ON SOME NONLINEAR SCHRÖDINGER EQUATIONS IN $\mathbb{R}^N$

JUNCHENG WEI AND YUANZE WU

Abstract. In this paper, we consider the following nonlinear Schrödinger equations with the critical Sobolev exponent and mixed nonlinearities:

$$\begin{cases} -\Delta u+\lambda u=t|u|^{q-2}u+|u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \\ u\in H^1(\mathbb{R}^N), \end{cases}$$

where $N\geq 3$, $t>0$, $\lambda>0$ and $2<q<2^*=\frac{2N}{N-2}$. Based on our recent study on the normalized solutions of the above equation in [31], we prove that

(1) the above equation has two positive radial solutions for $N=3$, $2<q<4$ and $t>0$ sufficiently large, which gives a rigorous proof of the numerical conjecture in [14];

(2) there exists $t_q^*>0$ for $2<q\leq 4$ such that the above equation has ground-states for $t\geq t_q^*$ in the case of $2<q<4$ and for $t>t_4^*$ in the case of $q=4$ while, the above equation has no ground-states for $0<t<t_q^*$ for all $2<q\leq 4$, which, together with the well-known results on ground-states of the above equation, almost completely solve the existence of ground-states to the above equation, except for $N=3$, $q=4$ and $t=t_4^*$.

Moreover, based on the almost completed study on ground-states to the above equation, we introduce a new argument to study the normalized solutions of the above equation to prove that there exists $0<\bar{t}_{a,q}<+\infty$ for $2<q<2+\frac{4}{N}$ such that the above equation has no positive normalized solutions for $t>\bar{t}_{a,q}$ with $\int_{\mathbb{R}^N}|u|^2dx=a^2$, which, together with our recent study in [31], gives a completed answer to the open question proposed by Soave in [30]. Finally, as applications of our new argument, we also study the following Schrödinger equation with a partial confinement:

$$\begin{cases} -\Delta u+\lambda u+(x_1^2+x_2^2)u=|u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\ u\in H^1(\mathbb{R}^3), \quad \int_{\mathbb{R}^3}|u|^2dx=r^2, \end{cases}$$

where $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, $\frac{10}{3}<p<6$, $r>0$ is a constant and (u,λ) is a pair of unknowns with λ being a Lagrange multiplier. We prove that the above equation has a second positive solution, which is also a mountain-pass solution, for $r>0$ sufficiently small. This gives a positive answer to the open question proposed by Bellazzini et al. in [7].

Keywords: Normalized solution; Ground state; Schrödinger equation; Power-type nonlinearity.

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1. Introduction

In the celebrated paper [16], the well-known Gidas-Ni-Nirenberg theorem asserts that the positive solution of the following equation,

$$\begin{cases} -\Delta u=f(u) \quad \text{in } \mathbb{R}^N, \\ u\to 0 \quad \text{as } |x|\to+\infty, \end{cases} \tag{1.1}$$ eqn0001

must be radially symmetric up to translations under some suitable conditions on the nonlinearities $f(u)$, where $N \geq 1$. Since then, an interesting and important problem is the uniqueness of the positive solution to (1.1). Kwong proved such uniqueness result in [20] for the power-type nonlinearities $f(u) = u^{p-1} - u$ with $2 < p < 2^*$, where 2^* is the critical Sobolev exponent given by $2^* = +\infty$ for $N = 1, 2$ and $2^* = 2N/(N-2)$ for $N \geq 3$ (see the earlier papers [12] for the cubic nonlinearity $f(u) = u^3 - u$ and [25–27] for general nonlinearities). The extension of Kwong's result can be found in [24, 28, 29] and so far, to out best knowledge, the most general extension of Kwong's result is due to Serrin and Tang in [29]: The positive solution of (1.1) is unique if there exists $b > 0$ such that $\frac{f(u)-u}{u-b} > 0$ for $u \neq b$ and the quotient $\frac{f'(u)u-u}{f(u)-u}$ is nonincreasing of $u \in (b, +\infty)$, which is not the case of the mixed nonlinearities $f(u) = \mu u^{q-1} + \nu u^{p-1} - \lambda u$ with $2 < q \neq p < 2^*$ and $\mu, \nu, \lambda > 0$. In this case, (1.1) reads as

$$\begin{cases} -\Delta u + \lambda u = \mu |u|^{q-2} u + \nu |u|^{p-2} u \quad \text{in } \mathbb{R}^N, \\ u \to 0 \quad \text{as } |x| \to +\infty. \end{cases} \tag{1.2}$$ eqn0002

By rescaling, (1.2) is equivalent to

$$\begin{cases} -\Delta u + \lambda u = t |u|^{q-2} u + |u|^{p-2} u \quad \text{in } \mathbb{R}^N, \\ u \to 0 \quad \text{as } |x| \to +\infty. \end{cases} \tag{1.3}$$ eqn0003

In an interesting paper [14], Davila et al. proved that for $N = 3$, $2 < q < 4$, $p < 6$ with sufficiently close to 6 and $t > 0$ sufficiently large, (1.3) has three positive radial solutions, which yields a rather striking result that Kwong's uniqueness result is in general not true for the mixed nonlinearities. Thus, the uniqueness of the positive radial solution of (1.3) (or more general, (1.1)) remains largely open. It is worth pointing out that the mentioned papers are all devoted to the Sobolev subcritical case for $N \geq 3$, that is, $\lim_{u \to +\infty} \frac{f(u)}{u^{2^*}} = 0$.

In the Sobolev critical case for $N \geq 3$, that is, $\lim_{u \to +\infty} \frac{f(u)}{u^{2^*}} > 0$, the well-known Gidas-Ni-Nirenberg theorem still holds, that is, positive solutions must be radially symmetric up to translations. However, for $N \geq 3$, compared to the Sobolev subcritical case (cf. [8]), the existence of positive solutions of (1.1) is more complicated in the Sobolev critical case. For example, for (1.3), the special case of (1.1), the existence of positive solutions is established in [2, 4, 6, 22, 33], which can be summarized as follows:

⟨thmn0001⟩ **Theorem 1.1.** *Let $N \geq 3$ and $p = 2^*$. Then (1.3) has a positive radial solution which is also a ground-state, provided that*

- (a) $N \geq 4$, $2 < q < 2^*$ *and* $t > 0$*;*
- (b) $N = 3$, $4 < q < 6$ *and* $t > 0$*;*
- (c) $N = 3$, $2 < q \leq 4$ *and* $t > 0$ *sufficiently large.*

Theorem 1.1 is proved by adapting the classical ideas of Brezís and Nirenberg in [9], that is, using the Aubin-Talanti bubbles (cf. (2.1)) as test functions to control the energy values so that the (PS) sequences of the associated functional, corresponding to (1.3) with $p = 2^*$, are compact at the ground-state level. This strategy is invalid for $N = 3$, $2 < q \leq 4$ and $t > 0$ not sufficiently large. Thus, whether (1.3) with $p = 2^*$ always has a positive radial solution is not clear. Note that according to the concentration-compactness principle (cf. [21]), the only possible way that the (PS) sequences of the associated functional loss the compactness at the ground-state level is that they concentrate at single points and behavior like a Aubin-Talanti bubble under some suitable scalings in passing to the limit. Thus, by the energy estimates in [2, 4, 6, 22, 33], it is reasonable to think that (1.3) with $p = 2^*$ has no ground-states for $N = 3$, $2 < q \leq 4$ and $t > 0$ not sufficiently large. On the other hand, the uniqueness of positive radial solutions to (1.3) with $p = 2^*$ seems also very complicated. If $3 \leq N \leq 6$ and $(N+2)/(N-2) < q < 2^*$ then Pucci and Serrin in [28] proved that (1.3) with $p = 2^*$ has at most one positive radial solution. Recently, Akahori et al. in [1,3,4] and Coles and Gustafson in [13] proved that the radial ground-state of (1.3) with $p = 2^*$ is unique and nondegenerate for all small $t > 0$ when $N \geq 5$ and $q \in (2, 2^*)$ or $N = 3$ and $q \in (4, 2^*)$; and for all large $t > 0$ when $N \geq 3$ and $2 + 4/N < q < 2^*$. However, the uniqueness of positive radial solutions seems not true for (1.3) with $p = 2^*$ in general, since it is suggested in [14] by the numerical evidence that (1.3) with $p = 2^*$ has two positive radial solutions for $N = 3$, $2 < q < 4$ and $t > 0$ sufficiently large. Moreover, Chen et al. in [10] proved the existence of arbitrary large number of bubble-tower positive solutions of (1.3) in the slightly supercritical case when $q < 2^* < p = 2^* + \varepsilon$ with $\varepsilon > 0$ sufficiently small. We also mention the paper [15], in which the authors proved the existence of positive radial solutions to (1.3) for $2 < q < 2^* \leq p$ with $t > 0$ sufficiently large and (1.3) has no positive solutions for $2 < q < 2^* < p$ with $t > 0$ sufficiently small via ODE's methods.

Inspired by the above facts, we shall explore the existence and nonexistence of positive solutions of (1.3) with $p = 2^*$ by studying the existence and nonexistence of ground-states of (1.3) for $N = 3$ and $2 < q \leq 4$. We shall also explore the uniqueness of positive solutions of (1.3) with $p = 2^*$ by giving a rigorous proof of the numerical conjecture in [14].

Let us first introduce some necessary notations. By classical elliptic estimates, for $N \geq 3$ and $p = 2^*$, (1.3) is equivalent to

$$\begin{cases} -\Delta u + \lambda u = t|u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.4}$$ eq0001

where $t > 0$, $\lambda > 0$ and $2 < q < 2^*$. Clearly, by rescaling if necessary, it is sufficiently to consider the case $\lambda = 1$ for (1.4). Let

$$m(t) = \inf_{v \in \mathcal{N}_t} \mathcal{E}_t(v), \tag{1.5}$$ eqnewnew0009

where

$$\mathcal{E}_t(v) = \frac{1}{2}(\|\nabla v\|_2^2 + \|v\|_2^2) - \frac{t}{q}\|v\|_q^q - \frac{1}{2^*}\|v\|_{2^*}^{2^*} \tag{1.6}$$ eqn0010

is the corresponding functional of (1.4) with $\lambda = 1$ and

$$\mathcal{N}_t = \{v \in H^1(\mathbb{R}^N) \backslash \{0\} \mid \mathcal{E}_t'(v)v = 0\}$$

is the usual Nehari manifold. Here, $\|\cdot\|_p$ is the usual norm in the Lebesgue space $L^p(\mathbb{R}^N)$.

Definition 1.1. *We say that u is a ground-state of (1.4) if u is a nontrivial solution of (1.4) with $\mathcal{E}_t(u) = m(t)$.*

Now, our main result is the following.

