j Z_i dz \\ &= \int_{\mathbb{R}^2_+} (\phi_{i-1} + \phi_{i+1})\frac{\partial w(y^i)}{\partial y_1}dy + O(\varepsilon^2) \\ &= O(\varepsilon^2) \end{aligned}$$

Moreover, since $|s_1 - s_2| = 2(1 + o(1))|\varepsilon \ln \varepsilon|$ and $|s_{k-1} - s_k| = 2(1 + o(1))|\varepsilon \ln \varepsilon|$, one can get that

$$\int_{\Omega_\varepsilon} \chi_\varepsilon(z - P_2)\phi_2 Z_1 dz = O(\varepsilon^2), \int_{\Omega_\varepsilon} \chi_\varepsilon(z - P_{k-1})\phi_{k-1} Z_k dz = O(\varepsilon^2). \tag{4.27}$$

Thus we have

$$|s_i| \le C\varepsilon^2. \tag{4.28}$$

Next we estimate ψ_{13}. Denote by

$$f_\varepsilon = L_\varepsilon(\psi - \sum_{i=1}^{k} \chi_\varepsilon(z - p_i)\phi_i - \varepsilon^2(\psi_{11} + \psi_{12})).$$

Claim:

$$\|f_\varepsilon\|_* \le C\varepsilon^2. \tag{4.29}$$

Proof of the Claim:

By the definition of f_ε, we have

$$
\begin{aligned}
&f_\varepsilon(z) = L_\varepsilon(\psi - \sum_{i=1}^{k} \chi_\varepsilon(z-P_i)\phi_i - \varepsilon^2(\psi_{11}-\psi_{12})) \\
&= E + N(\psi) + \sum_i c_i Z_i - \sum_i L_\varepsilon(\chi_\varepsilon(z-P_i)\phi_i) - \varepsilon^2 L(\psi_{11}+\psi_{12}) \\
&= (\sum_i \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i))^p - \sum_i w(z-P_i)^p + N(\psi) + \sum_i c_i Z_i \\
&- \sum_i \chi_\varepsilon(z-P_i)(\Delta_y \phi_i - \phi_i + p((\sum_i \mathcal{P}_{\Omega_\varepsilon} w(z-P_i))^{p-1} + O(\varepsilon))\phi_i) \\
&+ \sum_i (2\nabla\phi_i \nabla(\chi_\varepsilon(z-P_i)) + \phi_i \Delta\chi_\varepsilon(z-P_i)) - \varepsilon^2 L_\varepsilon(\psi_{11}+\psi_{12}) \\
&= (\sum_i \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i))^p - \sum_i w(z-P_i)^p + N(\psi) + \sum_i c_i Z_i \\
&- \sum_i \chi_\varepsilon(z-P_i)(p((\sum_i \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i))^{p-1} - w(y-P_i)^{p-1})\phi_i \\
&- pw(y-P_i)^{p-1}(w(y-P_{i-1}) + w(y-P_{i+1}) + \varepsilon v_{1i}(y)) + d_i Z_i) \\
&+ \sum_i O(\varepsilon)\phi_i + \sum_i (2\nabla\phi_i\nabla(\chi_\varepsilon(z-p_i)) + \phi_i\Delta\chi_\varepsilon(z-P_i)) - \varepsilon^2 L_\varepsilon(\psi_{11}+\psi_{12}).
\end{aligned}
$$

From the definition and estimates of ϕ_i, ψ_{11}, ψ_{12}, χ, and the configuration space, we know that $|c_i| = O(\varepsilon^2)$, so

$$\|f_\varepsilon\|_* \le C\varepsilon^2.$$

By the a priori estimate, we know that

$$\|\psi_{13}\|_* \le C,$$

thus we have

$$\|\psi_1\|_* \le C.$$

We thus finish the proof.

□

Given points P_j defined by (1.15), Proposition 3.2 guarantees the existence (and gives estimates) of a unique solution ψ, c_i, $i = 1, \ldots, k$, to problem (3.9). It is clear then that the function $u = U + \psi$ is an exact solution to our problem (1.1), with the required properties stated in Theorem 1.2 if we show that there exists a configuration for the points P_j that gives all the constants c_i in (3.9) equal to zero. In order to do so we first need to find the correct conditions on the points to get $c_i = 0$. This condition is naturally given by projecting in $L^2(\Omega_\varepsilon)$ the equation in (3.9) into the space spanned by Z_i, namely by multiplying the equation in (3.9) by Z_i and integrate all over Ω_ε. We will do it in details in the next section.

5. The Reduced Problem

In this section, we keep the notations and assumptions in the previous sections. As explained in the previous section, we have obtained a solution $u =$

$\sum_{i=1}^k \mathcal{P}_{\Omega_\varepsilon,P_i} w(z-P_i) + \sum_{i=1}^k \chi_\varepsilon(z-P_i)\phi_i + \varepsilon^2\psi_1$ of the following equation

$$\begin{cases} \Delta u - u + u^p = \sum_{i=1}^k c_i Z_i \text{ in } \Omega_\varepsilon \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_\varepsilon \end{cases} \tag{5.1}$$

In this section, we are going to solve $c_i = 0$ for all i by adjusting the position of the spikes, i.e. P_i. First, multiplying the above equation (5.1) by $Z_i, i = 1, \cdots, k$ and integrating over Ω_ε, we get that

$$M \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} \int_{\Omega_\varepsilon} (\Delta u - u + u^p) Z_1 \\ \int_{\Omega_\varepsilon} (\Delta u - u + u^p) Z_2 \\ \vdots \\ \int_{\Omega_\varepsilon} (\Delta u - u + u^p) Z_k \end{bmatrix} \tag{5.2}$$

Recall that M is invertible, so $c_i = 0, i = 1, \cdots, k$ is reduced to solve the following system:

$$\begin{bmatrix} \int_{\Omega_\varepsilon} (\Delta u - u + u^p) Z_1 \\ \int_{\Omega_\varepsilon} (\Delta u - u + u^p) Z_2 \\ \vdots \\ \int_{\Omega_\varepsilon} (\Delta u - u + u^p) Z_k \end{bmatrix} = 0. \tag{5.3}$$

We have the following estimates:

Lemma 5.1. *Under the assumption of Proposition 3.2, for ε small enough, the following expansion holds:*

$$\int_{\Omega_\varepsilon} (\Delta u - u + u^p) Z_1 dz = -\Psi(\frac{s_1 - s_2}{\varepsilon}) - \varepsilon^2 \nu_2 H'(\gamma(s_1)) + O(\varepsilon^3), \tag{5.4}$$

and for $i = 2, \cdots, k-1$,

$$\int_{\Omega_\varepsilon} (\Delta u - u + u^p) Z_i dz = \Psi(\frac{s_i - s_{i-1}}{\varepsilon}) - \Psi(\frac{s_i - s_{i+1}}{\varepsilon}) - \varepsilon^2 \nu_2 H'(\gamma(s_i)) + O(\varepsilon^3) \tag{5.5}$$

and

$$\int_{\Omega_\varepsilon} (\Delta u - u + u^p) Z_k dz = \Psi(\frac{s_k - s_{k-1}}{\varepsilon}) - \varepsilon^2 \nu_2 H'(\gamma(s_k)) + O(\varepsilon^3) \tag{5.6}$$

where $\nu_2 > 0$ is a constant defined in (5.12).

