On Sirakov’s open problem and related topics*

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Abstract

In the present paper, we make some progress on the Sirakov’s open problem (Comm. Math. Phys., 2007) about the existence of nontrivial nonnegative solution to the coupled nonlinear system

\[
\begin{align*}
\begin{cases}
-\Delta u + \lambda u &= \mu_1 u^3 + \beta uv^2 & \text{in } \mathbb{R}^N, \\
-\Delta v + v &= \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N, \\
u, v &> 0 & \text{in } \mathbb{R}^N, \quad N \leq 3.
\end{cases}
\end{align*}
\]

We also study some other properties for related questions, such as the uniqueness of the ground state solution, the asymptotic behavior of the least energy solution, nonexistence of the positive solution and the multiplicity of positive solutions, etc.

Key words: Schrödinger System; Ground State; Sirakov’s Open Problem.

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1 Introduction

The following Schrödinger systems attract a lot of people’s attention due to the background of quantum mechanics:

\[
\begin{cases}
\Delta u + \lambda_1 u = \mu_1 u^3 + \beta uv^2, & \text{in } \mathbb{R}^N, \\
\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2v, & \text{in } \mathbb{R}^N, \\
u, v > 0, & \text{in } \mathbb{R}^N, \quad N \leq 3.
\end{cases}
\] (1.1)

Note that \((u, v)\) is a positive solution to (1.1) if and only if

\[
\bar{u}(x) := \frac{1}{\sqrt{\lambda_2}} u \left( x/\sqrt{\lambda_2} \right), \quad \bar{v}(x) := \frac{1}{\sqrt{\lambda_2}} v \left( x/\sqrt{\lambda_2} \right),
\]

solve the following coupled nonlinear system with \(\lambda = \frac{\lambda_1}{\lambda_2} \):

\[
\begin{cases}
\Delta u + \lambda u = \mu_1 u^3 + \beta uv^2, & \text{in } \mathbb{R}^N, \\
\Delta v + v = \mu_2 v^3 + \beta u^2v, & \text{in } \mathbb{R}^N, \\
u, v > 0, & \text{in } \mathbb{R}^N, \quad N \leq 3.
\end{cases}
\] (1.2)

Based on this observation, we focus on the system (1.2) in this current paper. We search for nontrivial solutions of (1.2), or equivalently, for nontrivial critical points of the functional

\[
\Phi^\beta_{\lambda}(u, v) := \frac{1}{2} \left( |\nabla u|^2 + \lambda |u|^2 \right) + \frac{1}{2} \left( |\nabla v|^2 + |v|^2 \right) - \frac{1}{4} \left( \mu_1 |u|^4 + \beta |uv|^2 \right) \] (1.3)

on the energy space \(\mathcal{H} := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\). We denote by \(\mathcal{H}_{rad}\) the set of couples in \(\mathcal{H}\) who are radially symmetric with respect to a fixed point in \(\mathbb{R}^N\). As in [13, 14], etc, we consider the set

\[
\mathcal{M}_{\lambda}^\beta := \{ (u, v) \in \mathcal{H} : u \not\equiv 0, v \not\equiv 0, |\nabla u|^2 + \lambda |u|^2 = \mu_1 |u|^4 + \beta |uv|^2, |\nabla v|^2 + |v|^2 = \mu_2 |v|^4 + \beta |uv|^2 \} .
\] (1.4)

We also set

\[
\mathcal{A}_{\lambda}^\beta = \inf_{(u, v) \in \mathcal{M}_{\lambda}^\beta} \Phi^\beta_{\lambda}(u, v) = \inf_{(u, v) \in \mathcal{M}_{\lambda}^\beta} \frac{1}{4} \left( |\nabla u|^2 + \lambda |u|^2 + |\nabla v|^2 + |v|^2 \right),
\] (1.5)

and

\[
\mathcal{A}_{\lambda, rad}^\beta = \inf_{(u, v) \in \mathcal{H}_{rad} \cap \mathcal{M}_{\lambda}^\beta} \Phi^\beta_{\lambda}(u, v) = \inf_{(u, v) \in \mathcal{H}_{rad} \cap \mathcal{M}_{\lambda}^\beta} \frac{1}{4} \left( |\nabla u|^2 + \lambda |u|^2 + |\nabla v|^2 + |v|^2 \right).
\] (1.6)