⟨thm0002⟩ **Theorem 1.2.** *Let $\lambda = 1$, $N = 3$ and $2 < q \leq 4$. Then there exists $t_q^* > 0$, which may depend on q, such that*

(1) *(1.4) has ground-states for $t \geq t_q^*$ and has no ground-states for $0 < t < t_q^*$ in the case of $2 < q < 4$.*
(2) *(1.4) has ground-states for $t > t_4^*$ and has no ground-states for $0 < t < t_4^*$ in the case of $q = 4$.*

Moreover, if $2 < q < 4$ then there exists $t_q > 0$, which may depend on q, such that (1.4) has two positive radial solutions $u_{t,1}$ and $u_{t,2}$ for $t > t_q$, where $u_{t,1}$ is a ground-state with $\|u_{t,1}\|_\infty \sim t^{-\frac{1}{q-2}}$ and $u_{t,2}$ is a blow-up solution with

$$\|u_{t,2}\|_\infty \sim \begin{cases} t^{\frac{1}{4-q}}, & 3 < q < 4, \\ t \ln t, & q = 3, \\ t^{\frac{1}{q-2}}, & 2 < q < 3, \end{cases}$$

as $t \to +\infty$.

Remark 1.1. *Theorem 1.2, together with Theorem 1.1, almost completely solves the existence of ground-states to (1.4), except for $N = 3$, $q = 4$ and $t = t_4^*$. Moreover, Theorem 1.2 also verifies the numerical conjecture in [14].*

The proof of Theorem 1.2 is based on our very recent study on the normalized solution of (1.4) with the additional condition $\|u\|_2^2 = a^2$, where $a > 0$. We remark that we shall call u is a fixed-frequency solution of (1.4) if the frequency λ is fixed, since for the normalized solution of (1.4), the frequency λ is a part of unknowns, which appears as a Lagrange multiplier. Now, let us explain our ideas in proving Theorem 1.2. Let $\mu > 0$, $a > 0$ and (u_μ, λ_μ) be a normalized solution of (1.4) for $t = \mu$ with the additional condition $\|u_\mu\|_2^2 = a^2$, that is, (u_μ, λ_μ) is a solution of the following system:

$$\begin{cases} -\Delta u + \lambda u = \mu|u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad \|u\|_2^2 = a^2, \end{cases} \tag{1.7}$$ eqn1001

then by the Pohozaev identity satisfied by u_μ (cf. [31, (4.7)]),

$$\lambda_\mu a^2 = \lambda_\mu \|u_\mu\|_2^2 = (1 - \gamma_q)\mu\|u_\mu\|_q^q > 0, \tag{1.8}$$ eq0002

where $\gamma_q = \frac{N(q-2)}{2q}$. Let

$$v_\mu(x) = \lambda_\mu^{-\frac{N-2}{4}} u_\mu(\lambda_\mu^{-\frac{1}{2}} x), \tag{1.9}$$ eq0003

then by direct calculations, we know that v_μ is a fixed-frequency solution of (1.4) for $\lambda = 1$ and $t = \mu\lambda_\mu^{\frac{q\gamma_q - q}{2}}$. By (1.8), we also have

$$\lambda_\mu = \frac{(1-\gamma_q)\mu}{a^2} \lambda_\mu^{\frac{q\gamma_q - q}{2}} \|v_\mu\|_q^q.$$

Thus, by letting

$$t_\mu = \mu\lambda_\mu^{\frac{q\gamma_q - q}{2}}, \tag{1.10}$$ eq0010

we know that (v_μ, t_μ) solves the following system:

$$\begin{cases} -\Delta v + v = t|v|^{q-2}v + |v|^{2^*-2}v \quad \text{in } \mathbb{R}^N, \\ v \in H^1(\mathbb{R}^N), \quad t^{\frac{2}{q\gamma_q - q}-1} = \dfrac{1-\gamma_q}{a^2\mu^{\frac{2}{q-q\gamma_q}}}\|v\|_q^q. \end{cases} \tag{1.11}$$ eq0005

Clearly, if (v, t) is a solution of the system (1.11), then by letting

$$\lambda_\mu = \left(\frac{t}{\mu}\right)^{\frac{2}{q\gamma_q - q}} \quad \text{and} \quad u_\mu(x) = \lambda_\mu^{\frac{N-2}{4}} v(\lambda_\mu^{\frac{1}{2}}x), \tag{1.12}$$ eq0006

(u_μ, λ_μ) is also a normalized solution of (1.4) for $t = \mu$ with the additional condition $\|u_\mu\|_2^2 = a^2$, that is (u_μ, λ_μ) is also a normalized solution of (1.7). Thus, by our above observations, normalized solutions of (1.4) is equivalent to fixed-frequency solutions of (1.4) with another additional condition. Since we make a detail study on some special normalized solutions of (1.4) in [31], we could use these detail estimates to derive Theorem 1.2.

Our observations on the relations between fixed-frequency solutions and normalized solutions of (1.4) also bring in some new lights to study the normalized solutions of (1.4). Indeed, let v_t be a fixed-frequency solution of (1.4), then by the above observations, finding normalized solutions of (1.4) is equivalent to finding solutions of the following equation:

$$t^{\frac{2}{q\gamma_q - q}-1} - \frac{1-\gamma_q}{a^2\mu^{\frac{2}{q-q\gamma_q}}}\|v_t\|_q^q = 0. \tag{1.13}$$ eqn0020

This is a reduction, which heavily depends on the scaling technique and the Pohozaev identity, since we reduce the solvability of (1.4) in $H^1(\mathbb{R}^N)$ to the solvability of (1.13) in $\mathbb{R}^+$. Let

$$\mathcal{A}_\mu(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{\mu}{q}\|v\|_q^q - \frac{1}{2^*}\|v\|_{2^*}^{2^*}.$$

Then, $\mathcal{A}_\mu|_{S_a}(u)$ is the corresponding functional of (1.7), where $S_a = \{u \in H^1(\mathbb{R}^N) \mid \|u\|_2^2 = a^2\}$.

Definition 1.2. *We say that u is a normalized ground-state of (1.7) if u is a solution of (1.7) and $\mathcal{A}_\mu(u) \le \mathcal{A}_\mu(v)$ for any other solutions of (1.7).*

By (1.12), if (u_μ, λ_μ) is a solution of (1.7), then,

$$\mathcal{A}_\mu(u_\mu) + \frac{\lambda_\mu a^2}{2} = \mathcal{E}_{t_\mu}(v_\mu),$$

where (v_μ, t_μ) is a solution of (1.11). Thus, normalized ground-states of (1.7) must be generated by positive fixed-frequency ground-states of (1.4) through the equation (1.13). With these in minds, we can obtain the following results.

⟨thm0003⟩ **Theorem 1.3.** *Let $N \ge 3$ and $2 < q < 2 + \frac{4}{N}$. Then there exist $0 < \widehat{t}_{q,a} \le \bar{t}_{q,a} < +\infty$, which may depend on q and a, such that (1.4) has normalized ground-states with the additional condition $\|u\|_2^2 = a^2$ for $0 < t < \widehat{t}_{q,a}$ and (1.4) has no normalized ground-states with the additional condition $\|u\|_2^2 = a^2$ for $t > \bar{t}_{q,a}$.*

Remark 1.2. *Theorem 1.3, together with our recent study in [31], gives a completed answer to the open question proposed by Soave in [30].*

As an application of our new reduction in finding normalized solutions of (1.4), we shall also consider the following Schrödinger equation:

$$\begin{cases} -\Delta u+\lambda u+V(x)u=|u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\ u\in H^1(\mathbb{R}^3), \quad \|u\|_2^2=r^2, \end{cases} \tag{1.14}$$

where $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, $V(x)=x_1^2+x_2^2$, $\frac{10}{3}<p<6$ and $r>0$ is a constant. (1.14) is studied recently by Bellazzini et al. in [7], in which the authors proved that (1.14) has a ground-state normalized solution, which is also a local minimizer of the associated functional on the L^2-sphere $\|u\|_2^2=r^2$, with a negative Lagrange multiplier λ for $r>0$ sufficiently small. According to the geometry of the associated functional on the L^2-sphere $\|u\|_2^2=r^2$, Bellazzini et al. also conjecture in [7] that (1.14) has a second normalized solution, which is also a mountain-pass solution, for $r>0$ sufficiently small. In this paper, we prove this conjecture by obtaining the following result.

Theorem 1.4. *Let $\frac{10}{3}<p<6$. Then for $r>0$ sufficiently small, (1.14) has a second positive normalized solution $u_{r,2}$, which is also a mountain-pass solution, with a positive Lagrange multiplier*

$$\lambda_{r,2}=(1+o_r(1))\left[\frac{(6-p)\|w_\infty\|_p^p}{2pr^2}\right]^{\frac{2(p-2)}{3p-10}}\to+\infty \quad \text{as } r\to 0, \tag{1.15}$$

where w_∞ is the unique (up to translations) positive solution of the following equation:

$$\begin{cases} -\Delta w+w=|w|^{p-2}w \quad \text{in } \mathbb{R}^3, \\ w\in H^1(\mathbb{R}^3). \end{cases} \tag{1.16}$$

Moreover,

$$w_r(x)=\lambda_{r,2}^{-\frac{1}{p-2}}u_{r,2}(\lambda_{r,2}^{-\frac12}x)=w_\infty+o_r(1) \quad \text{in } H^1(\mathbb{R}^3) \quad \text{as } r\to 0. \tag{1.17}$$

To prove Theorem 1.4, we apply our new reduction argument to (1.14) by reducing finding normalized solutions of (1.14) to finding solutions of the following equation:

$$f(r,t):=r^2-t^{\frac{10-3p}{2(p-2)}}\Big(\frac{6-p}{2p}\|w_t\|_p^p-2t^{-2}\int_{\mathbb{R}^3}V(x)w_t^2dx\Big), \tag{1.18}$$

where w_t is a positive ground-state of the following equation:

$$\begin{cases} -\Delta w+w+t^{-2}V(x)w=|w|^{p-2}w \quad \text{in } \mathbb{R}^3, \\ w\in H^1(\mathbb{R}^3). \end{cases}$$

By the uniqueness and nondegeneracy of w_∞, we prove that the curve w_t is continuous for $t>0$ sufficiently large in a suitable space. Thus, (1.18) can be solved easily by the continuation method. We believe this method will be helpful in studying normalized solutions of other elliptic equations.

Notations. Throughout this paper, C and C' are indiscriminately used to denote various absolutely positive constants. $a \sim b$ means that $C'b \leq a \leq Cb$ and $a \lesssim b$ means that $a \leq Cb$.