Proof. First, by direct calculation, one can get the following expansion:

$$\begin{aligned}
&\Delta u - u + u^p \\
&= \Big[\Delta(U + \sum_{i=1}^k \chi_\varepsilon(z-P_i)\phi_i) - (U + \sum_{i=1}^k \chi_\varepsilon(z-P_i)\phi_i) + (U + \sum_i \chi_\varepsilon(z-P_i)\phi_i)^p\Big] \\
&+\Big[\varepsilon^2(\Delta\psi_1 - \psi_1 + p(U + \sum_{i=1}^k \chi_\varepsilon(z-P_i)\phi_i)^{p-1}\psi_1)\Big] \\
&+\Big[(U + \sum_{i=1}^k \chi_\varepsilon(z-P_i)\phi_i + \varepsilon^2\psi_1)^p - (U + \sum_{i=1}^k \chi_\varepsilon(z-P_i)\phi_i)^p - \\
&p(U + \sum_{i=1}^k \chi_\varepsilon(z-P_i)\phi_i)^{p-1}\varepsilon^2\psi_1\Big] \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

Next we calculate I_1 to I_3 term by term. First from the estimate on ψ_1 in (4.4)

$$\begin{aligned}
&\int_{\Omega_\varepsilon} I_2 Z_i = \varepsilon^2 \int_{\Omega_\varepsilon} (\Delta\psi_1 - \psi_1 + p(U + \sum_i \chi_\varepsilon(z-p_i)\phi_i)^{p-1}\psi_1)Z_i \\
&= \varepsilon^2 \int_{\Omega_\varepsilon} -pw(z-P_i)^{p-1}\frac{\partial w(z-P_i)}{\partial\tau}\psi_1 + p(U + \sum_i \chi_\varepsilon(z-P_i)\phi_i)^{p-1}Z_i\psi_1 \\
&= \varepsilon^2 \int_{\Omega_\varepsilon} p(p-1)w(z-P_i)^{p-2}\frac{\partial w(z-P_i)}{\partial\tau}\psi_1(\sum_{j\neq i}\frac{\partial w(z-P_j)}{\partial\tau} + O(\varepsilon))dz \\
&= O(\varepsilon^3). \qquad (5.7)
\end{aligned}$$

Moreover

$$\begin{aligned}
\int_{\Omega_\varepsilon} I_3 Z_i &= \int_{\Omega_\varepsilon} \Big[(U + \sum_i \chi_\varepsilon(z-P_i)\phi_i + \varepsilon^2\psi_1)^p - (U + \sum_i \chi_\varepsilon(z-P_i)\phi_i)^p \\
&\quad -p(U + \sum_i \chi_\varepsilon(z-P_i)\phi_i)^{p-1}\varepsilon^2\psi_1\Big]Z_i \\
&\leq C\int_{\Omega_\varepsilon} \varepsilon^4|\psi_1|^2|Z_i| = O(\varepsilon^3). \qquad (5.8)
\end{aligned}$$

Next from the equation satisfied by ϕ_i and the definition of the cutoff function χ, we get that

$$\begin{aligned}
\int_{\Omega_\varepsilon} I_1 Z_i &= \int_{\Omega_\varepsilon} (\Delta U - U + U^p)Z_i + \sum_j \int_{\Omega_\varepsilon} \chi_\varepsilon(z-P_j)(\Delta\phi_j - \phi_j + pU^{p-1}\phi_j)Z_i \\
&\quad +[(U + \sum_i \chi_\varepsilon(z-P_i)\phi_i)^p - U^p - pU^{p-1}\sum_i \chi_\varepsilon(z-p_i)\phi_i]Z_i + O(\varepsilon^3) \\
&= \int_{\Omega_\varepsilon} (\Delta U - U + U^p)Z_i + I_{11} + I_{12} + O(\varepsilon^3). \qquad (5.9)
\end{aligned}$$

In the following, we will show that, although ϕ_i is of $O(\varepsilon)$, but after projection with respect to Z_i, the terms containing ϕ_i is indeed $O(\varepsilon^3)$.

Similar to the estimate in (4.26), using the equation satisfied by ϕ_i, we have for $i = 2, \cdots, k-1$

$$\begin{aligned}
&\sum_j \int_{\Omega_\varepsilon} \chi_\varepsilon(z-P_j)(\Delta\phi_j - \phi_j + pU^{p-1}\phi_j)Z_i \\
&= \int_{\Omega_\varepsilon} \chi_\varepsilon(z-P_i)(\Delta\phi_j - \phi_j + pU^{p-1}\phi_j)Z_i \\
&+ \sum_{j\neq i} \int_{\Omega_\varepsilon} \chi_\varepsilon(z-P_j)(\Delta\phi_j - \phi_j + pU^{p-1}\phi_j)Z_i \\
&= \int_{\Omega_\varepsilon} p(U^{p-1} - w_i^{p-1})\phi_i Z_i + O(\varepsilon^3) \\
&+ \sum_{j=i-1,i+1} \int_{\Omega_\varepsilon} \chi(z-P_j)(\Delta\phi_j - \phi_j + pU^{p-1}\phi_j)Z_i + O(\varepsilon^3) \\
&= \int_{\Omega_\varepsilon} p(p-1)w_i^{p-2}(w_{i+1} + w_{i-1})\phi_i Z_i \\
&+ \sum_{j=i-1,i+1} \int_{\Omega_\varepsilon} \chi_\varepsilon(z-P_j)(\Delta\phi_j - \phi_j + pU^{p-1}\phi_j)Z_i + O(\varepsilon^3) \\
&= O(\varepsilon|\frac{s_i - s_{i-1}}{\varepsilon}| - |\frac{s_i - s_{i+1}}{\varepsilon}|| \min\{w(\frac{s_i - s_{i-1}}{\varepsilon}), w(\frac{s_i - s_{i+1}}{\varepsilon})\}) + O(\varepsilon^3) \\
&= O(\varepsilon^3)
\end{aligned}$$

and similarly we can always decompose

$$\sum_{j=1}^k \chi_\varepsilon(z-P_j)\phi_j = \varepsilon\psi_{1,i} + O(\varepsilon^2)$$

where $\psi_{1,i}$ is a function even in y_1^i, and by Proposition 2.2, we have

$$Z_i = \frac{\partial w(y^i)}{\partial y_1} + \varepsilon\eta_i + O(\varepsilon^2)$$

where η_i is odd in y_1^i. Thus we have

$$\begin{aligned}
I_{12} \quad &\leq C\int_{\Omega_\varepsilon} p(p-2)w_i^{p-2}(\sum_{j=1}^k \chi_\varepsilon(z-P_j)\phi_j)^2 Z_i dz + O(\varepsilon^3) \\
&\leq C\varepsilon^3.
\end{aligned}$$

For the case $i = 1, k$, recall that $w(\frac{s_1-s_2}{\varepsilon}), w(\frac{s_k-s_{k-1}}{\varepsilon}) = O(\varepsilon^2)$, one can also get that

$$I_{11} + I_{12} = O(\varepsilon^3). \tag{5.10}$$

Thus we have the following:

$$\int_{\Omega_\varepsilon} I_1 Z_i dz = \int_{\Omega_\varepsilon} (\Delta U - U + U^p)Z_i + O(\varepsilon^3).$$

Next for $i = 2, \cdots, k-1$

$$\begin{aligned}
&\int_{\Omega_\varepsilon} (\Delta U - U + U^p) Z_i \qquad (5.11)\\
&= \int_{\Omega_\varepsilon} \Big[(\sum_i \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i))^p - \sum_i w(z-P_i)^p \Big] Z_i\\
&= \int_{\Omega_\varepsilon} \Big[(w(z-P_i) + \varepsilon v_i^{(1)} + \varepsilon^2 (v_i^{(2)} + v_i^{(3)})\\
&+ \sum_{j\neq i} \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_j) + O(\varepsilon^3))^p - \sum_i w(z-P_i)^p \Big] Z_i\\
&= \int_{\Omega_\varepsilon} p w(z-P_i)^{p-1} \Big(\varepsilon v_i^{(1)} + \varepsilon^2 (v_i^{(2)} + v_i^{(3)})\\
&+ w(z-P_{i-1}) + w(z-P_{i+1}) \Big) \frac{\partial w(z-P_i)}{\partial \tau} + O(\varepsilon^3)\\
&= \int_{\mathbb{R}^2_+} p w(y) (w(y - \frac{s_{i-1}-s_i}{\varepsilon} e_1) + w(y - \frac{s_{i+1}-s_i}{\varepsilon} e_1)) \frac{\partial w(y)}{\partial y_1}\\
&+ \varepsilon^2 \int_{\mathbb{R}^2_+} p w(y)^{p-1} \frac{\partial w(y)}{\partial y_1} v_i^{(3)} + O(\varepsilon^3)
\end{aligned}$$

