Course 2- Homework Assignment 1 (Due Date: )

The purpose of this set of problems is to study the basic properties of the two profile functions: spikes (responsible for spotty pattern), and fronts (responsible for stripe pattern).

You only need to hand in solutions of the two problems from the following five problems. Extra credits will be given if you hand in more problems.

1. Consider the following ODE:
\[ w'' - w + w^p = 0, \quad w'(0) = 0, \quad w(\infty) = 0 \]

and its linearized operator
\[ L_0(\phi) = \phi'' - \phi + pw^{p-1}\phi \]

(a) Show that the principal eigenvalue and principal eigenfunction are given by
\[ \lambda_1 = \frac{(p-1)(p+3)}{4}, \quad \phi_1 = w^{\frac{p+1}{2}} \]

(b) As a consequence of (a), compute
\[ L_0^{-1}(w^{r-1}), \quad r = \frac{p+3}{2} \]

and
\[ \int w^{r-1}L_0^{-1}(w^{r-1}), \quad r = \frac{p+3}{2} \]

(c) Show that all eigenvalues of \( L_0 \) must be real.

2. Consider the following problem
\[ \Delta w - w + w^p = w_{rr} + \frac{N-1}{r}w_r - w + w^p = 0, \quad w'(0) = 0, \quad w(\infty) = 0 \]

Show that for \( p \geq \frac{N+2}{N-2} \) then \( w \equiv 0 \).

Hint: multiplying the equation by \( r^Nw' \) and \( r^{N-1}w \) respectively and integrating.

3. Consider the following minimization problem
\[ c = \inf E[u] \]

where
\[ E[u] = \frac{\int (|\nabla u|^2 + u^2)}{(\int |u|^{p+1})^{\frac{p+1}{p}}} \]

Let \( u \) be a minimizer and consider the following function
\[ \rho(t) := E[u + t\phi] \]
Compute \( \rho'(0) \) and \( \rho''(0) \)

4. Now we consider the front solution. Consider the following equation

\[
H'' + H - H^3 = 0, \quad H(-\infty) = -1, \quad H(+\infty) = 1
\]

(a) Show that a solution is given by

\[
H(t) = \tanh\left( \frac{t}{\sqrt{2}} \right)
\]

(b) Show that the equation in (a) is the Euler-Lagrange equation of the following energy functional

\[
E(u) = \int \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{4} (1 - u^2)^2 \right)
\]

(c) Now we consider the following Gierer-Meinhardt system with saturation

\[
H'' - H + \frac{H^2}{1 + aH^2} = 0
\]

Show that for \( 0 < a < \frac{1}{4} \) there are two positive constant steady states \( 0 < h_- < h_+ \). Now compute

\[
\int_{h_-}^{h_+} \left( -H + \frac{H^2}{1 + aH^2} \right)
\]

Find the equation for \( a = a_0 \) so that

\[
\int_{h_-}^{h_+} \left( -H + \frac{H^2}{1 + aH^2} \right) = 0
\]

Show that for such \( a_0 \) there exists a solution connecting 0 and the positive constant steady state \( h_+ \).

5. (a) Let \( H(t) = \tanh\left( \frac{t}{\sqrt{2}} \right) \). Under what conditions on \((a_1, ..., a_N, b)\) so that \( u(x) = H(a_1x_1 + ... + a_Nx_N + b) \) satisfies

\[
\Delta u + u - u^3 = 0 \text{ in } \mathbb{R}^N
\]

(b) Show that the equation in (b) is the Euler-Lagrange equation of the following energy functional

\[
E(u) = \int \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{4} (1 - u^2)^2 \right)
\]

(c) A solution \( u \) satisfying the equation in (b) is called stable if for any function with compact support \( \phi \) it holds

\[
\int \left( (|\nabla \phi|^2 + (3u^2 - 1)\phi^2) > 0
\]

Show that the function defined in (a) is stable. (The converse may not be true. Counterexample is given when \( N = 8 \).)