2. BLOW-UP SOLUTIONS FOR $N = 3$ AND $2 < q < 4$

It is well known that the Aubin-Talanti babbles,

$$U_\varepsilon(x) = [N(N-2)]^{\frac{N-2}{4}} \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{N-2}{2}}, \tag{2.1}$$

eq0095

is the only solutions to the following equation:

$$\begin{cases} -\Delta u = u^{2^*-1} \quad \text{in } \mathbb{R}^N, \\ u(0) = \max\limits_{x \in \mathbb{R}^N} u(x), \\ u(x) > 0 \quad \text{in } \mathbb{R}^N, \\ u(x) \to 0 \quad \text{as } |x| \to +\infty. \end{cases}$$

By [31, Theorem 1.2], for $\mu > 0$ sufficiently small, (1.7) has a positive radial solution $\widetilde{u}_\mu$ with the Lagrange multiplier $\widetilde{\lambda}_\mu > 0$ such that $\varepsilon_\mu^{\frac{1}{2}} \widetilde{u}_\mu(\varepsilon_\mu x) \to U_{\varepsilon_0}$ strongly in $D^{1,2}(\mathbb{R}^3)$ for some $\varepsilon_0 > 0$ as $\mu \to 0$ up to a subsequence, where U_{ε_0} is given by (2.1) and ε_μ satisfies

$$\mu \sim \begin{cases} \varepsilon_\mu^{\frac{q}{2}-1}, \quad 3 < q < 6, \\ \dfrac{\varepsilon_\mu^{\frac{1}{2}}}{\ln(\frac{1}{\varepsilon_\mu})}, \quad q = 3, \\ \varepsilon_\mu^{5-\frac{3q}{2}}, \quad 2 < q < 3. \end{cases} \tag{2.2}$$

eq0007

Moreover, by [31, Lemma 4.1], we have

$$1 \sim \begin{cases} \dfrac{\mu \sigma_\mu^{\frac{6-q}{2}}}{\widetilde{\lambda}_\mu}, \quad 3 < q < 6, \\ \dfrac{\mu \sigma_\mu^{\frac{3}{2}}}{\widetilde{\lambda}_\mu} \ln\left(\dfrac{1}{\sqrt{\widetilde{\lambda}_\mu} \sigma_\mu}\right), \quad q = 3, \\ \dfrac{\mu \sigma_\mu^{\frac{q}{2}}}{\widetilde{\lambda}_\mu^{\frac{5-q}{2}}}, \quad 2 < q < 3. \end{cases} \tag{2.3}$$

eq0012

On the other hand, in the proof of [31, Proposition 4.2], we also show that

$$\sigma_\mu \sim \varepsilon_\mu \quad \text{as} \quad \mu \to 0. \tag{2.4}$$

eq0060

⟨propn0001⟩ **Proposition 2.1.** *Let $\lambda = 1$, $N = 3$ and $2 < q < 4$. Then there exists $t_q > 0$, which may depend on q, such that (1.4) has two positive radial solutions $u_{t,1}$ and $u_{t,2}$ for $t > t_q$, where $u_{t,1}$ is a ground-state with $\|u_{t,1}\|_\infty \sim t^{-\frac{1}{q-2}}$ and $u_{t,2}$ is a* *blow-up solution with*

$$\|u_{\mu,2}\|_\infty \sim \begin{cases} t^{\frac{1}{4-q}}, & 3<q<4,\\ t\ln t, & q=3,\\ t^{\frac{1}{q-2}}, & 2<q<3, \end{cases}$$

as $t \to +\infty$.

Proof. By (1.9) and (1.10), $(\widetilde{v}_\mu, \widetilde{t}_\mu)$ is a solution of (1.11). In particular, $\widetilde{v}_\mu$ is a solution of (1.4) for $\lambda = 1$ and $t = \widetilde{t}_\mu = \mu\widetilde{\lambda}_\mu^{\frac{q\gamma_q - q}{2}}$. By the well-known Gidas-Ni-Nirenberg theorem [16], $\widetilde{v}_\mu$ is radial and decreasing for $r = |x|$ up to translations. Thus, without loss of generality, we may assume that $\widetilde{v}_\mu(0) = \max_{x\in\mathbb{R}^N} \widetilde{v}_\mu$. Recall that $\varepsilon_\mu^{\frac12}\widetilde{u}_\mu(\varepsilon_\mu x) \to U_{\varepsilon_0}$ strongly in $D^{1,2}(\mathbb{R}^3)$ for some $\varepsilon_0 > 0$ as $\mu \to 0$ up to a subsequence, by the classical elliptic regularity and the Sobolev embedding theorem, $\varepsilon_\mu^{\frac12}\widetilde{u}_\mu(\varepsilon_\mu x) \to U_{\varepsilon_0}$ strongly in $C^{1,\alpha}_{loc}(\mathbb{R}^3)$ for some $\alpha \in (0,1)$ as $\mu \to 0$ up to a subsequence. In particular, $\varepsilon_\mu^{\frac12}\widetilde{u}_\mu(0) \to U_{\varepsilon_0}(0)$ as $\mu \to 0$ up to a subsequence. Thus, by (2.1),

$$\widetilde{v}_\mu(0) = \widetilde{\lambda}_\mu^{-\frac14}\widetilde{u}_\mu(0) \sim \widetilde{\lambda}_\mu^{-\frac14}\varepsilon_\mu^{-\frac12} \quad \text{as } \mu \to 0 \text{ up to a subsequence.} \tag{2.5}$$

eq0008

In the following, let us estimates $\widetilde{v}_\mu(0)$ and $\widetilde{t}_\mu$ as $\mu \to 0$. We begin with the estimate of $\widetilde{t}_\mu$. We first consider the case $2 < q < 3$. In this case, by (2.2), (2.3) and (2.4), $\widetilde{\lambda}_\mu \sim \varepsilon_\mu^2$, which, together with (1.10), implies

$$\widetilde{t}_\mu \sim \varepsilon_\mu^{\frac{10-3q}{2}}(\varepsilon_\mu^2)^{\frac{q-6}{4}} = \varepsilon_\mu^{2-q} \to +\infty \quad \text{as } \mu \to 0.$$

For $q = 3$, by (2.2), (2.3) and (2.4),

$$\widetilde{\lambda}_\mu \sim \varepsilon_\mu^2 \frac{\ln(\frac{1}{\sqrt{\widetilde{\lambda}_\mu}\varepsilon_\mu})}{\ln(\frac{1}{\varepsilon_\mu})} \gtrsim \varepsilon_\mu^2.$$

It follows that

$$\ln(\frac{1}{\varepsilon_\mu}) \lesssim \ln(\frac{1}{\sqrt{\widetilde{\lambda}_\mu}\varepsilon_\mu}) \lesssim \ln(\frac{1}{\varepsilon_\mu}).$$

Thus, we also have $\widetilde{\lambda}_\mu \sim \varepsilon_\mu^2$ for $q = 3$. By (1.10) and (2.2),

$$\widetilde{t}_\mu \sim \varepsilon_\mu^{\frac12}\frac{1}{\ln(\frac{1}{\varepsilon_\mu})}(\varepsilon_\mu^2)^{-\frac34} = \varepsilon_\mu^{-1}\frac{1}{\ln(\frac{1}{\varepsilon_\mu})} \to +\infty \quad \text{as } \mu \to 0.$$

For $3 < q < 4$, by (2.2), (2.3) and (2.4), $\widetilde{\lambda}_\mu \sim \varepsilon_\mu^2$. Now, by (1.10),

$$\widetilde{t}_\mu \sim \varepsilon_\mu^{\frac{q-2}{2}}(\varepsilon_\mu^2)^{\frac{q-6}{4}} = \varepsilon_\mu^{q-4} \to +\infty \quad \text{as } \mu \to 0.$$

Thus, for all $2 < q < 4$, we always have

$$\widetilde{\lambda}_\mu \sim \varepsilon_\mu^2 \quad \text{and} \quad \widetilde{t}_\mu \sim \begin{cases} \varepsilon_\mu^{q-4}, & 3<q<4,\\ \varepsilon_\mu^{-1}\frac{1}{\ln(\frac{1}{\varepsilon_\mu})}, & q=3,\\ \varepsilon_\mu^{2-q}, & 2<q<3, \end{cases} \tag{2.6}$$

eq0014

as $\mu \to 0$. Now, by (2.2)–(2.4) and (2.5), we have

$$\widetilde{v}_\mu(0) \sim \begin{cases} \mu^{-\frac{2}{q-2}}, & 3 < q < 4, \\ \left(\dfrac{1}{\mu|\ln \mu|}\right)^2, & q = 3, \\ \mu^{-\frac{2}{10-3q}}, & 2 < q < 3. \end{cases}$$

It follows from (2.2) and (2.6) that

$$\widetilde{v}_\mu(0) \sim \begin{cases} \widetilde{t}_\mu^{\frac{1}{4-q}}, & 3 < q < 4, \\ \widetilde{t}_\mu \ln \widetilde{t}_\mu, & q = 3, \\ \widetilde{t}_\mu^{\frac{1}{q-2}}, & 2 < q < 3. \end{cases}$$

Thus, by (2.6), $\widetilde{v}_\mu$ is a blow-up solution of (1.4) for $N = 3$, $\lambda = 1$, $2 < q < 4$ and $t = \widetilde{t}_\mu$. Note that by [23, Theorem 2.2], the ground-states of (1.4) for $\lambda = 1$, say $\overline{v}_t$, satisfies $\|\overline{v}_t\|_\infty \sim t^{-\frac{1}{q-2}}$ as $t \to +\infty$. For $\mu > 0$ sufficiently small, $\widetilde{v}_\mu$ is a second positive radial solution of (1.4) with $N = 3$, $\lambda = 1$, $2 < q < 4$ and $t > 0$ sufficiently large. □

Remark 2.1. *Let $\widetilde{v}_\mu$ be given in the proof of Proposition 2.1 and define*

$$\widetilde{w}_\mu(x) = \widetilde{t}_\mu^{\frac{1}{q-2}} \widetilde{v}_\mu(x),$$

then $\widetilde{w}_\mu$ satisfies the following equation:

$$\begin{cases} -\Delta w + w = |w|^{q-2} w + \widetilde{t}_\mu^{-\frac{2^*-2}{q-2}} |w|^{2^*-2} w \quad \text{in } \mathbb{R}^N, \\ v \in H^1(\mathbb{R}^N), \end{cases} \tag{2.7}$$
eq0024

where $t = \widetilde{t}_\mu$ is also given in the proof of Proposition 2.1. By similar arguments as that used for [14, Lemma 5.3], (2.7) has a unique bounded positive radial solution for $t > 0$ sufficiently large. However, by (1.10) and (2.5),

$$\widetilde{w}_\mu(0) \sim \mu^{\frac{1}{q-2}} \widetilde{\lambda}_\mu^{-\frac{1}{q-2}} \varepsilon_\mu^{-\frac{1}{2}} \quad \text{as } \mu \to 0. \tag{2.8}$$
eq0011

By (2.2), (2.6) and (2.8),

$$\widetilde{w}_\mu(0) \sim \begin{cases} \varepsilon_\mu^{-\frac{2}{q-2}}, & 2 < q < 3, \\ \varepsilon_\mu^{-2} \dfrac{1}{\ln(\frac{1}{\varepsilon_\mu})}, & q = 3, \\ \varepsilon_\mu^{-\frac{2}{q-2}}, & 3 < q < 4. \end{cases}$$

Thus, $\widetilde{w}_\mu$ is also a blow-up solution of (2.7) as $\widetilde{t}_\mu \to +\infty$.

3. Ground-states for $N = 3$ and $2 < q \leq 4$

The associated fibering map of (1.6) for every $v \neq 0$ in $H^1(\mathbb{R}^3)$ is given by

$$E(s) = \frac{s^2}{2}(\|\nabla v\|_2^2 + \|v\|_2^2) - \frac{ts^q}{q}\|v\|_q^q - \frac{s^6}{6}\|v\|_6^6. \tag{3.1}$$
eq0075

Since $q > 2$, it is standard to show that for every $v \neq 0$ in $H^1(\mathbb{R}^3)$, there exists a unique $s_0 > 0$ such that $E(s)$ is strictly increasing for $0 < s < s_0$ and strictly decreasing for $s > s_0$.