Similarly, one has for $i = 1, k$,

$$\begin{aligned}
&\int_{\Omega_\varepsilon} (\Delta U - U + U^p) Z_1\\
&= \int_{\mathbb{R}^2_+} p w(y) w(y - \frac{s_2 - s_1}{\varepsilon} e_1) \frac{\partial w(y)}{\partial y_1}\\
&+ \varepsilon^2 \int_{\mathbb{R}^2_+} p w(y)^{p-1} \frac{\partial w(y)}{\partial y_1} v_1^{(3)} + O(\varepsilon^3),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Omega_\varepsilon} (\Delta U - U + U^p) Z_k\\
&= \int_{\mathbb{R}^2_+} p w(y) w(y - \frac{s_{k-1} - s_k}{\varepsilon} e_1) \frac{\partial w(y)}{\partial y_1}\\
&+ \varepsilon^2 \int_{\mathbb{R}^2_+} p w(y)^{p-1} \frac{\partial w(y)}{\partial y_1} v_k^{(3)} + O(\varepsilon^3).
\end{aligned}$$

Next by the definition of $v_i^{(3)}$, we can get that

$$\begin{aligned}
\int_{\mathbb{R}^2_+} p w(y)^{p-1} \frac{\partial w(y)}{\partial y_1} v_i^{(3)} dy &= \int_{\mathbb{R}^2_+} -(\Delta - 1) \frac{\partial w(y)}{\partial y_1} v_i^{(3)}\\
&= - \int_{\partial \mathbb{R}^2_+} \frac{\partial w(y)}{\partial y_1} \frac{\partial v_i^{(3)}}{\partial y_2} - v_i^{(3)} \frac{\partial}{\partial y_2} \frac{\partial w(y)}{\partial y_1} dy\\
&= -\frac{1}{3} \int_{\mathbb{R}} (\frac{w'(|y|)}{|y|})^2 \rho^{(3)}(P_i) y_1^4 dy_1\\
&= -\nu_2 \rho^{(3)}(P_i) = -\nu_2 H'(\gamma(s_i)), \qquad (5.12)
\end{aligned}$$

where $\nu_2 = \frac{1}{3}\int_{\mathbb{R}}(\frac{w'}{|y|})^2 y_1^4 > 0$ is a positive constant.

Recall that the interaction function is defined by

$$\Psi(s) = -\int_{\mathbb{R}^2_+} pw(y-(s,0))w(y)^{p-1}\frac{\partial w(y)}{\partial y_1}dy, \tag{5.13}$$

Combining (5.9), (5.10),(5.11), (5.12) and (5.13), we know that

$$\int_{\Omega_\varepsilon} I_1 Z_1 dz = -\Psi(|\frac{s_1-s_2}{\varepsilon}|) - \varepsilon^2\nu_2 H'(\gamma(s_1)) + O(\varepsilon^3) \tag{5.14}$$

and for $i = 2, \cdots, k-1$

$$\int_{\Omega_\varepsilon} I_1 Z_i dz = \Psi(|\frac{s_i-s_{i-1}}{\varepsilon}|) - \Psi(|\frac{s_i-s_{i+1}}{\varepsilon}|) - \varepsilon^2\nu_2 H'(\gamma(s_i)) + O(\varepsilon^3) \tag{5.15}$$

and

$$\int_{\Omega_\varepsilon} I_1 Z_k dz = \Psi(|\frac{s_k-s_{k-1}}{\varepsilon}|) - \varepsilon^2\nu_2 H'(\gamma(s_i)) + O(\varepsilon^3). \tag{5.16}$$

The results follows from (5.7), (5.8) and (5.14)-(5.16).

□

From Lemma 5.1, the problem (5.3) is reduced to the following system:

$$\begin{cases} \Psi_1(|\frac{s_1-s_2}{\varepsilon}|) + \varepsilon^2 H'(\gamma(s_1)) = O(\varepsilon^3), \\ \Psi_1(|\frac{s_3-s_2}{\varepsilon}|) - \Psi_1(|\frac{s_2-s_1}{\varepsilon}|) + \varepsilon^2 H'(\gamma(s_2)) = O(\varepsilon^3), \\ \qquad \vdots \\ \Psi_1(|\frac{s_k-s_{k-1}}{\varepsilon}|) - \Psi_1(|\frac{s_{k-1}-s_{k-2}}{\varepsilon}|) + \varepsilon^2 H'(\gamma(s_{k-1})) = O(\varepsilon^3), \\ -\Psi_1(|\frac{s_k-s_{k-1}}{\varepsilon}|) + \varepsilon^2 H'(\gamma(s_k)) = O(\varepsilon^3). \end{cases}$$

where we denote by

$$\Psi_1(s) = \nu_2^{-1}\Psi(s). \tag{5.17}$$

By summing up the first i equations, one has

$$\begin{cases} \Psi_1(\frac{s_{i+1}-s_i}{\varepsilon}) + \sum_{j=1}^{i}\varepsilon^2 H'(\gamma(s_j)) = O(i\varepsilon^3) \text{ for } i = 1, \cdots, k-1, \\ \sum_{i=1}^{k}\varepsilon^2 H'(\gamma(s_i)) = O(k\varepsilon^3). \end{cases} \tag{5.18}$$

We need to find a solution of (5.18) in (1.15).

6. Solving the nonlinear system

Our aim in the rest of this paper is to find a solution $\{s_i\}$ to the above non-linear system (5.18) in (1.15).

Observe that the linearized matrix of the above system at main order is degenerate, thus the terms containing $H'(\gamma(s))$ will play an important role. We will explain how we solve system (5.18). The novelty of this paper is to consider the above system as a discretization of an ODE system. In order to explain this idea, we first introduce some notations.

Let

$$s = G(b)$$

be the solution of $\Psi_1(s) = b$. Since $\Psi_1(s) = C_n s^{-\frac{1}{2}} e^{-s}(1 + o(1))$ as $s \to \infty$, using this asymptotic behaviour of Ψ_1, one has the following:

$$G(b) = -(1 + O(\frac{\ln(-\ln b)}{\ln b}))\ln b, \text{ as } b \to 0. \tag{6.1}$$

Then the above reduced system (5.18) is equivalent to the following system:

$$\begin{cases} s_{i+1} - s_i = \varepsilon G(-\sum_{j=1}^{i} \varepsilon^2 H'(\gamma(s_j)) + O(\varepsilon^3 i)), \text{ for } i = 1, \cdots, k-1 \\ s_k - s_{k-1} = \varepsilon G(\varepsilon^2 H'(\gamma(s_k)) + O(\varepsilon^3 k)). \end{cases} \tag{6.2}$$