Let \(U\) be the unique positive solution (cf. [12]) to

\[
- \Delta u + u = u^3 \text{ in } \mathbb{R}^N; \quad u(x) \to 0 \text{ as } |x| \to \infty;
\] (1.7)
then
\[ S|u|^2 \leq (|\nabla u|_2^2 + |u|_2^2) \text{ for all } u \in H^1(\mathbb{R}^N), \] (1.8)
where the sharp constant
\[ S = (|\nabla U|_2^2 + |U|_2^2)^{\frac{1}{2}} = |U|_4. \] (1.9)
We note that by [5, Lemma 3.3], (1.2) has no solution for \( \lambda \leq 0 \). Hence, we always assume \( \lambda > 0 \) in the present paper. Let \( S_\lambda \) be the sharp constant of
\[ S_\lambda |u|^2 \leq (|\nabla u|_2^2 + \lambda |u|_2^2) \text{ for all } u \in H^1(\mathbb{R}^N). \] (1.10)
Then we have
\[ S_1 = S \text{ and } S_\lambda = \lambda^{\frac{4-N}{2}} S. \] (1.11)
Setting
\[ U_{\lambda,\mu}(x) = \frac{\sqrt{\lambda}}{\sqrt{\mu}} U(\sqrt{\lambda}x), \quad \lambda > 0, \mu > 0; \]
it is easy to check that \( U_{\lambda,\mu} \) is the unique positive solution to the following equation:
\[-\Delta u + \lambda u = \mu u^3 \text{ in } \mathbb{R}^N, \quad \lambda > 0, \text{ } u(x) \to 0 \text{ as } |x| \to \infty. \] (1.12)
Let \( m_{\lambda,\mu} \) be the corresponding least energy, i.e.,
\[ m_{\lambda,\mu} := \frac{1}{2} \left( |\nabla U_{\lambda,\mu}|_2^2 + \lambda |U_{\lambda,\mu}|_2^2 \right) - \frac{\mu}{4} |U_{\lambda,\mu}|_4^2 = \frac{\lambda^{\frac{4-N}{4}}}{4\mu} S^2 = \frac{1}{4\mu} S_\lambda^2. \] (1.13)
Hence, we have that
\[ (|\nabla u|_2^2 + \lambda |u|_2^2) \geq 2 \sqrt{m_{\lambda,\mu}} \left( \int_{\mathbb{R}^N} \mu u^4 \right)^{\frac{1}{2}} = \frac{1}{\sqrt{\mu}} S_\lambda \left( \int_{\mathbb{R}^N} \mu u^4 \right)^{\frac{1}{2}} \] (1.14)
for all \( u \in H^1(\mathbb{R}^N). \) As in [4, (1.6)] we introduce the function \( \tau : \mathbb{R}^+ \to \mathbb{R}^+ \) defined by
\[ \tau(s) := \inf_{\phi \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla \phi|^2 + s \phi^2)}{\int_{\mathbb{R}^N} U_\phi^2 \phi^2}. \] (1.15)
Define the functions (see also [5, (3.4)])
\[ \beta_1(\lambda) = \mu_1 \tau(1/\lambda) \quad \text{and} \quad \beta_2(\lambda) = \mu_2 \tau(\lambda) \quad \text{for } \lambda > 0. \] (1.16)
Let
\[ J_\lambda^\beta(u, v) := \langle \Phi_\lambda^\beta(u, v), (u, v) \rangle = |\nabla u|_2^2 + |\nabla v|_2^2 + \lambda |u|_2^2 + |v|_2^2 - \mu_1 |u|_4^4 - \mu_2 |v|_4^4 - 2\beta |uv|_2^2. \]
We also consider the another kind of Nehari manifold
\[ \mathcal{M}_\lambda^2 := \{(u, v) \in \mathcal{H} \setminus \{(0, 0)\} : J_\lambda^\beta(u, v) = 0 \}. \] (1.17)
and define $\tilde{A}_1^\beta, \tilde{A}_{\lambda,rad}^\beta$ similar to (1.5), (1.6).