⟨lem0001⟩ **Lemma 3.1.** *Let $N = 3$, $\lambda = 1$ and $2 < q \leq 4$. Then $m(t) = \frac{1}{3}S^{\frac{3}{2}}$ for $t > 0$ sufficiently small, where $m(t)$ is given by (1.5).*

Proof. We argue in the contrary by supposing that there exists $t_n \to 0$ as $n \to \infty$ such that $m(t_n) < \frac{1}{3}S^{\frac{3}{2}}$. Then, it is standard to show (cf. [6]) that $m(t_n)$ is attained by a positive and radial function, which is also a solution of (1.4) with $\lambda = 1$, $N = 3$ and $t = t_n$. We denote this solution by v_{t_n}. Since $t_n \to 0$ as $n \to \infty$, it is also standard to show that

$$\|\nabla v_{t_n}\|_2^2 = \|v_{t_n}\|_6^6 + o_n(1) = S^{\frac{3}{2}} + o_n(1) \quad \text{as } n \to \infty. \tag{3.2}$$ eq0065

Thus, $\{v_{t_n}\}$ is a minimizing sequence of the Sobolev inequality. By Lions' result (cf. [32, Theorem 1.41]), up to a subsequence, there exists $\sigma_n > 0$ such that for some $\varepsilon_* > 0$,

$$w_{t_n}(x) = \sigma_n^{\frac{1}{2}} v_{t_n}(\sigma_n x) \to U_{\varepsilon_*} \text{ strongly in } D^{1,2}(\mathbb{R}^3) \text{ as } n \to \infty.$$

Clearly, by direct computations, we know that w_{t_n} satisfies the following equation:

$$-\Delta w_{t_n} + \sigma_n^2 w_{t_n} = t_n \sigma_n^{3-\frac{q}{2}} w_{t_n}^{q-1} + w_{t_n}^5 \quad \text{in } \mathbb{R}^3. \tag{3.3}$$ eq0021

Since v_{t_n} is positive and radial, w_{t_n} is also positive and radial. Thus, by the boundedness of $\{w_{t_n}\}$ in $D^{1,2}(\mathbb{R}^3)$, the Sobolev embedding theorem and Strusss radial lemma (cf. [8, Lemma A.2]),

$$w_{t_n} \lesssim r^{-\frac{1}{2}} \quad \text{for all } r \geq 1 \text{ uniformly as } n \to \infty.$$

On the other hand, since $w_{t_n} \to U_{\varepsilon_*}$ strongly in $D^{1,2}(\mathbb{R}^3)$ as $n \to \infty$, by applying the Moser iteration in a standard way and using the Sobolev embedding theorem, we know that $w_{t_n} \to U_{\varepsilon_*}$ strongly in $C^{1,\alpha}_{loc}(\mathbb{R}^3)$ as $n \to \infty$ for some $\alpha \in (0,1)$. Thus,

$$w_{t_n} \lesssim (1+r)^{-\frac{1}{2}} \quad \text{for all } r \geq 0 \text{ uniformly as } n \to \infty.$$

Now, we can adapt the ODE's argument in [5, 17, 19] as that in the proof of [31, Lemma 4.1] to obtain

$$w_{t_n} \lesssim \frac{1}{(1+r^2)^{\frac{1}{2}}} \quad \text{for all } r \geq 0 \text{ uniformly as } n \to \infty. \tag{3.4}$$ eq0020

On the other hand, since $N = 3$, it is easy to check that $r^{-1}e^{-\sigma_n r}$ is a subsolution of $-\Delta u + \sigma_n^2 u = 0$ for $r \geq 1$. Thus, by the fact that $w_{t_n} \to U_{\varepsilon_*}$ strongly in $C^{1,\alpha}_{loc}(\mathbb{R}^3)$ as $n \to \infty$ for some $\alpha \in (0,1)$, we can use the maximum principle in a standard way to show that

$$w_{t_n} \gtrsim r^{-1}e^{-\sigma_n r} \quad \text{for } r \geq 1 \text{ uniformly as } n \to \infty.$$

It follows that

$$\|w_{t_n}\|_q^q \gtrsim \int_1^{\frac{1}{\sigma_n}} r^{2-q} e^{-q\sigma_n r} dr \sim \begin{cases} \sigma_n^{q-3}, & 2 \leq q < 3, \\ |\ln \sigma_n|, & q = 3, \\ 1, & 3 < q < 6. \end{cases} \tag{3.5}$$ eq0061

Since $t_n \to 0$ as $n \to \infty$, by (3.4), for $r \gtrsim \left(\frac{1}{\sigma_n}\right)^{\frac{1}{2}}$, (3.3) reads as

$$-\Delta w_{t_n} + \frac{1}{4}\sigma_n^2 w_{t_n} \leq 0 \quad \text{in } \mathbb{R}^3.$$

Thus, by (3.4), we can use the maximum principle in a standard way again to obtain

$$w_{t_n} \lesssim r^{-1}e^{-\frac{\sigma_n}{4}r} \quad \text{for } r \gtrsim \left(\frac{1}{\sigma_n}\right)^{\frac{1}{2}} \text{ uniformly as } n \to \infty.$$

On the other hand, since $\|w_{t_n}\|_6^6 = \|v_{t_n}\|_6^6 = S^{\frac{3}{2}} + o_n(1)$, by (3.8) and the Hölder inequality,

$$\sigma_n^2\|w_{t_n}\|_2^2 \lesssim t_n\sigma_n^{3-\frac{q}{2}}\|w_{t_n}\|_q^q \lesssim t_n\sigma_n^{3-\frac{q}{2}}\|w_{t_n}\|_2^{\frac{6-q}{2}},$$

which implies

$$\sigma_n\|w_{t_n}\|_2 \lesssim t_n^{\frac{2}{q-2}}.$$

Since $w_{t_n} \to U_{\varepsilon_*}$ strongly in $D^{1,2}(\mathbb{R}^3)$ as $n \to \infty$ and $U_{\varepsilon_*} \not\in L^2(\mathbb{R}^3)$, by the Fatou lemma,

$$\liminf_{n\to\infty} \|w_{t_n}\|_2 = +\infty.$$

Thus, by $t_n \to 0$ as $n \to \infty$, we have $\sigma_n \to 0$ as $n \to \infty$. It follows from (3.4) once more that

$$\|w_t\|_q^q \lesssim 1 + \int_1^{\frac{1}{\sigma_n}} r^{2-q}dr + \int_{(\frac{1}{\sigma_n})^{\frac{1}{2}}}^{+\infty} r^{2-q}e^{-\frac{q}{4}\sigma_n r}dr \sim \begin{cases} \sigma_n^{q-3}, & 2 \leq q < 3, \\ |\ln\sigma_n|, & q = 3, \\ 1, & 3 < q < 6. \end{cases} \qquad \boxed{\mathtt{eq0062}} \tag{3.6}$$

Thus, by (3.5) and (3.6), we have

$$\|w_{t_n}\|_q^q \sim \begin{cases} \sigma_n^{q-3}, & 2 \leq q < 3, \\ |\ln\sigma_n|, & q = 3, \\ 1, & 3 < q < 6. \end{cases} \qquad \boxed{\mathtt{eq0063}} \tag{3.7}$$

Note that as that of (1.8), by the Pohozaev identity, we have

$$\sigma_n^2\|w_{t_n}\|_2^2 = (1-\gamma_q)t_n\sigma_n^{3-\frac{q}{2}}\|w_{t_n}\|_q^q. \qquad \boxed{\mathtt{eq0040}} \tag{3.8}$$

Thus, by (3.7),

$$\sigma_n \sim \begin{cases} t_n\sigma_n^{\frac{q}{2}}, & 2 < q < 3, \\ t_n\sigma_n^{\frac{3}{2}}|\ln\sigma_n|, & q = 3, \\ t_n\sigma_n^{3-\frac{q}{2}}, & 3 < q < 6, \end{cases}$$

which implies

$$t_n \sim \begin{cases} \sigma_n^{\frac{2-q}{2}}, & 2 < q < 3, \\ \sigma_n^{-\frac{1}{2}}\dfrac{1}{|\ln\sigma_n|}, & q = 3, \\ \sigma_n^{\frac{q-4}{2}}, & 3 < q < 6. \end{cases} \qquad \boxed{\mathtt{eq0041}} \tag{3.9}$$

(3.9) contradicts the facts that $t_n, \sigma_n \to 0$ as $n \to \infty$ for $2 < q \leq 4$. It follows that $m(t) \geq \frac{1}{3}S^{\frac{3}{2}}$ for $t > 0$ sufficiently small in the case of $2 < q \leq 4$. On the other hand, since $m(t)$ is the minimum of $\mathcal{E}_t(v)$ on the Nehari manifold $\mathcal{N}_t$, it is standard (cf. [31, Lemma 3.3]) to use the fibering maps (3.1) to show that $m(t)$ is nonincreasing for $t > 0$. Note that it is well known that $m(0) = \frac{1}{3}S^{\frac{3}{2}}$, thus, $m(t) \leq \frac{1}{3}S^{\frac{3}{2}}$ for all $t > 0$. It follows that $m(t) = \frac{1}{3}S^{\frac{3}{2}}$ for $t > 0$ sufficiently small in the case of $2 < q \leq 4$. □

Let

$$t_q^* = \sup\{t > 0 \mid m_t = \frac{1}{3}S^{\frac{3}{2}}\}. \tag{3.10}$$ eqn0090

Then by Lemma 3.1, $t_q^* > 0$ for $2 < q \leq 4$. Since it is well known (cf. [6]) that $m(t) < \frac{1}{3}S^{\frac{3}{2}}$ for $t > 0$ sufficiently large in the case of $2 < q \leq 4$, we have $0 < t_q^* < +\infty$ for all $2 < q \leq 4$. Since $m(t) < \frac{1}{3}S^{\frac{3}{2}}$ for $t > t_q^*$, it is standard (cf. [6]) to show that $m(t)$ is attained for $t > t_q^*$. Let v_t be a ground-state of (1.4), which is radial and positive for $t > t_q^*$ in the case of $2 < q < 4$. Then, we have the following.

⟨prop0001⟩ **Proposition 3.1.** *Let $N = 3$, $\lambda = 1$ and $2 < q < 4$. Then, $\|v_t\|_q^q \sim 1$ as $t \to t_q^*$.*

Proof. The conclusion $\|v_t\|_q^q \lesssim 1$ as $t \to t_q^*$ is standard so we omit it. For the conclusion $\|v_t\|_q^q \gtrsim 1$ as $t \to t_q^*$, we argue in the contrary. Then there exists $t_n \to t_q^*$ as $n \to \infty$ such that $\|v_{t_n}\|_q^q \to 0$ as $n \to \infty$. Similar to that of (3.2), we also have

$$\|\nabla v_{t_n}\|_2^2 = \|v_{t_n}\|_6^6 + o_n(1) = S^{\frac{3}{2}} + o_n(1) \quad \text{as } n \to \infty.$$

Thus, $\{v_{t_n}\}$ is a minimizing sequence of the Sobolev inequality. By Lions' result (cf. [32, Theorem 1.41]), up to a subsequence, there exists $\sigma_n' > 0$ such that for some $\varepsilon_* > 0$,

$$w_n(x) = (\sigma_n')^{\frac{1}{2}} v_{t_n}(\sigma_n' x) \to U_{\varepsilon_*} \text{ strongly in } D^{1,2}(\mathbb{R}^3) \text{ as } n \to \infty.$$

Now, repeating the arguments for (3.9), we will arrive at

$$t_q^* \sim \begin{cases} (\sigma_n')^{\frac{2-q}{2}}, & 2 < q < 3, \\ (\sigma_n')^{-\frac{1}{2}} \dfrac{1}{|\ln \sigma_n'|}, & q = 3, \\ (\sigma_n')^{\frac{q-4}{2}}, & 3 < q < 4. \end{cases}$$

This is impossible since $\sigma_n' \to 0$ as $n \to \infty$ by similar arguments as that used for σ_n in the proof of Lemma 3.1. Thus, we must have $\|v_t\|_q^q \gtrsim 1$ as $t \to t_q^*$. □

Now, we are arriving at the following.