Let $h = -\varepsilon \ln \varepsilon$ be the boot size, if we denote by $s_i = x(t_i)$ where $t_i = (i-1)h$, then from the above system (6.2),

$$\begin{cases} \frac{x(t_{i+1}) - x(t_i)}{h} = -\frac{1}{\ln \varepsilon} G(-\frac{\varepsilon}{\ln \varepsilon}(-\sum_{j=1}^{i} H'(\gamma(x(t_j)))h) + O(\varepsilon^3 i)), \\ \frac{x(t_k) - x(t_{k-1})}{h} = -\frac{1}{\ln \varepsilon} G(\varepsilon^2 H'(\gamma(x(t_k))) + O(\varepsilon^3 k)). \end{cases} \tag{6.3}$$

In order the solve the above system, we consider the limiting case of the above system, i.e. view $\frac{x(t_{i+1}) - x(t_i)}{h}$ as $x'(t)$ and $\sum_{j=1}^{i} H'(\gamma(x(t_j)))h$ as $\int_0^t H'(\gamma(x(t)))dt$, and introduce the following ODE:

$$\begin{cases} \frac{dx}{dt} = -\frac{1}{\ln \varepsilon} G(\frac{\varepsilon}{\ln \varepsilon}\rho(t)), \\ \frac{d\rho}{dt} = H'(\gamma(x(t))), \\ \rho(0) = 0, \ \rho(b_\varepsilon) = \rho_b, \\ x'(b_\varepsilon) = -\frac{1}{\ln \varepsilon} G(\varepsilon^2 H'(\gamma(x(b_\varepsilon)))), \end{cases} \tag{6.4}$$

where $b_\varepsilon = (k-1)h = [\frac{b}{h}]h = b + O(h)$.

One can see that the above second order ODE has three initial conditions. Besides the two end point initial values, there is an extra condition, i.e. the last equation of (6.4), which in fact comes from the last equation of (6.2). This ODE with extra initial condition is not always solvable. It turns out that this extra condition corresponds to some balancing condition of the curvature of the segment γ. In order to solve this ODE, we need assumption (H_1) on γ. For this ODE, we have the following existence result:

Lemma 6.1. *Under the assumption (H_1), there exists $\varepsilon_0 > 0$, such that for every $\varepsilon < \varepsilon_0$, there exist $\rho_b = \rho_b(\varepsilon) < 0$, such that the above ODE (6.4) is solvable. Moreover, ρ_b satisfies the following asymptotic behaviour:*

$$\rho_b = -(H'(\gamma(b_\varepsilon)) + O(\frac{\ln(-\ln \varepsilon)}{\ln \varepsilon}))h. \tag{6.5}$$

Proof. From the asymptotic behaviour of G, we know that the first equation of (6.4) is

$$\begin{aligned} \frac{dx}{dt} &= -\frac{1}{\ln \varepsilon} G(\frac{\varepsilon}{\ln \varepsilon}\rho(t)) \\ &= (1 + O(\frac{\ln(-\ln \varepsilon)}{\ln \varepsilon}))(a_1 \ln(-\rho(t)) + a_2) \end{aligned}$$

where

$$a_1 = \frac{1}{\ln \varepsilon}, \ a_2 = 1 - \frac{\ln(-\ln \varepsilon)}{\ln \varepsilon}.$$

Integrating the above equation from b_ε to t, one has

$$\begin{aligned} x(t)-x(b_\varepsilon) &= \int_{b_\varepsilon}^{t} -\frac{1}{\ln\varepsilon}G(\frac{\varepsilon}{\ln\varepsilon}\rho(t))dt \\ &= (1+O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}))[a_2(t-b_\varepsilon)+a_1\int_{b_\varepsilon}^{t}\ln(-\rho(t))dt]. \end{aligned}$$

Plugging the expression for $x(t)$ into the second equation,

$$\rho'(t)=H'(\gamma(x(b_\varepsilon))+(1+O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}))[a_2(t-b_\varepsilon)+a_1\int_{b_\varepsilon}^{t}\ln(-\rho(t))dt]). \quad (6.6)$$

By the boundary condition $\rho(0)=0, \rho(b_\varepsilon)=\rho_b$, we have

$$\int_{b_\varepsilon}^{0} H'(\gamma(x(b_\varepsilon)+(1+O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}))[a_2(t-b_\varepsilon)+a_1\int_{b_\varepsilon}^{t}\ln(-\rho(t))dt]))dt=-\rho_b. \quad (6.7)$$

By Taylor's expansion,

$$\begin{aligned} &H'(\gamma(x(b_\varepsilon)+(1+O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}))[a_2(t-b_\varepsilon)+a_1\int_{b}^{t}\ln(-\rho(t))dt])) \\ &=H'(\gamma(x(b_\varepsilon)+a_2(t-b)+O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}))) \\ &=H'(\gamma(x(b_\varepsilon)+a_2(t-b_\varepsilon)))+O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}). \end{aligned}$$

So from (6.7) and the above equation, we have

$$\begin{aligned} \int_{b_\varepsilon}^{0} H'(\gamma(x(t)))dt &= \int_{b_\varepsilon}^{0} H'(\gamma(x(b_\varepsilon)+a_2(t-b)))dt+O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}) \\ &= H(\gamma(x(b_\varepsilon)-a_2b_\varepsilon))-H(\gamma(x(b_\varepsilon)))+O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}) \\ &= \rho_b. \end{aligned} \quad (6.8)$$

Since by the third boundary condition

$$x'(b_\varepsilon)=-\frac{1}{\ln\varepsilon}G(\varepsilon^2H'(\gamma(x(b_\varepsilon)))), \quad (6.9)$$

one can get that

$$\rho_b=H'(\gamma(x(b_\varepsilon)))\varepsilon\ln\varepsilon. \quad (6.10)$$

We assume that

$$\rho_b=(H'(\gamma(b_\varepsilon))+\rho_\varepsilon)\varepsilon\ln\varepsilon, \quad (6.11)$$

then

$$x(b_\varepsilon)=b_\varepsilon+\frac{(1+o(1))\rho_\varepsilon}{H''(\gamma(b_\varepsilon))}. \quad (6.12)$$

Using (6.12), (6.8) is reduced to the following:

$$\begin{aligned} &H(\gamma(0))-H(\gamma(b_\varepsilon))+\frac{H'(\gamma(0))-H'(\gamma(b_\varepsilon))}{H''(\gamma(b_\varepsilon))}\rho_\varepsilon \quad (6.13) \\ &+o(\rho_\varepsilon)+O(\rho_\varepsilon^2)=O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}). \end{aligned}$$

By the assumption (H_1)

$$H(\gamma(0))=H(\gamma(b)),\ H'(\gamma(0))\neq H'(\gamma(b)), \quad (6.14)$$

and

$$H''(\gamma) \geq c_0 > 0,\ b_\varepsilon = b + O(h), \tag{6.15}$$

the above equation is uniquely solvable with

$$\rho_\varepsilon = O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}). \tag{6.16}$$

So there exists unique $\rho_b = (H'(\gamma(b_\varepsilon)) + O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}))\varepsilon\ln\varepsilon$ such that (6.4) is solvable, and we have

$$x(0) = O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}),\ x(b_\varepsilon) = b_\varepsilon + O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}). \tag{6.17}$$

□

We will use the solution of the ODE to approximate the solution of (6.2). In order to obtain a good approximate solution, one need to control the error of

$$\sum_{j=1}^{i} H'(\gamma(x(t_j)))h - \int_0^{t_{i+1}} H'(\gamma(x(t)))dt.$$

So we will use the midpoint Riemann sum approximation of integrals which will give us

$$\sum_{j=1}^{i} H'(\gamma(x(t_j)))h - \int_0^{t_{i+1}} H'(\gamma(x(t)))dt = O(h^2). \tag{6.18}$$

To be more specific, we will choose the approximate solution to be the following:

$$x_i^0 = x(\bar{t}_i),\ \bar{t}_i = \frac{t_i + t_{i+1}}{2}, i = 1, \cdots, k-1, \tag{6.19}$$

and

$$x_k^0 = x_{k-1}^0 + \varepsilon G(\frac{\varepsilon}{\ln\varepsilon}\rho_b) \tag{6.20}$$

where $x(t)$ is the solution determined by the ODE (6.4).