Then values $\beta_i(\lambda)(i = 1, 2)$ defined by (1.16) are the important numbers to determine the geometric structure of $\Phi_1^\beta$ at the semi-trivial solutions $(U_{\lambda,\mu_1}, 0)$ and $(0, U_{1,\mu_2})$ on $\tilde{M}_1^\beta$. Take $i = 1$ as an example, if $\beta < \beta_1(\lambda)$, then $(U_{\lambda,\mu_1}, 0)$ is a strict local minima of $\Phi_1^\beta$ on $\tilde{M}_1^\beta$. While $\beta > \beta_1(\lambda)$, $(U_{\lambda,\mu_1}, 0)$ is a saddle point of $\Phi_1^\beta$ on $\tilde{M}_1^\beta$. A similar result holds for $i = 2$ (see [1, Lemma 4]).

1.1 Ambrosetti and Colorado’s conjecture

Theorem A (cf.[1, Theorem 1 and Theorem 2])

(i) System (1.2) has a positive radially symmetric solution $(u^*, v^*)$ of mountain pass type for any $0 < \beta < \min\{\beta_1(\lambda), \beta_2(\lambda)\}$;

(ii) System (1.2) has a radial ground state $(u^*, v^*)$ provided $\beta > \max\{\beta_1(\lambda), \beta_2(\lambda)\}$ and the corresponding energy satisfies

$$A_1^\beta = A_{\lambda,rad}^\beta = \tilde{A}_1^\beta = \tilde{A}_{\lambda,rad}^\beta < \min\{m_{\lambda,\mu_1}, m_{1,\mu_2}\}.$$ 

Ambrosetti and Colorado’s conjecture: In [1, Remark 5], Ambrosetti and Colorado conjectured that the solution $(u^*, v^*)$ obtained in Theorem A-(i) is also a least energy solution. In other word, $(u^*, v^*)$ attains $A_1^\beta$.

In [14], Sirakov studied the existence of the least energy solution and made some progress on the Ambrosetti and Colorado’s conjecture. Precisely, if $\lambda = 1$, let

$$u_\beta(x) = \sqrt{\frac{\beta - \mu_2}{\beta^2 - \mu_1\mu_2}} U(x), \quad v_\beta(x) = \sqrt{\frac{\beta - \mu_1}{\beta^2 - \mu_1\mu_2}} U(x).$$

One can check that $(u_\beta, v_\beta)$ is a mountain pass solution of (1.2). By [14, Theorem 1-(i)], $(u_\beta, v_\beta)$ attains $A_1^\beta$ provided $0 < \beta < \min\{\mu_1, \mu_2\}$. Noting that $\beta_i(1) = \mu_i, i = 1, 2$, we see that Ambrosetti and Colorado’s conjecture is solved for $\lambda = 1$. However, the general case with $\lambda \neq 1$ is much more complicated. In [14], Sirakov also made some progress on Ambrosetti and Colorado’s conjecture via estimating the value of $\beta_i(\lambda), i = 1, 2$.

Theorem B (cf.[14, Theorem 2]) Suppose $\lambda \leq 1$ in (1.2) and set $\nu_1 = \mu_1\lambda^{1 - \frac{N}{2}}, \nu_2 = \mu_2\lambda^{\frac{N}{2} - 1}$.

(i) Let $\nu_0$ be the smaller root of the equation

$$\lambda^{\frac{N}{2}} t^2 - (\nu_1 + \nu_2)t + \nu_1\nu_2 = 0.$$ 

Then $A_\lambda^\beta = A_{\lambda,rad}^\beta$ is attained by a solution of (1.2) provided $0 < \beta < \nu_0$. 

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(ii) If \( \mu_2 \leq \beta \leq \mu_1 \) and \( \mu_2 < \mu_1 \), then system (1.2) has no solution.

(iii) If \( \beta > \max\{\mu_1 \lambda^{-1}, \mu_2 \lambda^{1-\frac{2}{N}}\} \), then \( A^\beta = A^\beta_{\lambda,rad} \) is attained by a solution of (1.2).

From the process of the proof of theorems in [14], one can see that
\[ \nu_0 < \min\{\beta_1(\lambda), \beta_2(\lambda)\}, \]
and the solution given in Theorem B-(i) is of mountain pass type. So Theorem B-(i) gives a partial answer to Ambrosetti and Colorado’s conjecture.

In 2011, Ikoma and Tanaka obtained another progress on Ambrosetti and Colorado’s conjecture.