⟨propn0002⟩ **Proposition 3.2.** *Let $\lambda = 1$, $N = 3$ and $2 < q \leq 4$. Then*

(1) *(1.4) has ground-states for $t \geq t_q^*$ and has no ground-states for $0 < t < t_q^*$ in the case of $2 < q < 4$.*
(2) *(1.4) has ground-states for $t > t_4^*$ and has no ground-states for $0 < t < t_4^*$ in the case of $q = 4$.*

Here, t_q^* is given by (3.10).

Proof. We first prove that there is no ground-states of (1.4) for $0 < t < t_q^*$ in the case of $2 < q \leq 4$. Suppose the contrary that (1.4) has a ground-state for some $0 < t < t_q^*$ in the case of $2 < q \leq 4$. Then $m(t)$ is attained. Now, by use the fibering maps (3.1) in a standard way (cf. [31, Lemma 3.3]), we have $m(t') < m(t)$ for all $t' > t$. It follows that $m(t') < \frac{1}{3}S^{\frac{3}{2}}$ for all $t' > t$, which contradicts the definition of t_q^* given by (3.10). Thus, there is no ground-states of (1.4) for $0 < t < t_q^*$ in the case of $2 < q \leq 4$. It remains to prove that (1.4) has a ground-state for $t = t_q^*$ in the case of $2 < q < 4$, which is equivalent to prove that $m(t_q^*)$ is attained for $2 < q < 4$. Let v_t be a ground-state of (1.4), which is radial and positive for $t > t_q^*$ in the case of $2 < q < 4$ such that $t \to t_q^*$. By Proposition 3.1, $\|v_t\|_q^q \gtrsim 1$ as $t \to t_q^*$. Since v_t is radial, it is standard to show that $v_t \to v_{t_q^*} \neq 0$ strongly in $H^1(\mathbb{R}^3)$ as $t \to t_q^*$ up to a subsequence. Thus, $m(t_q^*)$ is attained by $v_{t_q^*}$, which is also a ground-state of (1.4) for $t = t_q^*$ in the case of $2 < q < 4$. □

Remark 3.1. *Upon to Theorem 1.2, the existence of ground-states of (1.4) is almost completely solved, except for $N = 3$, $q = 4$ and $t = t_4^*$. In this case, we believe that there is no ground-states of (1.4). Indeed, let $\mu > 0$, $a > 0$ and (u_μ, λ_μ) be a normalized solution of (1.7), then by (1.9) and (1.10), $\widetilde{v}_\mu$ is a solution of (1.4) with $\lambda = 1$ and $t = \widetilde{t}_\mu = \mu\widetilde{\lambda}_\mu^{\frac{q\gamma_q - q}{2}}$. By (2.2) and (2.6),*

$$\widetilde{t}_\mu \sim \mu\varepsilon_\mu^2 \sim \mu^{\frac{2(q-4)}{q-2}} \quad \text{as } \mu \to 0 \text{ for } 4 \leq q < 6.$$

Thus, $\widetilde{t}_\mu \to 0$ as $\mu \to 0$ for $4 < q < 6$ and $\widetilde{t}_\mu \sim 1$ as $\mu \to 0$ for $q = 4$. Note that $\widetilde{v}_\mu$, generated by $\widetilde{u}_\mu$ though (1.9), is a solution of (1.4) for $t = \widetilde{t}_\mu$ and by [31, Theorem 1.2],

$$\|\nabla\widetilde{v}_\mu\|_2^2 = \|\nabla\widetilde{u}_\mu\|_2^2 = S^{\frac{3}{2}} + o_\mu(1) \quad \text{as } \mu \to 0.$$

It seems that $\widetilde{v}_\mu$ will approximate the ground-state level $m(t) = \frac{1}{3}S^{\frac{3}{2}}$ for $N = 3$, $\lambda = 1$, $q = 4$ and $t = t_4^*$ as $\mu \to 0$, which suggests that the concentration phenomenon will happen at the ground-state level $m(t) = \frac{1}{3}S^{\frac{3}{2}}$ for $N = 3$, $\lambda = 1$, $q = 4$ and $t = t_4^*$.

We close this section by the proof of Theorem 1.2.

Proof of Theorem 1.2: It follows from Propositions 2.1 and 3.2. □

4. Normalized ground-states for $2 < q < 2 + 4/N$

Let

$$t_q^{**} = \begin{cases} 0, & N \geq 4, \\ t_q^*, & N = 3, \end{cases} \tag{4.1}$$ eqn0091

where t_q^* is given by (3.10). Then, by [6, Theorem 1.2] and Theorem 1.2, (1.4) has a ground-state v_t for $t > t_q^{**}$ and $2 < q < 2 + \frac{4}{N}$, which is positive and radial. By (1.11) and (1.12), (u_t, λ_t) is a positive normalized solution of (1.7) if and only if

$$F(t, \mu) := t^{\frac{2}{q\gamma_q - q} - 1} - \frac{1 - \gamma_q}{a^2\mu^{\frac{2}{q - q\gamma_q}}}\|v_t\|_q^q = 0.$$

Clearly, for every $t > t_q^{**}$, there exists a unique

$$\mu_t = a^{q\gamma_q - q}\bigg[(1-\gamma_q)\|v_t\|_q^q t^{\frac{q-q\gamma_q+2}{q-q\gamma_q}}\bigg]^{\frac{q-q\gamma_q}{2}} \tag{4.2}$$

such that $F(t, \mu_t) = 0$. Let

$$\overline{\mu}_{q,a} = \sup\{\mu_t > 0 \mid t > t_q^{**}\}.$$

Then, (1.7) has a positive normalized solution if and only if $\mu < \overline{\mu}_{q,a}$ and $\mu = \mu_t$. Now, we are prepared for the proof of Theorem 1.3.

Proof of Theorem 1.3: By [31, Theorem 1.1] and [18, Theorem 1.6], (1.7) has a normalized ground-state for $\mu > 0$ sufficiently small. Thus, we only need to prove (1.7) has no normalized ground-states for $\mu > 0$ sufficiently large, which is equivalent to show that $\overline{\mu}_{q,a} < +\infty$. Recall that $\gamma_q = \frac{N(q-2)}{2q}$, we always have $q > q\gamma_q$. It follows from (4.2) that $\mu_t \to 0$ as $t \to t_q^{**}$ for $N \geq 4$ since $t_q^{**} = 0$ for $N \geq 4$. For $N = 3$, we have $t_q^{**} = t_q^* > 0$ and $\|v_t\|_q^q \sim 1$ as $t \to t_q^*$ by Proposition 3.1. Thus, $\mu_t \lesssim 1$ as $t \to t_q^{**}$ for all $N \geq 3$. Since v_t is a ground-state of (1.4) with the least energy $m(t)$ on the Nehari manifold $\mathcal{N}_t$, by standard arguments (cf. [11, Lemma 2.2]),

$$m(t) = \frac{1}{N}S^{\frac{N}{2}} - \int_{t_q^{**}}^{t}\frac{1}{q}\|v_\tau\|_q^q d\tau \quad \text{for all } t > t_q^{**} \tag{4.3}$$

and

$$m'(t) = -\frac{1}{q}\|v_t\|_q^q \quad \text{for a.e. } t > t_q^{**}. \tag{4.4}$$

As that of (1.8), by the Pohozaev identity, we have

$$\|\nabla v_t\|_2^2 = \gamma_q t\|v_t\|_q^q + \|v_t\|_{2^*}^{2^*} \quad \text{and} \quad \|\nabla v_t\|_2^2 = Nm(t). \tag{4.5}$$

Thus, by (4.3) and (4.4),

$$Nm(t) + q\gamma_q m'(t)t \geq 0 \qquad \text{for a.e. } t > t_q^{**},$$

which implies $m(t)t^{\frac{N}{q\gamma_q}}$ is increasing for $t > t_q^{**}$. Now, let $t_0 > t_q^{**}$ with $t_0 - t_q^{**} > 0$ sufficiently small such that $\mu_t \lesssim 1$ for $t < t_0$, then

$$m(t) \gtrsim t^{-\frac{N}{q\gamma_q}} \quad \text{for } t \geq t_0. \tag{4.6}$$

On the other hand, by the definition of t_q^{**} given by (4.1), [6, Theorem 1.2] and Theorem 1.2, $m(t) < \frac{1}{N}S^{\frac{N}{2}}$ for $t > t_q^{**}$. Thus, it is standard to apply the classical elliptic estimates to show that $\|v_t\|_\infty \lesssim 1$ for all $t \geq t_0$. By (4.4) and (4.5),

$$Nm(t) = \|\nabla v_t\|_2^2 \leqslant (1 + O(\frac{1}{t}))\gamma_q\|v_t\|_q^q t = -(1 + O(\frac{1}{t}))q\gamma_q m'(t)t \qquad \text{for a.e. } t \geq t_0,$$

which implies that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that $m(t) \lesssim t^{-\frac{N}{q\gamma_q+\varepsilon}}$ for $t \geq t_\varepsilon$. It follows from (4.5) once more that

$$\|v_t\|_q^q \lesssim t^{-\frac{N}{q\gamma_q+\varepsilon}-1} \quad \text{for } t \geq t_\varepsilon.$$

Thus, by (4.5) and $\|v_t\|_\infty \lesssim 1$ for all $t \geq t_0$, we have

$$Nm(t) = \|\nabla v_t\|_2^2 \leqslant \gamma_q\|v_t\|_q^q t + C_0 t^{-\frac{N}{q\gamma_q+\varepsilon}-1} \qquad \text{for } t \geq t_\varepsilon,$$

which implies $m(t)t^{\frac{N}{q\gamma_q}} - C_1 t^{-\frac{N}{q\gamma_q+\varepsilon}}$ is decreasing for $t \geq t_\varepsilon$. Therefore, $m(t) \lesssim t^{-\frac{N}{q\gamma_q}}$ for $t > 0$ sufficiently large, which, together with (4.6), implies that

$$m(t) \sim t^{-\frac{N}{q\gamma_q}} \quad \text{as } t \to +\infty.$$

It follows from (4.5) and $\|v_t\|_\infty \lesssim 1$ for all $t \geq t_0$ that

$$\|v_t\|_q^q \sim t^{-\frac{N}{q\gamma_q}-1} \quad \text{as } t \to +\infty.$$

Since

$$\frac{2}{q - q\gamma_q} - \frac{N}{q\gamma_q} = \frac{2N(q-2-\frac{4}{N})}{(q-2)(2N-q(N-2))} < 0 \quad \text{for } 2 < q < 2 + \frac{4}{N},$$

by (4.2), $\overline{\mu}_{q,a} < +\infty$ for $2 < q < 2 + \frac{4}{N}$. $\square$

5. An application

In this section, we shall apply our above strategy to study the Schrödinger equation (1.14). Since there is an additional condition $\|u\|_2^2 = r^2$ in (1.14), λ in (1.14) is not fixed but appears as a Lagrange multiplier.