We want to find the solution to (6.2) of the form

$$s_i = x_i^0 + y_i. \tag{6.21}$$

Then y_i will satisfy the following equation:

$$\begin{cases} y_{i+1} - y_i = -E_i + \varepsilon\Big(G(-\varepsilon^2\sum_{j=1}^{i} H'(\gamma(x_j^0 + y_j)) + O(\varepsilon^3 i)) - G(-\varepsilon^2\sum_{j=1}^{i} H'(\gamma(x_j^0)))\Big), \\ \qquad \text{for } i = 1, \cdots, k-1 \\ \varepsilon^2\sum_{j=1}^{k} H''(\gamma(x_j^0))y_j + O(\varepsilon^2)\sum_{j=1}^{k}|y_j|^2 = -E_k + O(\varepsilon^3 k), \end{cases} \tag{6.22}$$

where

$$E_i = x_{i+1}^0 - x_i^0 - \varepsilon G(-\varepsilon^2\sum_{j=1}^{i} H'(\gamma(x_j^0)))$$

for $i = 1, \cdots, k-1$, and

$$E_k = \varepsilon^2\sum_{j=1}^{k} H'(\gamma(x_j^0)).$$

First we show that the approximate solution we choose is indeed a good approximate solution, i.e. the error E_i is small enough. In fact, we have the following error estimate:

Lemma 6.2.

$$E_i = x_{i+1}^0 - x_i^0 - \varepsilon G(-\varepsilon^2 \sum_{j=1}^{i} H'(\gamma(x_j^0))) = O(\varepsilon) \tag{6.23}$$

for $i = 1, \cdots, k-1$, *and*

$$E_k = \varepsilon^2 \sum_{j=1}^{k} H'(\gamma(x_j^0)) = O(\varepsilon^2 \frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}). \tag{6.24}$$

Moreover, the following estimate holds:

$$\sum_{i=1}^{k-1} |E_i| = O(\varepsilon). \tag{6.25}$$

Proof. First for $i = k-1$, we have

$$\begin{aligned} & x_k^0 - x_{k-1}^0 - \varepsilon G(-\varepsilon^2 \sum_{j=1}^{k-1} H'(\gamma(x_j^0))) \\ &= \varepsilon G(\frac{\varepsilon}{\ln\varepsilon}\rho_b) - \varepsilon G(-\varepsilon^2 \sum_{j=1}^{k-1} H'(\gamma(x_j^0))) \\ &= O(\frac{\varepsilon}{\rho_b})|\rho_b - \sum_{j=1}^{k-1} H'(\gamma()x_j^0)h| \end{aligned}$$

Since we choose the midpoint approximation, we have for $i = 1, \cdots, k-2$,

$$\rho(t_{i+1}) - \sum_{j=1}^{i} H'(\gamma(x_j^0))h = O(h^2), \tag{6.26}$$

and

$$\begin{aligned} \sum_{j=1}^{k} H'(\gamma(x_j^0))h &= (\sum_{j=1}^{k-1} H'(\gamma(x_j^0))h - \rho(t_k)) + (H'(\gamma(x_k^0))h + \rho(t_k)) \\ &= O(h^2) + O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon})h = O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon})h. \end{aligned} \tag{6.27}$$

By (6.26) and (6.27), and recall that $\rho_b = O(h)$, one can obtain that

$$E_{k-1} = x_k^0 - x_{k-1}^0 - \varepsilon G(-\varepsilon^2 \sum_{j=1}^{k-1} H'(\gamma(x_j^0))) = O(\varepsilon h)$$

and

$$E_k = O(\varepsilon^2 \frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}). \tag{6.28}$$

Next by the equation satisfied by $\rho(t)$, we can get that

$$\rho(t_i) = O(\min\{i, k-i+1\}h) \tag{6.29}$$

so for $i = 1, \cdots, k-2$

$$\begin{aligned}
&x_{i+1}^0 - x_i^0 - \varepsilon G(-\varepsilon^2 \sum_{j=1}^{i} H'(\gamma(x_j^0))) \\
&= \int_{\bar{t}_i}^{\bar{t}_{i+1}} -\frac{1}{\ln\varepsilon} G(\frac{\varepsilon}{\ln\varepsilon}\rho(t))dt - \varepsilon G(-\varepsilon^2 \sum_{j=1}^{i} H'(\gamma(x_j^0))) \\
&= -\frac{1}{\ln\varepsilon} G(\frac{\varepsilon}{\ln\varepsilon}\rho(t_{i+1}))h - \varepsilon G(-\varepsilon^2 \sum_{j=1}^{i} H'(\gamma(x_j^0))) + O(\frac{\rho''\rho - (\rho')^2}{|\ln\varepsilon|\rho^2}(t_{i+1}))h^3 \\
&= \varepsilon(G(\frac{\varepsilon}{\ln\varepsilon}\rho(t_{i+1})) - G(-\varepsilon^2 \sum_{j=1}^{i} H'(\gamma(x_j^0)))) + O(\frac{\rho''\rho - (\rho')^2}{|\ln\varepsilon|\rho^2}(t_{i+1}))h^3 \\
&= O(\frac{\varepsilon}{\rho(t_{i+1})})(\rho(t_{i+1}) - \sum_{j=1}^{i} H'(\gamma(x_j^0))h) + O(\frac{\rho''\rho - (\rho')^2}{|\ln\varepsilon|\rho^2}(t_{i+1}))h^3 \\
&= O(\frac{\varepsilon h}{\min\{i, k-i+1\}}) + O(\varepsilon)(\frac{1}{\min\{i, k-i+1\}^2} + \frac{h}{\min\{i, k-i+1\}}) \\
&= O(\varepsilon).
\end{aligned}$$

Morevore, from the above estimate, we have

$$\sum_{j=1}^{i} E_j = O(\varepsilon), \text{ for } i = 1, \cdots, k-1.$$

□

Finally, we will show that equation (6.22) is solvable.

Lemma 6.3. *There exists $\varepsilon_0 > 0$, such that for $\varepsilon < \varepsilon_0$, there exists a solution $\{y_i\}_{1\le i\le k}$ to (6.22) such that*

$$\|y\|_\infty \le C\varepsilon \ln(-\ln\varepsilon). \tag{6.30}$$

Proof. For $\|y\|_\infty << \varepsilon|\ln\varepsilon|$, we have

$$\begin{aligned}
&\varepsilon G(-\varepsilon^2 \sum_{j=1}^{i} H'(\gamma(x_j^0 + y_j)) + O(\varepsilon^3 i)) - \varepsilon G(-\varepsilon^2 \sum_{j=1}^{i} H'(\gamma(x_j^0))) \\
&= -\varepsilon(\frac{\sum_{j=1}^{i} H''(\gamma(x_j^0))y_j}{\sum_{j=1}^{i} H'(\gamma(x_j^0))}) + O(\frac{\varepsilon i|y|_{j\le i}^2}{\sum_{j=1}^{i} H'(\gamma(x_j^0))}) + O(\frac{\varepsilon^2 i}{\sum_{j=1}^{i} H'(\gamma(x_j^0))})
\end{aligned}$$

The equations (6.22) for y_i can be rewritten as follows:

$$\begin{cases}
y_{i+1} - y_i + \varepsilon\frac{\sum_{j=1}^{i} H''(\gamma(x_j^0))y_j}{\sum_{j=1}^{i} H'(\gamma(x_j^0))} = -E_i + O(\frac{\varepsilon i|y|_{j\le i}^2}{\sum_{j=1}^{i} H'(\gamma(x_j^0))}) + O(\frac{\varepsilon^2 i}{\sum_{j=1}^{i} H'(\gamma(x_j^0))}) \\
\qquad \text{for } i = 1, \cdots, k-1 \\
\sum_{j=1}^{k} H''(\gamma(x_j^0))y_j + \sum_{j=1}^{k} H'''(\gamma(x_j^0))y_j^2 = O(\varepsilon k) + O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}).
\end{cases} \tag{6.31}$$

We will show that one can first solve y_2 to y_k in terms of y_1 from the first $k-1$ equations, and finally solve y_1 by the k-th equation of (6.31).