**Theorem C** (cf. [11, Proposition 2.3, Proposition 2.5 and Remark 2.6]) For any \( 0 < \beta < \min\{\beta_1(\lambda), \beta_2(\lambda), \sqrt{\mu_1 \mu_2}\} \), \( (u^*, v^*) \) obtained in Theorem A-(i) is in fact a least energy solution with
\[
\Phi^\beta_\lambda(u^*, v^*) = A^\beta_\lambda > \max\{m_{\lambda, \mu_1}, m_{1, \mu_2}\}. \tag{1.18}
\]

**Remark 1.1.** By a direct computation, one can check that for \( \lambda \leq 1 \), \( \nu_0 \) given in Theorem B-(i) satisfies \( \nu_0 \leq \sqrt{\mu_1 \mu_2} \), and “=” holds only for \( \lambda = 1 \). We see that \( \nu_0 < \min\{\beta_1(\lambda), \beta_2(\lambda), \sqrt{\mu_1 \mu_2}\} \). Hence, Theorem C is an improvement of Theorem B-(i), and also a partial answer to the Ambrosetti and Colorado’s conjecture.

Recently, Bartsch, Zhong and Zou [5] also make some partial progress on the Ambrosetti and Colorado’s conjecture for \( N = 3 \).

**Theorem D** (cf. [5, Theorem 2.5]) Let \( \tau_0 = \lim_{s \to 0} \tau(s) \), with \( \tau(s) \) given by (1.15).

a) Problem (1.2) has at most one positive solution for \( \lambda > 0 \) small or for \( \lambda \) large.

b) If \( \beta \leq \tau_0 \mu_2 \) then (1.2) has a unique positive solution for \( \lambda > 0 \) small. If \( \beta \leq \tau_0 \mu_1 \), then (1.2) has a unique positive solution for \( \lambda \) large.

**Remark 1.2.** By [5, Corollary 4.5], for \( N = 3 \), we have that \( \inf_{\lambda \in (0, \infty)} \beta_i(\lambda) = \tau_0 \mu_i, i = 1, 2 \). We see that \( \beta \) in Theorem D-b) satisfies \( \beta < \min\{\beta_1(\lambda), \beta_2(\lambda)\} \). Thus, by the uniqueness, \( (u^*, v^*) \) given by Theorem A-(i) is a least energy solution. We note that \( \beta \) in Theorem D-b) is allowed to be \( \beta > \sqrt{\mu_1 \mu_2} \). Therefore, it is a progress on the Ambrosetti and Colorado’s conjecture in some sense.

To carry out our following study, we shall also give a better estimate on \( \beta_i(\lambda), i = 1, 2 \), see Lemma 2.6. Especially, for \( N = 1 \), we can give the precise value of \( \beta_i(\lambda) \), see Appendix.

As a by-product, we improve Theorem B as following:

**Theorem 1.3.** Suppose \( \lambda \leq 1 \) in (1.2), then \( A^\beta_\lambda = A^\beta_{\lambda,rad} \) is attained by a solution of (1.2) provided \( 0 < \beta \leq \min\{\mu_1 \lambda^{-\frac{N-4}{4}}, \mu_2 \lambda^{\frac{4-N}{4}}\} \).
1.2 Nonexistence of positive solution