Let (u_r, λ_r) be a solution of (1.14). Since $V(x) = x_1^2 + x_2^2$, we have $\nabla V(x) \cdot x = 2V(x)$. Thus, the Pohozaev identity of (1.14) (cf. [7]) is given by

$$\frac{1}{6}\|\nabla u_r\|_2^2 + \frac{\lambda_r r^2}{2} + \frac{5}{6}\int_{\mathbb{R}^3} V(x)u_r^2 dx = \frac{1}{p}\|u_r\|_p^p,$$

which, combining the equation (1.14), implies that

$$\lambda_r r^2 = \frac{6-p}{2p}\|u_r\|_p^p - 2\int_{\mathbb{R}^3} V(x)u_r^2 dx. \qquad \boxed{\text{eq0069}} \tag{5.1}$$

We define

$$w_r(x) = \lambda_r^{-\frac{1}{p-2}} u_r(\lambda_r^{-\frac{1}{2}} x) \quad \text{and} \quad t_r = \lambda_r \qquad \boxed{\text{eq0068}} \tag{5.2}$$

Then by $V(x) = x_1^2 + x_2^2$ and (5.1), (w_r, t_r) is a solution of the following equation:

$$\begin{cases} -\Delta w + w + t^{-2}V(x)w = |w|^{p-2}w \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \quad r^2 = t^{\frac{10-3p}{2(p-2)}}\left(\frac{6-p}{2p}\|w\|_p^p - 2t^{-2}\int_{\mathbb{R}^3} V(x)w^2 dx\right). \end{cases} \qquad \boxed{\text{eq0067}} \tag{5.3}$$

Clearly, if (w_r, t_r) is a solution of (5.3), then, by (5.2), (u_r, λ_r) is also a solution of (1.14).

With these basic observations in hands, to find normalized solutions of (1.14) with positive Lagrange multipliers, it is equivalent to study the existence of solutions of (5.3). For this purpose, let us first consider the following equation:

$$\begin{cases} -\Delta w + w + t^{-2}V(x)w = |w|^{p-2}w \quad \text{in } \mathbb{R}^3, \\ w \in H^1(\mathbb{R}^3). \end{cases} \qquad \boxed{\text{eq0070}} \tag{5.4}$$

The corresponding functional of (5.4) is given by

$$\mathcal{J}_t(w) = \frac{1}{2}(\|\nabla w\|_2^2 + \|w\|_2^2 + \int_{\mathbb{R}^3} t^{-2}V(x)w^2 dx) - \frac{1}{p}\|w\|_p^p.$$

By [7, Lemma 2.1] and the Sobolev embedding theorem, this functional is well defined and of class C^2 in the Hilbert space

$$X = \{w \in H^1(\mathbb{R}^3) \mid \int_{\mathbb{R}^3} V(x)w^2 dx < +\infty\} \tag{5.5}$$

eq0078

with the norm

$$\|w\|_X = (\|\nabla w\|_2^2 + \int_{\mathbb{R}^3} V(x)w^2 dx)^{\frac{1}{2}}.$$

We also define the usual Nehari manifold of $\mathcal{J}_t(w)$ as follows:

$$\mathcal{M}_t = \{w \in X\backslash\{0\} \mid \mathcal{J}_t'(w)w = 0\}.$$

The associated fibering map for every $w \neq 0$ in X is given by

$$J(s) = \frac{s^2}{2}(\|\nabla w\|_2^2 + \|w\|_2^2 + \int_{\mathbb{R}^3} t^{-2}V(x)w^2 dx) - \frac{s^p}{p}\|w\|_p^p. \tag{5.6}$$

eq0076

Since $p > 2$, it is standard to show that for every $w \neq 0$ in X, there exists a unique $s_0' > 0$ such that $J(s)$ is strictly increasing for $0 < s < s_0'$ and is strictly decreasing for $s > s_0'$. Let

$$\mathfrak{m}(t) = \inf_{v \in \mathcal{M}_t} \mathcal{J}_t(v).$$

Definition 5.1. *We say that w is a ground-state of (5.4) if w is a nontrivial solution of (5.4) with $\mathcal{J}_t(w) = \mathfrak{m}(t)$.*

We also need the following equation:

$$\begin{cases} -\Delta u + tu + V(x)u = |u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases} \tag{5.7}$$

eq0077

The corresponding functional of (5.7) is given by

$$\mathcal{I}_t(u) = \frac{1}{2}(\|\nabla u\|_2^2 + t\|u\|_2^2 + \int_{\mathbb{R}^3} V(x)u^2 dx) - \frac{1}{p}\|u\|_p^p.$$

This functional is well defined and of class C^2 in the Hilbert space X, which is given by (5.5). We define the usual Nehari manifold of $\mathcal{I}_t(u)$ by

$$\mathcal{P}_t = \{u \in X\backslash\{0\} \mid \mathcal{I}_t'(u)u = 0\}.$$

The associated fibering map for every $u \neq 0$ in X is given by

$$I(s) = \frac{s^2}{2}(\|\nabla u\|_2^2 + t\|u\|_2^2 + \int_{\mathbb{R}^3} V(x)u^2 dx) - \frac{s^p}{p}\|u\|_p^p. \tag{5.8}$$

eq0079

Since $p > 2$, it is standard to show that for every $u \neq 0$ in X, there exists a unique $s_* > 0$ such that $I(s)$ is strictly increasing for $0 < s < s_*$ and is strictly decreasing for $s > s_*$. Let

$$\mathbb{M}(t) = \inf_{v \in \mathcal{P}_t} \mathcal{I}_t(v).$$

Definition 5.2. *We say that u is a ground-state of (5.7) if u is a nontrivial solution of (5.7) with $\mathcal{I}_t(u) = \mathbb{M}(t)$.*

Now, we have the following result of (5.4).

⟨prop0002⟩ **Proposition 5.1.** *Let* $\frac{10}{3} < p < 6$*, then* (5.4) *has a positive ground-state* w_t *for all* $t > 0$ *satisfying* $\|w_t\|_2^2 \sim t^{\frac{3p-10}{2(p-2)}}$ *as* $t \to 0$ *and* $w_t \to w_\infty$ *strongly in* $H^1(\mathbb{R}^3)$ *as* $t \to +\infty$*, where* w_∞ *is the unique (up to translations) positive solution of the following equation:*

$$\begin{cases} -\Delta w + w = |w|^{p-2}w & \text{in } \mathbb{R}^3, \\ w \in H^1(\mathbb{R}^3). \end{cases} \tag{5.9}$$ eq0072

Moreover, w_t *is unique for* $t > 0$ *sufficiently large.*

Proof. The proof is standard so we only sketch it here. We first prove the existence of ground-states of (5.4). By the discussion in [7, 4.2 Symmetry of minimizers], we know that for the energy level $\mathfrak{m}(t)$, there exists a minimizing sequence $\{w_n\}$ on the Nehari manifold $\mathcal{M}_t$ such that w_n is real and positive. Moreover, w_n is radial and decreasing w.r.t. (x_1, x_2) for all x_3 and w_n is even and decreasing w.r.t. x_3 for all (x_1, x_2). Since $\frac{10}{3} < p < 6$, it is standard to use the fibering maps (5.6) to show that $\mathfrak{m}(t) > 0$ on $\mathcal{M}_t$. Thus, by [7, Lemma 3.4], there exists $\{z_n\} \in \mathbb{R}$ such that

$$w_n(x_1, x_2, x_3 - z_n) \rightharpoonup w_0 \neq 0 \quad \text{weakly in } X \text{ as } n \to \infty.$$

Since $\frac{10}{3} < p < 6$, the fibering map of every $w \neq 0$ in X, see (5.6), has a unique maximum point s_0' and it interacts the Nehari manifold $\mathcal{M}_t$ only at the unique maximum point s_0'. Thus, we can use standard arguments (cf. [31, Proposition 3.1]) to show that w_0 is a positive ground-state of (5.4). We next prove the convergent conclusion for $t \to +\infty$. Let w_t be a positive ground-state of (5.4) for $t > 0$. Since $V(x) \geq 0$, $t > 0$ and w_t is positive, we know that w_t satisfies

$$-\Delta w_t + w_t \leq w_t^{p-1} \quad \text{in } \mathbb{R}^3. \tag{5.10}$$ eq0071

By using the fibering maps (5.6) in a standard way (cf. [31, Lemma 3.2]), we know that $\mathfrak{m}(t)$ is decreasing w.r.t. $t > 0$. Thus, $\{w_t\}$ is bounded in $H^1(\mathbb{R}^3)$. It follows from (5.10) and the classical elliptic estimates that

$$w_t \lesssim (1 + |x|)^{-1} e^{-\frac{1}{2}|x|} \quad \text{in } \mathbb{R}^3 \text{ for } t \geq 1. \tag{5.11}$$ eqnew0001

Thus, by $V(x) = x_1^2 + x_2^2$,

$$\int_{\mathbb{R}^3} V(x) w_t^2 dx \lesssim 1 \quad \text{for all } t \geq 1,$$

which implies that

$$t^{-2} \int_{\mathbb{R}^3} V(x) w_t^2 dx = o_t(1) \quad \text{as } t \to +\infty. \tag{5.12}$$ eq0097

Now, using the fibering maps (5.6) in a standard way, we know that $\mathfrak{m}(t) \geq \mathfrak{m} + o_t(1)$ as $t \to +\infty$, where

$$\mathfrak{m} = \inf_{v \in \mathcal{M}} \mathcal{J}(v)$$

with

$$\mathcal{J}(w) = \frac{1}{2}(\|\nabla w\|_2^2 + \|w\|_2^2) - \frac{1}{p}\|w\|_p^p$$

and

$$\mathcal{M} = \{w \in H^1(\mathbb{R}^3) \backslash \{0\} \mid \mathcal{J}'(w)w = 0\}.$$

On the other hand, it is well known that (5.9) has a unique (up to translations) positive radial solution w_∞, which exponentially decays to zero at infinity. Thus, using w_∞ as a test function and adapting the property of the fibering maps (5.6) in a standard way, we also have $\mathfrak{m}(t) \leq \mathfrak{m} + o_t(1)$ as $t \to +\infty$. It follows that $\mathfrak{m}(t) = \mathfrak{m} + o_t(1)$ as $t \to +\infty$, which implies that $\|w_t\|_p^p = \|w_\infty\|_p^p + o_t(1)$. Now, by standard arguments and the uniqueness of w_∞, we can show that $w_t \to w_\infty$ strongly in $H^1(\mathbb{R}^3)$ as $t \to +\infty$. We now turn to the proof of the convergent conclusion for $t \to 0$. For every $t > 0$, let w_t be a positive ground-state of (5.4), then by (5.2), u_t is a positive solution of (5.7). Moreover, by direct calculations,

$$\mathcal{J}_t(w_t) = t^{\frac{p-6}{2(p-2)}}\mathcal{I}_t(u_t) \quad \text{and} \quad \mathcal{J}_t'(w_t)w_t = t^{\frac{p-6}{2(p-2)}}\mathcal{I}_t'(u_t)u_t.$$