For $1 \le l \le i_0 = (1-\delta)k$ where $\delta > 0$ is a small number to be determined later, we have

$$\begin{aligned}
&y_{l+1} - y_1 + \varepsilon \sum_{i=1}^{l} \frac{\sum_{j=1}^{i} H''(\gamma(x_j^0))y_j}{\sum_{j=1}^{i} H'(\gamma(x_j^0))} \\
&= \sum_{i=1}^{l} E_i + \sum_{i=1}^{l} \frac{\varepsilon i}{\sum_{j=1}^{i} H'(\gamma(x_j^0))} |y|_{i\le l}^2 + \sum_{i=1}^{l} \frac{\varepsilon^2 i}{\sum_{j=1}^{i} H'(\gamma(x_j^0))} \\
&= O(\varepsilon) + \sum_{i=1}^{l} \frac{\varepsilon i |y|_{i\le l}^2}{\min\{i, k-i+1\}} + \sum_{i=1}^{l} O(\frac{\varepsilon^2 i}{\min\{i, k-i+1\}}) \\
&= O(\varepsilon) + O(\frac{\varepsilon l}{\delta})|y|_{i\le l}^2.
\end{aligned}$$

where we denote by

$$|y|_{i_1\le i\le i_2} = \sup_{i_1\le i\le i_2} |y_i|.$$

Moreover,

$$\begin{aligned}
\varepsilon \sum_{i=1}^{l} \frac{\sum_{j=1}^{i} H''(\gamma(x_j^0))y_j}{\sum_{j=1}^{i} H'(\gamma(x_j^0))} &= \varepsilon \sum_{i=1}^{l} O(\frac{i|y|_{i\le l}}{\min\{i, k-i+1\}}) \\
&= O(\frac{\varepsilon l |y|_{i\le l}}{\delta}) = o(1)|y|_{i\le l}.
\end{aligned}$$

Thus one can get that for $l \le i_0$

$$y_l = y_1 + o(1)|y|_{i\le i_0} + o(1)|y|_{i\le i_0}^2 + O(\varepsilon) \tag{6.32}$$

So we can get that

$$y_i = (1+o(1))y_1 + O(\varepsilon), i = 2, \cdots, i_0. \tag{6.33}$$

For $l > i_0$, we have the following:

$$\begin{aligned}
y_{l+1} - y_1 &= -\varepsilon \sum_{i=i_0+1}^{l} \frac{\sum_{j=1}^{i} H''(\gamma(x_j^0))y_j}{\sum_{j=1}^{i} H'(\gamma(x_j^0))} \\
&+ O(\varepsilon) + O(|y|_{i_0<i\le l}^2) + O(\frac{\varepsilon l}{\delta})|y|_{i\le i_0}^2 + o(1)|y|_{i\le i_0} \\
&= C_0\delta|y|_{i_0<i\le l} + O(|y|_{i_0<i\le l}^2) + O(|y|_{i\le i_0}) + O(\varepsilon)
\end{aligned}$$

for some C_0 independent of ε and δ. So for $i_0 < i \le k$,

$$y_i = O(y_1) + C_0\delta|y|_{i_0<i\le l} + O(|y|_{i_0<i\le l}^2) + O(\varepsilon) \tag{6.34}$$

If $\delta > 0$ is small such that $C_0\delta < \frac{1}{4}$, then the above system is solvable with

$$y_i = O(y_1) + O(\varepsilon). \tag{6.35}$$

From the last equation, we have

$$\begin{aligned}&\sum_{i=1}^{i_0} H''(\gamma(x_i^0))y_i + \sum_{i=i_0+1}^{k} H''(\gamma(x_i^0))y_i + O(k|y_1|^2) + O(k\varepsilon^2)\\&= (\sum_{i=1}^{i_0} H''(\gamma(x_i^0))(1+o(1))y_1 + O(\delta k|y_1|) + O(k\varepsilon) + O(k|y_1|^2)\\&= O(\frac{\ln(-\ln\varepsilon)}{\ln\varepsilon}).\end{aligned}$$

Thus by the assumption (H_1), the equation is reduced to

$$y_1 = o(1)y_1 + O(\delta)|y_1| + O(|y_1|^2) + O(\varepsilon\ln(-\ln\varepsilon))$$

If we further choose δ small enough but independent of ε such that $O(\delta)|y_1| < \frac{1}{2}|y_1|$, it is easy to see that by contraction mapping, the above equation has a solution and satisfies

$$y_1 = O(\varepsilon\ln(-\ln\varepsilon)). \tag{6.36}$$

Thus we get that there exists a solution to (6.22) with

$$\|y\|_\infty \le C\varepsilon\ln(-\ln\varepsilon) << \varepsilon|\ln\varepsilon|.$$

Thus we have proved the existence of solution to (6.22). □

7. Appendix: Proof of Proposition 3.1

In this appendix, we shall give a proof of Proposition 3.1. The proof is rather standard. It follows from argument in [5] and [29]. It is based on Fredholm Alternative Theorem for compact operator and an a-priori estimate.

First we need an estimate on the following matrix M defined by

$$M_{ij} = \int_{\Omega_\varepsilon} Z_i Z_j dz,\ i,j = 1,\cdots,k. \tag{7.1}$$

Lemma 7.1. *For ε sufficiently small, given any vector $\vec{b} \in \mathbb{R}^k$, there exists a unique vector $\vec{\beta} \in \mathbb{R}^k$, such that $M\vec{\beta} = \vec{b}$. Moreover,*

$$\|\vec{\beta}\|_\infty \le C\|\vec{b}\|_\infty \tag{7.2}$$

for some constant C independent of ε.

Proof. To prove the existence, it is sufficient to prove the a priori estimate (7.2). Suppose that $|\beta_i| = \|\beta\|_\infty$, we have

$$\sum_{i=1}^{k} M_{ij}\beta_j = b_i.$$

For the entries M_{ij}, from the definition of Λ_k, and the exponential decay property of Z_i, we know that

$$M_{ii} = \int_{\Omega_\varepsilon} Z_i^2 dz = (1+o(1))\int_{\mathbb{R}^2_+} (\frac{\partial w}{\partial y_1})^2 dy > c_0 > 0,$$

and

$$\sum_{j\neq i} |M_{ij}| \le C\sum_{j\neq i} e^{-\frac{|P_i-P_j|}{2}} = o(1).$$

Hence for ε small, we have

$$c_0\|\vec{\beta}\|_\infty \le c_0|\vec{\beta}_i| \le \sum_{j\neq i}|M_{ij}||\vec{\beta}_j| + |b_i| \le o(1)\|\vec{\beta}\|_\infty + \|\vec{b}\|_\infty$$

from which the desired result follows. □

Next we need the following a priori estimate:

Lemma 7.2. *Let $h \in L^2(\Omega_\varepsilon)$ with $\|h\|_*$ bounded and assume that $(\psi, \{c_i\})$ is a solution to (3.4). Then there exist positive numbers ε_0 and C, such that for all $\varepsilon \le \varepsilon_0$, for any points P_i, $i = 1, \dots, k$ given by (1.15) , one has*

$$\|\psi\|_* \le C\|h\|_*. \tag{7.3}$$

Proof. We argue by contradiction. Assume there exists ψ solution to (3.4) and

$$\|h\|_* \to 0, \quad \|\psi\|_* = 1.$$

We prove that

$$c_i \to 0 \text{ for } i = 1, \cdots, k. \tag{7.4}$$

Multiplying the equation in (3.4) against Z_j and integrating in Ω_ε, we get

$$\int_{\Omega_\varepsilon} L_\varepsilon \psi Z_j(z) = \int_{\Omega_\varepsilon} hZ_j + M(c_j),$$

By the exponentially decay of Z_i, we first know that

$$|\int_{\Omega_\varepsilon} hZ_j| \le C\|h\|_*.$$

Here and in what follows, C stands for a positive constant independent of ε, as $\varepsilon \to 0$. Secondly, by the equation satisfied by $\mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i)$,

$$\begin{aligned}
&\int_{\Omega_\varepsilon} L_\varepsilon \psi Z_i dz = \int_{\Omega_\varepsilon} (\Delta\psi - \psi + p(\sum_{i=1}^k \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i))^{p-1}\psi) Z_i dz \\
&= \int_{\Omega_\varepsilon} (\Delta Z_i - Z_i + p(\sum_{i=-k}^k \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i))^{p-1} Z_i)\psi dz \\
&= \int_{\Omega_\varepsilon} p(\sum_{i=1}^k [\mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i))^{p-1} \frac{\partial \mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_i)}{\partial\tau} - pw(z-P_i)^{p-1}\frac{\partial w(z-P_i)}{\partial\tau}]\psi dz \\
&\le C\int_{B_{|\ln\varepsilon|}(P_i)} |\frac{\partial w(z-P_i)}{\partial\tau}||O(\varepsilon) + w(z-P_i)^{p-2}\sum_{j\neq i}\mathcal{P}_{\Omega_{\varepsilon,P_i}} w(z-P_j)||\psi| dz \\
&+ \int_{\Omega_\varepsilon/B_{|\ln\varepsilon|}(P_i)} \frac{\partial |w(z-p_i)}{\partial\tau}|[\sum_{j=1}^k w_j^{p-1} + O(\varepsilon)\sum_{j=1}^k e^{-\mu|z-P_j|}]|\psi| dz \\
&\le C\|\psi\|_*(O(\varepsilon) + O(\varepsilon^{\frac{p-\eta}{2}})) \\
&\le C\varepsilon\|\psi\|_*
\end{aligned}$$

if we choose η small enough such that $p - \eta > 2$. This can be done since $p > 2$.

Since M is invertible and $\|M^{-1}\| \le C$, we get that

$$|c_i| \le C(\|h\|_* + O(\varepsilon)\|\psi\|_*). \tag{7.5}$$

Thus we get the validity of (7.4), since we are assuming $\|\psi\|_* = 1$ and $\|h\|_* \to 0$.

Let now $\mu \in (0,1)$. It is easy to check that the function

$$W := \sum_{i=1}^{k} e^{-\mu\,|\cdot - P_i|},$$

satisfies

$$L_\varepsilon\, W \le \frac{1}{2}\,(\mu^2 - 1)\, W\,,$$

in $\Omega_\varepsilon \setminus \cup_{j=1,\dots,k} B(P_j, R)$ provided R is fixed large enough (independently of ε). Hence the function W can be used as a barrier to prove the pointwise estimate

$$|\phi|(x) \le C\,\left(\|L_\varepsilon\,\psi\|_* + \sup_j \|\psi\|_{L^\infty(B(p_j,R)\cap\Omega_\varepsilon)}\right) W(x)\,, \tag{7.6}$$

for all $z \in \Omega_\varepsilon \setminus \cup_j B(P_j, R)$.

Granted these preliminary estimates, the proof of the result goes by contradiction. Let us assume there exist a sequence of $\varepsilon \to 0$ and a sequence of solutions of (3.4) for which the inequality is not true. The problem being linear, we can reduce to the case where we have a sequence $\varepsilon^{(n)}$ tending to 0 and sequences $h^{(n)}$, $\psi^{(n)}, c^{(n)}$ such that

$$\|h^{(n)}\|_* \to 0, \quad \text{and} \quad \|\psi^{(n)}\|_* = 1.$$

But (7.4) implies that we also have

$$\|c^{(n)}\|_* \to 0\,.$$

Then (7.6) implies that there exists $P_i^{(n)}$ such that

$$\|\psi^{(n)}\|_{L^\infty(B(P_i^{(n)},R))} \ge C, \tag{7.7}$$

for some fixed constant $C > 0$. Using elliptic estimates together with Ascoli-Arzela's theorem, we can find a sequence $P_i^{(n)}$ and we can extract, from the sequence $\psi_i^{(n)}(\cdot - P_i^{(n)})$ a subsequence which will converge (on compact) to ψ_∞ a solution of

$$\begin{cases} \left(\Delta - 1 + p\,w^{p-1}\right)\,\psi_\infty = 0 \text{ in } \mathbb{R}^2_+, \\ \frac{\partial \psi_\infty}{\partial y_2} = 0, \text{ on } \partial\mathbb{R}^2_+ \end{cases}$$

which is bounded by a constant times $e^{-\mu\,|x|}$, with $\mu > 0$. Moreover, since $\psi_i^{(n)}$ satisfies the orthogonality conditions in (3.4), the limit function ψ_∞ also satisfies

$$\int_{\mathbb{R}^2_+} \psi_\infty\, \frac{\partial w}{\partial y_1}\, dx = 0\,.$$

But the solution w being non-degenerate, this implies that $\psi_\infty \equiv 0$, which is certainly in contradiction with (7.7) which implies that ψ_∞ is not identically equal to 0.

Having reached a contradiction, this completes the proof of the lemma. □

We can now prove Proposition 3.1.

Proof of Proposition 3.1. Consider the space

$$\mathcal{H} = \{u \in H^2_N(\Omega_\varepsilon)\,:\, \int_{\Omega_\varepsilon} u Z_i = 0, \quad i = 1, \dots, k\}.$$

Notice that the problem (3.4) in ψ gets re-written as

$$\psi + K(\psi) = \bar{h} \quad \text{in} \quad \mathcal{H} \tag{7.8}$$

where $\bar{h}$ is defined by duality and $K : \mathcal{H} \to \mathcal{H}$ is a linear compact operator. Using Fredholm's alternative, showing that equation (7.8) has a unique solution for each $\bar{h}$ is equivalent to showing that the equation has a unique solution for $\bar{h} = 0$, which in turn follows from Proposition 7.2. The estimate (3.6) follows directly from Proposition 7.2. This concludes the proof of Proposition (3.1).

References

[1] Ambrosetti, A; Malchiodi A; Ni, W.-M. Singularly perturbed elliptic equations with symmetry: Existence of solutions concentrating on spheres, I, *Comm. Math. Phys.* **235** (2003), 427 466. MR 1974510

[2] Ambrosetti, A; Malchiodi A; Ni, W.-M. Singularly perturbed elliptic equations with symmetry: Existence of solutions concentrating on spheres, II, *Indiana Univ. Math. J.*,**53** (2004), no.2, 297-329.

[3] Ao, W., Musso, M. and Wei, J. On spikes concentrating on line segments to a semilinear Neumann problem, *Journal of Differential Equations*, 251(2011), no. 4-5, 881-901.

[4] Ao, W. , Musso, M. and Wei, J. Triple junction solutions for a singularly perturbed Neumann problem, *SIAM Journal on Mathematical Analysis*, 43(2011), no. 6, 2519-2541.