By Theorem A we know that when \((\lambda, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+\) satisfying
\[
\beta \in \left(0, \min\{\beta_1(\lambda), \beta_2(\lambda)\}\right) \cup \left(\max\{\beta_1(\lambda), \beta_2(\lambda)\}, \infty\right),
\]
then the system (1.2) possesses a positive solution. However, much less is known that whether there exists a solution to system (1.2) if
\[
\beta \in \left(0, \min\{\beta_1(\lambda), \beta_2(\lambda)\}\right) \cup \left(\max\{\beta_1(\lambda), \beta_2(\lambda)\}, \infty\right).
\]
Sirakov [14, Theorem 2-(ii)] claims that problem (1.2) has no solution as long as the three constants \((\lambda - 1), (\beta - \mu_1), (\mu_2 - \beta)\) are of the same sign or zero, and one of them is not zero. Indeed, suppose we have a solution \((u, v)\) of system (1.2), a direct computation deduce
\[
\int_{\mathbb{R}^N} uv[(\lambda - 1) + (\beta - \mu_1)u^2 + (\mu_2 - \beta)v^2] = 0. \tag{1.19}
\]
Equality (1.19) can yield the conclusion above. For the case of \(\lambda = 1\), we have \(\beta_i(1) = \mu_i, i = 1, 2\) and system (1.2) has no solution for \(\beta \in [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]\) if \(\mu_1 \neq \mu_2\), see also [3]. While \(\mu_2 = \mu_1 := \mu\) and \(\beta = \mu\), system (1.2) possesses a family of positive solutions
\[
(u, v) = (\cos \theta, \sin \theta) U_{1, \mu}, \quad \theta \in (0, \frac{\pi}{2}).
\]
However, the case of \(\lambda \neq 1\) is far from known. In particular, without loss of generality, we consider the case of \(\mu_2 \leq \mu_1\). Then equality (1.19) yields that if
\[
(\lambda, \beta) \in (0, 1) \times [\mu_2, \mu_1],
\]
problem (1.2) has no solution. By [5, Corollary 4.5], we see that \([\mu_2, \mu_1] \subseteq [\beta_2(\lambda), \beta_1(\lambda)]\) due to the fact
\[
\beta_1(\lambda) > \mu_1 \geq \mu_2 > \beta_2(\lambda), \quad \lambda < 1.
\]
The remain range is open for a long time, especially for the case \(\lambda > 1\) (under the premise of \(\mu_2 \leq \mu_1\)). Recently, Bartsch, Zhong and Zou [5] make a progress on it.

**Theorem E** (cf.[5, Theorem 2.4])

a) For \(\beta \geq \mu_1\) there exists \(\eta_1(\beta) > 0\) such that (1.2) has no positive solution if \(\lambda > \eta_1(\beta)\).

b) For \(\beta \geq \mu_2\) there exists \(\eta_2(\beta) > 0\) such that (1.2) has no positive solution if \(\lambda < \eta_2(\beta)\).

We note that by [5, Corollary 4.5], one can see that \(\beta \in (\beta_2(\lambda), \beta_1(\lambda))\) for \(\lambda\) small and \(\beta \in (\beta_1(\lambda), \beta_2(\lambda))\) for \(\lambda\) large. We also note that Theorem E-a) involves the case of \(\lambda > 1\). In present paper, we shall make another progress for the case of \(\lambda > 1\).

**Theorem 1.4.** Assume that \(\mu_2 < \mu_1\), the system (1.2) has no solution if
\[
(\lambda, \beta) \in \left[1, \sqrt{\frac{\mu_1}{\mu_2}}\right] \times \left[\frac{\mu_1}{\lambda}, \frac{\mu_1}{\lambda}\right].
\]
1.3 Sirakov’s open problem

Basing on the results above, we see that the best range for the existence is

\[ \beta \in \left(0, \min\{\mu_1, \mu_2\}\right) \cup \left(\max\{\mu_1, \mu_2\}, \infty\right), \quad \text{if } \lambda = 1. \]

However, the case of \( \lambda \neq 1 \) is not solved. This is the so-called

Sirakov’s open problem: When \( \lambda \neq 1 \), what is the optimal range for the existence of positive radial solution to the system (1.2)? See [14, Remark 4].

In the following, we consider a weaker problem: what is the optimal ranges for the existence of ground state solution to the system (1.2)? In order to understand better the open problem, we define

\[ \bar{\beta}_1(\lambda) := \sup\{\beta' > 0, \text{ system (1.2) has least energy solutions for all } 0 < \beta < \beta'\}, \]

\[ \bar{\beta}_2(\lambda) := \inf\{\beta' > 0, \text{ system (1.2) has least energy solutions for all } \beta > \beta'\}. \]

We observe that finding the values of \( \bar{\beta}_1(\lambda) \) and \( \bar{\beta}_2(\lambda) \), is one part of the Sirakov’s open problem. For \( \beta \in [\bar{\beta}_1(\lambda), \bar{\beta}_2(\lambda)] \), whether system (1.2) possesses nontrivial solution or not is also unknown. By Theorem A(ii), we see that

\[ \bar{\beta}_2(\lambda) \leq \max\{\beta_1(\lambda), \beta_2(\lambda)\}. \]

(1.20)

We have that finding the values of \( \bar{\beta}_1(\lambda) \) and \( \bar{\beta}_2(\lambda) \), is one part of the Sirakov’s open problem. For \( \beta \in [\bar{\beta}_1(\lambda), \bar{\beta}_2(\lambda)] \), whether system (1.2) possesses nontrivial solution or not is also unknown. By Theorem A(ii), we see that

\[ \bar{\beta}_2(\lambda) \leq \max\{\beta_1(\lambda), \beta_2(\lambda)\}. \]