Thus, u_t is a positive ground-state of (5.7) for all $t > 0$. On the other hand, by [7, Lemma 2.1], Hölder and Sobolev inequalities,

$$\|u\|_p^p \lesssim \|u\|_2^{\frac{6-p}{2}}\|\nabla u\|_2^{\frac{3p-6}{2}} \lesssim \|u\|_X^p \quad \text{for all } u \in X.$$

Thus, by using the fibering maps (5.8) in a standard way, we know that $\mathbb{M}(0) > 0$. By similar arguments as that used above to compare the energy levels $\mathbb{M}(0)$ and $\mathbb{M}(t)$, we can obtain that $\mathbb{M}(t) = \mathbb{M}(0) + o_t(1)$ as $t \to 0$. It follows that $\{u_t\}$ is bounded in X and $\|u_t\|_p^p \sim 1$ as $t \to 0$. By [7, Lemma 2.1], $\{u_t\}$ is also bounded in $H^1(\mathbb{R}^3)$ as $t \to 0$. Now, by the Lions' lemma (cf. [21, Lemma I.1] or [32, Lemma 1.21]), we can conclude that $\|u_t\|_2^2 \sim 1$ as $t \to 0$. It follows from (5.2) that $\|w_t\|_2^2 \sim t^{\frac{3p-10}{2(p-2)}}$ as $t \to 0$. We close this proof by showing the uniqueness of w_t for $t > 0$ sufficiently large. Let w_t and w_t' be two different positive ground-states of (5.4) and we define $\phi_t = \frac{w_t - w_t'}{\|w_t - w_t'\|_{L^\infty(\mathbb{R}^3)}}$. Then by the Taylor expansion,

$$-\Delta\phi_t + \phi_t + t^{-2}V(x)\phi_t = (p-1)(w_t + \theta(w_t - w_t')))^{p-2}\phi_t, \quad \text{in } \mathbb{R}^3,$$

where $\theta \in (0,1)$. Since $V(x) \geq 0$, by (5.11),

$$-\Delta(\phi_t)^2 + \frac{3}{2}(\phi_t)^2 \leq 0, \quad \text{in } \mathbb{R}^3.$$

Thus, by the maximum principle, $|\phi_t| \lesssim e^{-\frac{1}{2}|x|}$ for $|x| \geq 1$. It is standard to show that $\phi_t \to \phi$ strongly in any compact sets as $t \to +\infty$ and

$$-\Delta\phi + \phi = (p-1)w_\infty^{p-2}\phi, \quad \text{in } \mathbb{R}^3.$$

Note that w_t and w_t' are radial w.r.t. (x_1, x_2) for all x_3 and even w.r.t. x_3 for all (x_1, x_2). Thus, ϕ_t is also radial w.r.t. (x_1, x_2) for all x_3 and even w.r.t. x_3 for all (x_1, x_2). Now, by the well-known nondegeneracy of w_∞, we have $\phi_\infty \equiv 0$. It, together with $|\phi_t| \lesssim e^{-\frac{1}{2}|x|}$ for $|x| \geq 1$, contradicts $\|\phi_t\|_{L^\infty(\mathbb{R}^3)} = 1$. Therefore, w_t is unique for $t > 0$ sufficiently large. $\square$

Let w_t be a positive ground-state of (5.4) given by Proposition 5.1 and we define

$$f(r,t) := r^2 - t^{\frac{10-3p}{2(p-2)}}(\frac{6-p}{2p}\|w_t\|_p^p - 2t^{-2}\int_{\mathbb{R}^3} V(x)w_t^2 dx).$$

By Proposition 5.1, for every $t > 0$ sufficiently large, there exists a unique

$$r_t = (t^{\frac{10-3p}{2(p-2)}}(\frac{6-p}{2p}\|w_t\|_p^p - 2t^{-2}\int_{\mathbb{R}^3} V(x)w_t^2 dx))^{\frac{1}{2}} > 0 \tag{5.13}$$ eq0081

such that $f(r_t, t) = 0$. Thus, by (5.2), (u_{r_t}, t) is a positive normalized solution of (1.14) with a positive Lagrange multiplier $t > 0$. We are now prepared for the proof of Theorem 1.4.

Proof of Theorem 1.4: By the uniqueness of w_t given by Proposition 5.1 for $t > 0$ sufficiently large, say $t > T_*$. It is standard to show that $\int_{\mathbb{R}^3} V(x)w_t^2$ is continuous for $t > T_*$. Note that by Proposition 5.1,

$$(\frac{6-p}{2p}\|w_t\|_p^p - 2t^{-2}\int_{\mathbb{R}^3} V(x)w_t^2 dx) = \frac{6-p}{2p}\|w_\infty\|_p^p + o_t(1).$$

Thus, by $\frac{10}{3} < p < 6$, for every $r < (T_*^{\frac{10-3p}{2(p-2)}}(\frac{6-p}{2p}\|w_{T_*}\|_p^p - 2T_*^{-2}\int_{\mathbb{R}^3} V(x)w_{T_*}^2 dx))^{\frac{1}{2}}$, $f(r, t) = 0$ has a solution $t_r > T_*$. This, together with [7, Theorem 2], implies that (1.14) has a second positive normalized solution $u_{r,2}$ with a positive Lagrange multiplier $\lambda_{r,2}$. The asymptotic behavior of $u_{r,2}$ and $\lambda_{r,2}$ is obtained by (5.2) and (5.13). It remains to show that $u_{r,2}$ is a mountain-pass solution of (1.14) for $r > 0$ sufficiently small. As that in [7, Remark 1.10], we introduce the mountain-pass level

$$\alpha(r) = \inf_{g\in\Gamma_r}\max_{t\in[0,1]} \mathcal{Y}(g[t]),$$

where $\mathcal{Y}(u) = \frac{1}{2}\|u\|_X^2 - \frac{1}{p}\|u\|_p^p$ and

$$\Gamma_r = \{g[s] \in C([0,1], \mathcal{S}_r) \mid g[0] = u_{r,1} \quad \text{and} \quad \mathcal{Y}(g[1]) < \mathcal{Y}(g[0])\}$$

with $u_{r,1}$ being a local minimizer of $\mathcal{Y}(u)$ in $\mathcal{S}_r$ found in [7] and $\mathcal{S}_r = \{u \in X \mid \|u\|_2^2 = r^2\}$. Let

$$B_{\rho,X,t} = \{u \in X \mid \|u\|_{X,t}^2 \leq \rho^2\},$$

where $\|u\|_{X,t}$ is a norm in X given by

$$\|w\|_{X,t} = (\|\nabla w\|_2^2 + \|w\|_2^2 + t^{-2}\int_{\mathbb{R}^3} V(x)w^2 dx)^{\frac{1}{2}}.$$

Then by [7, Lemma 2.1] and the Sobolev inequality, for a fixed $\rho > 0$ sufficiently small, it can be proved by using $\frac{10}{3} < p < 6$ in a standard way that

$$\mathfrak{m}(t) = \inf_{h\in\Theta}\max_{t\in[0,1]} \mathcal{J}_t(h[s]),$$

where

$$\Theta = \{h[t] \in C([0,1], X) \mid h[0] \in B_{\rho,X,t} \quad \text{and} \quad \mathcal{J}_t(h[1]) < \frac{1}{4}\rho^2\}.$$

Now, for every $g[s] \in \Gamma_r$, we define $g^*[s] = \lambda_{r,2}^{-\frac{1}{p-2}} g[s](\lambda_{r,2}^{-\frac{1}{2}}x)$. Then

$$\mathcal{J}_{\lambda_{r,2}}(g^*[s]) = \lambda_{r,2}^{\frac{p-6}{2(p-2)}}(\mathcal{Y}(g[s]) + \frac{\lambda_{r,2}r^2}{2}).$$

By [7, Theorem 1] and (1.15),

$$\|g^*[0]\|_{X,\lambda_{r,2}}^2 \lesssim r^2\lambda_{r,2}^{\frac{p-6}{2(p-2)}} \sim \lambda_{r,2}^{-1} \to 0 \quad \text{as } r \to 0.$$

Thus, $g^*[0] \in B_{\rho,X,\lambda_{r,2}}$ for $r > 0$ sufficiently small and $\mathcal{J}_t(g^*[0]) \to 0$ as $r \to 0$. By the definition of $g[t]$, we also have $\mathcal{J}_t(g^*[1]) < \frac{1}{4}\rho^2$. It follows that $g^*[t] \in \Theta$, which implies

$$\mathfrak{m}(\lambda_{r,2}) \leq \lambda_{r,2}^{\frac{p-6}{2(p-2)}}(\alpha(r) + \frac{\lambda_{r,2}r^2}{2}).$$

On the other hand, the fibering map of $\mathcal{Y}(u)$ at $u_{r,2}$ is given by

$$\mathcal{T}_{u_{r,2}}(\tau)=\frac{\tau^2}{2}\|\nabla u_{r,2}\|_2^2+\frac{1}{2\tau^2}\int_{\mathbb{R}^3}V(x)u_{r,2}^2dx-\frac{\tau^{p\gamma_p}}{p}\|u_{r,2}\|_p^p.$$

By direct calculations,

$$\mathcal{T}_{u_{r,2}}'(\tau)=\tau\|\nabla u_{r,2}\|_2^2-\frac{1}{\tau^3}\int_{\mathbb{R}^3}V(x)u_{r,2}^2dx-\gamma_p\tau^{p\gamma_p-1}\|u_{r,2}\|_p^p$$

and

$$\mathcal{T}_{u_{r,2}}''(\tau)=\|\nabla u_{r,2}\|_2^2+\frac{3}{\tau^4}\int_{\mathbb{R}^3}V(x)u_{r,2}^2dx-\gamma_p(p\gamma_p-1)\tau^{p\gamma_p-2}\|u_{r,2}\|_p^p.$$

Clearly, $\mathcal{T}_{u_{r,2}}'(1)=0$. Moreover, by (1.17), (5.12) and the Pohozaev identity of w_∞,

$$\mathcal{T}_{u_{r,2}}''(1)=\lambda_{r,2}^{\frac{6-p}{2(p-2)}}(\gamma_p\|w_\infty\|_p^p(2-p\gamma_p)+o_r(1))<0$$

for $r>0$ sufficiently small. Now, let $h(\tau)=\tau^4\|\nabla u_{r,2}\|_2^2-\gamma_p\tau^{p\gamma_p+2}\|u_{r,2}\|_p^p$, then,

$$\max_{\tau\geq0}h(\tau)=\left[\frac{4\|\nabla u_{r,2}\|_2^2}{\gamma_p(p\gamma_p+2)\|u_{r,2}\|_p^p}\right]^{\frac{4}{p\gamma_p-2}}\frac{p\gamma_p-2}{p\gamma_p+2}\|\nabla u_{r,2}\|_2^2>\int_{\mathbb{R}^3}V(x)u_{r,2}^2dx.$$