[5] Ao, W.W., Wei, J.C. and Zeng, J. An optimal bound on the number of interior spike solutions for the Lin-Ni-Takagi problem, *Journal of Functional Analysis*, 265(2013), no.7, 1324-1356.

[6] Bates, P.; Fusco, G. Equilibria with many nuclei for the Cahn-Hilliard equation. *J. Diff. Eqns* **160** (2000), 283-356.

[7] Bates, P.; Dancer, E.N.; Shi, J. Multi-spike stationary solutions of the Cahn-Hilliard equation in higher-dimension and instability. *Adv. Diff. Eqns* **4** (1999), 1-69.

[8] Badiale M. and D'Aprile T., Concentration around a sphere for a singularly perturbed Schrödinger equation , *Nonlinear Anal. Ser. A: Theory Methods* **49** (2002), 947-985.

[9] Benci V. and D'Aprile T., The semiclassical limit of the nonlinear Schrödinger equation in a radial potential, *J. Differential Equations* **184** (2002), 109-138.

[10] Butscher A. and Mazzeo R., CMC hypersurfaces condensing to geodesic segments and rays in Riemannian Manifolds, *Ann. Sc. Norm. Super. Pisa. Cl. Sci.***(5)**, Vol. XI(2012), 653-706.

[11] D'Aprile T., On a class of solutions with non-vanishing angular momentum for nonlinear Schrödinger equations, *Differential Integral Equations* **16** (2003), 349 384.

[12] T. D'Aprile and A. Pistoia, On the existence of some new positive interior spike solutions to a semilinear Neumann problem, *J. Diff. Eqns.* 248(2010), 556-573.

[13] A. Butscher, CMC surfaces in Riemannian manifolds Condensing to a compact network of curves, arXiv:0910.4442.

[14] del Pino, M.; Felmer, P.; Wei, J.-C. On the role of mean curvature in some singularly perturbed Neumann problems. *SIAM J. Math. Anal.* **31** (1999), 63-79.

[15] del Pino, M.; Felmer, P.; Wei, J.-C. On the role of distance function in some singularly perturbed problems. *Comm. PDE* **25** (2000), 155-177.

[16] del Pino, M.; Felmer, P.; Wei, J.-C. Mutiple peak solutions for some singular perturbation problems. *Cal. Var. PDE* **10** (2000), 119-134.

[17] Dancer, E.N.; Yan, S. Multipeak solutions for a singular perturbed Neumann problem. *Pacific J. Math.* **189** (1999), 241-262.

[18] Mahmoudi, F. and Malchiodi, A. Concentration on minimal sub manifolds for a singularly perturbed Neumann problem. *Adv. in Math*, 209-2 (2007), 460-525.

[19] Mahmoudi,F. and Malchiodi, A. Concentration at manifolds of arbitrary dimension for a singularly perturbed Neumann problem. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.* 17 (2006), no. 3, 279-290.

[20] A. Gierer and H. Meinhardt, *A theory of biological pattern formation*, Kybernetik (Berlin) 12 (1972), 30-39.

[21] Gui, C.-F.; Wei, J.-C. Multiple interior spike solutions for some singular perturbed Neumann problems. *J. Diff. Eqns.* **158** (1999), 1-27.

[22] Gui, C.-F.; Wei, J.-C. On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems. *Can. J. Math.* **52** (2000), 522-538.

[23] Gui, C.-F.; Wei, J.-C.; Winter, M. Multiple boundary peak solutions for some singularly perturbed Neumann problems. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **17** (2000), 249-289.

[24] Grossi, M.; Pistoia, A.; Wei, J.-C. Existence of multi-peak solutions for a semi-linear Neumann problem via non-smooth critical point theory. *Cal. Var. PDE* **11** (2000), 143-175.

[25] Kwong, M.K. Uniquness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$. *Arch. Rational Mech. Anal.* **105** (1991), 243-266.

[26] E.F. Keller and L.A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theor. Biol.* 26, 1970, 399-415.

[27] Li, Y.-Y. On a singularly perturbed equation with Neumann boundary condition. *Comm. P.D.E.* **23** (1998), 487-545.

[28] Li, Y.-Y.; Nirenberg, L. The Dirichlet problem for singularly perturbed elliptic equations. *Comm. Pure Appl. Math.* **51** (1998), 1445-1490.

[29] F.H. Lin, W.M. Ni and J.C. Wei, On the number of interior peak solutions for a singularly perturbed Neumann problem, *Comm. Pure Appl. Math.* 60 (2007), 252-281.

[30] Malchiodi, A. Concentration at curves for a singularly perturbed Neumann problem in three-dimensional domains. *Geom. Funct. Anal.* 15(2005), no.6, 1162-1222.

[31] Malchiodi, A., Some new entire solutions of semilinear elliptic equations in $\mathbb{R}^N$, *Adances in Math.* (2009), no. 6, 1843-1909.

[32] Malchiodi, A.; Montenegro, M. Boundary concentration phenomena for a singularly perturbed ellptic problem. *Comm. Pure Appl. Math.* **55** (2002), 1507-1508.

[33] Malchiodi, A.; Montenegro, M. Multidimensional boundary layers for a singularly perturbed Neumann problem. *Duke Math. J.* **124** (2004), no. 1, 105-143.

[34] Malchiodi, A.; Montenegro, M. Boundary layers of arbitrary dimension for a singularly perturbed Neumann problem. *Mat. Contemp.* 27 (2004), 117146.

[35] Malchiodi, A.; Ni, W.-M.; Wei, J.-C. Multiple clustered layer solutions for semi-linear Neumann problems on a ball. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **22** (2005), no.2, 143-163.

[36] Ni, W.-M. Diffusion, cross-diffusion, and their spike-layer steady states. *Notices of Amer. Math. Soc.* **45** (1998), 9-18.

[37] W.-M. Ni, *Qualitative properties of solutions to elliptic problems.* Stationary partial differential equations. Vol. I, 157–233, Handb. Differ. Equ., North-Holland, Amsterdam, 2004.

[38] Ni, W.-M.; Takagi, I. On the shape of least energy solution to a semilinear Neumann problem. *Comm. Pure Appl. Math.* **41** (1991), 819-851.

[39] Ni, W.-M.; Takagi, I. Locating the peaks of least energy solutions to a semilinear Neumann problem. *Duke Math. J.* **70** (1993), 247-281.

[40] Ni, W.-M.; Wei, J.-C. On the location and profile of spike-layer solutions to singularly perturbed semi-linear Dirichlet problems. *Comm. Pure Appl. Math.* **48** (1995), 731-768.

[41] Wei, J.-C. On the boundary spike layer solutions of singularly perturbed semilinear Neumann problem. *J. Diff. Eqns.* **134** (1997), 104-133.

[42] J. Wei, *Existence and Stability of Spikes for the Gierer-Meinhardt System*, in HANDBOOK OF DIFFERENTIAL EQUATIONS, Stationary Partial Differential Equations, volume 5 (edited by M. Chipot), pp. 487-585. North-Holland, Amsterdam, 2008.

[43] Wei, J.-C.; Winter, M. Stationary solutions for the Cahn-Hilliard equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **15** (1998), 459-492.

[44] Wei, J.-C.; Winter, M. Multiple boundary spike solutions for a wide class of singular perturbation problems. *J. London Math. Soc.* **59** (1999), 585-606.

[45] Wei, J.-C.; Yang, Jun, Concentration on lines for a singularly perturbed Neumann problem in two-dimensional domains, *Indiana Univ. Math. J.* 56(2007), 3025-3073.

[46] Wei, J.-C.; Yang, Jun Toda system and interior clustering line concentration for a singularly perturbed Neumann problem in two dimensional domain. *Discrete Contin. Dyn. Syst.* 22 (2008), no. 3, 465-508

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