(1.21)

By Theorem C above, we know that

\[ \bar{\beta}_1(\lambda) \geq \min\{\beta_1(\lambda), \beta_2(\lambda), \sqrt{\mu_1\mu_2}\}. \]

(1.22)

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\[ \bar{\beta}_1(\lambda) \geq \min\{\beta_1(\lambda), \beta_2(\lambda), \sqrt{\mu_1\mu_2}\}. \]

(1.23)

So far, the first (and the best) answer to the above Sirakov’s open problem is given by Chen and Zou [9].

**Theorem F** (cf.[9, Theorem 1.2]) Suppose \( \lambda \neq 1 \) and \( (1 - \lambda)(\mu_2 - \mu_1) \leq 0 \), then

\[ \bar{\beta}_1(\lambda) = \min\{\beta_1(\lambda), \beta_2(\lambda)\}. \]

We note that under the assumption \( (1 - \lambda)(\mu_2 - \mu_1) \leq 0 \), we must have that

\[ \min\{\beta_1(\lambda), \beta_2(\lambda)\} < \min\{\mu_1, \mu_2\} \leq \sqrt{\mu_1\mu_2}. \]

This property plays a crucial role in [9]’s arguments, since they need the fact that system (1.2) has no nontrivial nonnegative solutions for any

\[ \beta \in [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}], \]

which requires that the three constants \( (\lambda - 1), (\beta - \mu_1), (\mu_2 - \beta) \) are of the same sign or zero, and \( \lambda - 1 \neq 0 \).

Firstly, we shall study the Sirakov’s open problem for the small \( \beta \) case. Precisely, we shall find \( \bar{\beta}_1(\lambda) \) by removing the assumption \( (1 - \lambda)(\mu_2 - \mu_1) \leq 0 \). Our result will cover the case \( \min\{\mu_1, \mu_2\} \leq \min\{\bar{\beta}_1(\lambda), \bar{\beta}_2(\lambda)\} < \sqrt{\mu_1\mu_2}. \)
Thus, in Appendix, if defined by \( (1.2) \) and \( \beta \) is open for \( N \), then \( \bar{\beta}_1, \lambda \) defined by \( (1.20) \) equals \( \min \{ \beta_1(\lambda), \beta_2(\lambda) \} \). In other words, \( \lambda = \min \{ \beta_1(\lambda), \beta_2(\lambda) \} \). Furthermore, let \( (u_{\lambda, \beta}, v_{\lambda, \beta}) \) be the solution obtained in Theorem 1.5 to system \((1.2)\), then we have the following asymptotic behavior:

1. If \( \beta_2(\lambda) < \beta_1(\lambda) \), then \( (u_{\lambda, \beta}, v_{\lambda, \beta}) \to (0, U_{1, \mu_2}) \) as \( \beta \to \beta_2(\lambda) \);
2. If \( \beta_1(\lambda) < \beta_2(\lambda) \), then \( (u_{\lambda, \beta}, v_{\lambda, \beta}) \to (U_{\lambda, \mu_2}, 0) \) as \( \beta \to \beta_1(\lambda) \).

By [5, Corollary 4.5], there exists a unique \((\lambda^*, \beta^*)\) such that

\[
\beta_1(\lambda^*) = \beta_2(\lambda^*) =: \beta^* \tag{1.24}
\]

Then we shall also study \( \bar{\beta}_2(\lambda) \) in case \( \beta > \beta^* \) in present paper, which seems to be new in this direction. Here comes our second main result.

**Theorem 1.6.** Let \( \lambda > 0 \) and \( \bar{\beta}_2(\lambda) \) be defined by \( (1.21) \), then we have

\[
\bar{\beta}_2(\lambda) = \begin{cases} 
\min \{ \beta_1(\lambda), \beta_2(\lambda) \}, & \text{if } \lambda \in [\min\{\lambda^*, (\frac{\mu_1}{\mu_2})^{\frac{2}{\lambda}}\}, \max\{\lambda^*, (\frac{\mu_1}{\mu_2})^{\frac{2}{\lambda}}\}]
\cup (0, \min\{\lambda^*, (\frac{\mu_1}{\mu_2})^{\frac{2}{\lambda}}\}) \cup (\max\{\lambda^*, (\frac{\mu_1}{\mu_2})^{\frac{2}{\lambda}}\}, \infty)
\end{cases}
\]