It follows that there exists $\tau_r<1$ such that $\mathcal{T}_{u_{r,2}}'(\tau_r)=0$ and $\mathcal{T}_{u_{r,2}}''(\tau_r)>0$. We claim that $\tau_r\to0$ as $r\to0$. If not, then, there exists $r_n\to0$ such that $\tau_{r_n}\gtrsim1$ as $n\to\infty$. Without loss of generality, we may assume $\tau_r\gtrsim1$ for all $r>0$ sufficiently small. By $\mathcal{T}_{u_{r,2}}'(\tau_r)=0$, (1.17), (5.12) and the Pohozaev identity of w_∞,

$$\begin{aligned}\mathcal{T}_{u_{r,2}}''(\tau)&=\lambda_{r,2}^{\frac{6-p}{2(p-2)}}(4\|\nabla w_\infty\|_2^2-\gamma_p(p\gamma_p+2)\|w_\infty\|_p^p\tau_r^{p\gamma_p-2}+o_r(1))\\&=\lambda_{r,2}^{\frac{6-p}{2(p-2)}}\gamma_p(4-(p\gamma_p+2)\tau_r^{p\gamma_p-2}+o_r(1))\|w_\infty\|_p^p,\end{aligned}$$

which implies $\tau_r<(\frac{4}{p\gamma_p+2})^{\frac{1}{p\gamma_p-2}}<1$. Without loss of generality, we may assume that $\tau_r\to\tau_0$ as $r\to0$. Then by $\mathcal{T}_{u_{r,2}}'(\tau_r)=0$, (5.12) and the fact that w_∞ solves (1.16), we must have $\tau_0=0$. It is impossible. Thus, we must have $\tau_r\to0$ as $r\to0$. By (1.15) and (1.17),

$$\frac{1}{\tau_r^4\lambda_{r,2}^2}(\int_{\mathbb{R}^3}V(x)w_\infty^2dx+o_r(1))=\|\nabla w_\infty\|_2^2+o_r(1).$$

It follows from (1.15) and (1.17) that

$$\|(u_{r,2})_{\tau_r}\|_X^2=\tau_r^2\|\nabla u_{r,2}\|_2^2+\tau_r^{-2}\int_{\mathbb{R}^3}V(x)u_{r,2}^2dx\sim\lambda_{r,2}^{\frac{10-3p}{2(p-2)}}\sim r^2$$

as $r\to0$, where $(u_{r,2})_{\tau_r}=\tau_r^{\frac{3}{2}}u_{r,2}(\tau_rx)$. Thus, $(u_{r,2})_{\tau_r}\in B_{r\chi,X,1}$ for a fixed and large $\chi>0$. Since $B_{r\chi,X,1}$ is connected, we can find a continuous path $\Upsilon:[0,1]$ with $\Upsilon(0)=u_{r,1}$ and $\Upsilon(1)=(u_{r,2})_{\tau_r}$. Now, we define

$$h^{**}[s]=\begin{cases}\Upsilon[(2s)], & 0\leq s\leq\dfrac{1}{2},\\(u_{r,2})_{2(1-s)\tau_r+(2s-1)\tau_{r,*}}, & \dfrac{1}{2}\leq s\leq1,\end{cases}$$

where we choose $\tau_{r,*} > 1$ such that $\mathcal{T}_{u_{r,2}}(\tau_{r,*}) < \mathcal{Y}(u_{r,1})$. Note that $\mathcal{Y}(u) \lesssim r^2$ in $B_{r\chi,X,1}$ and $\mathcal{Y}(u_{r,2}) \gtrsim 1$ by (1.15) and (1.17). Thus, for $r > 0$ sufficiently small, $h^{**}[s] \in \Gamma_r$ and

$$\alpha(r) \leq \max_{0\leq s\leq 1} h^{**}[s] = \mathcal{T}_{u_{r,2}}(1) = \mathfrak{m}(\lambda_{r,2})\lambda_{r,2}^{\frac{6-p}{2(p-2)}} - \frac{\lambda_{r,2}r^2}{2}.$$

Therefore, $\mathfrak{m}(\lambda_{r,2})\lambda_{r,2}^{\frac{6-p}{2(p-2)}} - \frac{\lambda_{r,2}r^2}{2} = \alpha(r)$ and $u_{r,2}$ is a mountain-pass solution of (1.14) for $r > 0$ sufficiently small. $\square$

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References

AIK20 [1] T. Akahori, S. Ibrahim, H. Kikuchi, Linear instability and nondegeneracy of ground state for combined power-type nonlinear scalar field equations with the Sobolev critical exponent and large frequency parameter, *Proc. Roy. Soc. Edinburgh Sect. A,* **150** (2020), 2417–2441.

AIKN12 [2] T. Akahori, S. Ibrahim, H. Kikuchi and H. Nawa, Existence of a ground state and blow-up problem for a nonlinear Schrödinger equation with critical growth, *Differ. Integral Equ.,* **25** (2012), 383–402.

AIKN2021 [3] T. Akahori, S. Ibrahim, H. Kikuchi, H. Nawa, Global dynamics above the ground state energy for the combined power type nonlinear Schrodinger equations with energy critical growth at low frequencies, arXiv:1510.08034 [Math. AP](to appear in Memoirs of the AMS).

AIIKN19 [4] T. Akahori, S. Ibrahim, N. Ikoma, H. Kikuchi, H. Nawa, Uniqueness and nondegeneracy of ground states to nonlinear scalar field equations involving the Sobolev critical exponent in their nonlinearities for high frequencies, *Calc. Var. PDEs,* **58** (2019), 120.

AP86 [5] F. Atkinson, L. Peletier, Emden-Fowler equations involving critical exponents, *Nonlinear Anal.,* **10**(1986), 755–776.

ASM12 [6] C.O. Alves, M.A.S. Souto, M. Montenegro, Existence of a ground state solution for a nonlinear scalar field equation with critical growth, *Calc. Var. PDEs,* **43**(2012), 537–554.

BBJV17 [7] J. Bellazzini, N. Boussaid, L. Jeanjean, N. Visciglia, Existence and Stability of Standing Waves for Supercritical NLS with a Partial Confinement, *Commun. Math. Phys.,* **353**(2017), 229–251.

BL83 [8] H. Berestycki, P.-L. Lions, Nonlinear scalar field equations. I: Existence of a ground state, *Arch. Ration. Mech. Anal.,* **82**(1983), 313–345.

BN83 [9] H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents. *Commun. Pure Appl. Math.,* **36**(1983), 437–477.

CDG16 [10] W. Chen, J. Dávila, I. Guerra, Bubble tower solutions for a supercritical elliptic problem in $\mathbb{R}^N$, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5),* **15** (2016), 85–116.

CZ12 [11] Z. Chen, W. Zou, On the Brezís-Nirenberg problem in a ball, *Differ. Integral Equ.,* **25**(2012), 527–542.

C72 [12] C. Coffman, Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions. *Arch. Rational Mech. Anal.,* **46** (1972), 81–95.

CG20 [13] M. Coles, S. Gustafson, Solitary Waves and Dynamics for Subcritical Perturbations of Energy Critical NLS, *Publ. Res. Inst. Math. Sci.,* **56** (2020), 647–699.

DPG13 [14] J. Dávila, M. del Pino, I. Guerra, Non-uniqueness of positive ground states of non-linear Schrödinger equations, *Proc. Lond. Math. Soc.,* **106**(2013), 318–344.

FG03 [15] A. Ferrero, F. Gazzola, On subcriticality assumptions for the existence of ground states of quasilinear elliptic equations, *Adv. Differential Equations,* **8** (2003), 1081–1106.

GNN81 [16] B. Gidas, W. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$, in: L. Nachbin (Ed.), Math. Anal. Appl. Part A, Advances in Math. Suppl. Studies, vol. 7A, Academic Press, 1981, pp. 369–402.

GS03 [17] F. Gazzola, J. Serrin, Asymptotic behavior of ground states of quasilinear elliptic problems with two vanishing parameters, *Ann. Inst. H. Poincaré Anal. Non Linéaire,* **19**(2002), 477–504.

JL201 [18] L. Jeanjean, T. Le, Multiple normalized solutions for a Sobolev critical Schrödinger equation, *Math. Ann.*, DOI: 10.1007/s00208-021-02228-0.

KP89 [19] M. Knaap, L. Peletier, Quasilinear elliptic equations with nearly critical growth, *Comm. PDEs,* **14**(1989), 1351–1383.

K89 [20] M. Kwong, Uniqueness of positive solution of $\Delta u - u + u^p = 0$ in $\mathbf{R}^N$. *Arch. Rational Mech. Anal.,* **105** (1989), 243–266.

L84 [21] P.L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part 2, *Ann. Inst. H. Poincaré Anal. Non Linéaire,* **1** (1984), 223-283.

LLT17 [22] J. Liu, J.-F. Liao, C.-L. Tang, Ground state solution for a class of Schrödinger equations involving general critical growth term, *Nonlinearity,* **30** (2017), 899–911.

MM21 [23] S. Ma, V. Moroz, Asymptotic profiles for a nonlinear Schrödinger equation with critical combined powers nonlinearty, arXiv2108.01421 [Math. AP].

M93 [24] K. McLeod, Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^N$. II, *Trans. Amer. Math. Soc.,* **339** (1993), 495–505.

MS87 [25] K. McLeod, J. Serrin, Uniqueness of positive radial solutions of $\Delta u + f(u) = 0$ in $\mathbb{R}^N$, *Arch. Ration. Mech. Anal.,* **99** (1987), 115–145.

PS83 [26] L. Peletier, J. Serrin, Uniqueness of positive solutions of semilinear equations in $\mathbb{R}^N$, *Arch. Ration. Mech. Anal.,* **81** (1983), 181–197.

PS86 [27] L. Peletier, J. Serrin, Uniqueness of nonnegative solutions of semilinear equations in $\mathbb{R}^N$, *J. Differential Equations,* **61** (1986), 380–397.

PS98 [28] P. Pucci, J. Serrin, Uniqueness of ground states for quasilinear elliptic operators, *Indiana Univ. Math. J.,* **47** (1998), 501–528.

ST00 [29] J. Serrin, M. Tang, Uniqueness of ground states for quasilinear elliptic equations, *Indiana Univ. Math. J.,* **49** (2000), 897–923.

S20 [30] N. Soave, Normalized ground states for the NLS equation with combined nonlinearities: The Sobolev critical case. *J. Funct. Anal.,* **279**(2020), 108610.

WW21 [31] J. Wei, Y. Wu, Normalized solutions for Schrodinger equations with critical Sobolev exponent and mixed nonlinearities, arXiv:2102.04030 [Math. AP].

W96 [32] M. Willem, Minimax Theorems. Birkhäuser, Boston, 1996.

ZZ12 [33] J. Zhang, W. Zou, A Berestycki-Lions theorem revisited, *Commun. Contemp. Math.,* **14** (2012), 1250033.

Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada, V6T 1Z2

E-mail address: `jcwei@math.ubc.ca`

School of Mathematics, China University of Mining and Technology, Xuzhou, 221116, P.R. China

E-mail address: `wuyz850306@cumt.edu.cn`