or we can write it as

\[
\bar{\beta}_2(\lambda) = \begin{cases} 
\beta_1(\lambda), & \text{if } 0 < \lambda < (\frac{\mu_1}{\mu_2})^{\frac{2}{\lambda}}
\min\{\beta_1(\lambda), \beta_2(\lambda)\}, & \text{if } \lambda = (\frac{\mu_1}{\mu_2})^{\frac{2}{\lambda}}
\beta_2(\lambda), & \text{if } \lambda > (\frac{\mu_1}{\mu_2})^{\frac{2}{\lambda}}
\end{cases}
\]

In other words, the least energy is attained only by semi-trivial solution. Furthermore, let \( \beta \to \bar{\beta}_2(\lambda) \) and \( (u_\lambda, v_\lambda) \) be a positive radial ground state solution to system \((1.2)\) with \( \beta = \bar{\beta}_2(\lambda) \), then we have that, up to a subsequence,

\[
(u_\lambda, v_\lambda, \beta_\lambda) \to \begin{cases} 
(U_{\lambda, \mu_1}, 0, \beta_2(\lambda)) & \text{if } \lambda < (\frac{\mu_1}{\mu_2})^{\frac{2}{\lambda}} \text{ or } \lambda = (\frac{\mu_1}{\mu_2})^{\frac{2}{\lambda}} > \lambda^*,
(0, U_{1, \mu_2}, \beta_2(\lambda)) & \text{if } \lambda > (\frac{\mu_1}{\mu_2})^{\frac{2}{\lambda}} \text{ or } \lambda = (\frac{\mu_1}{\mu_2})^{\frac{2}{\lambda}} < \lambda^*.
\end{cases}
\tag{1.25}
\]

**Remark 1.7.** In Appendix, if \( N = 1 \), we shall compute that

\[
\beta_1(\lambda) = \frac{\mu_1}{2} \left( \frac{1}{\lambda} + \sqrt{\frac{1}{\lambda}} \right), \quad \beta_2(\lambda) = \frac{\mu_2}{2} (\lambda + \sqrt{\lambda}).
\]

Thus,

\[
\lambda^* = \left( \frac{\mu_1}{\mu_2} \right)^{\frac{2}{\lambda}}
\]

and \( \bar{\beta}_2(\lambda) \) is continuous. However, for the technical reason, the precise value of \( \lambda^* \) is open for \( N = 2, 3 \). By Theorem 1.6, we see that \( \bar{\beta}_2(\lambda) \) is not continuous at \( \lambda = (\frac{\mu_1}{\mu_2})^{\frac{2}{\lambda}} \) if \( \lambda^* \neq (\frac{\mu_1}{\mu_2})^{\frac{2}{\lambda}} \), \( N = 2, 3 \). Basing on Theorem 1.5 and Theorem 1.6, if \( \lambda^* \neq (\frac{\mu_1}{\mu_2})^{\frac{2}{\lambda}} \), we can obtain the following multiplicity result as a product.
**Theorem 1.8.** Assume that \( \lambda^* \neq (\frac{\mu_1}{\mu_2})^{\frac{2}{4-N}} \), \( N = 2 \) or \( N = 3 \), then for any \((\lambda, \beta) \in D_\lambda \times D_\beta\), where

\[
D_\lambda := \left[ \min\left\{ \left(\frac{\mu_1}{\mu_2}\right)^{\frac{2}{4-N}} \mu, \lambda^* \right\}, \max\left\{ \left(\frac{\mu_1}{\mu_2}\right)^{\frac{2}{4-N}} \mu, \lambda^* \right\} \right] \setminus \{\lambda^*\}
\]

and

\[
D_\beta := \left( \min\left\{ \beta_1(\lambda), \beta_2(\lambda) \right\}, \max\left\{ \beta_1(\lambda), \beta_2(\lambda) \right\} \right),
\]

system (1.2) possesses at least two different positive solutions.

The paper is organized as follows. In the next section we shall devote to study the Sirakov’s open problem and prove Theorems 1.5 and 1.6. Based on these results, we prove Theorem 1.8 in Section 3 and prove Theorem 1.4 in Section 4. A uniqueness result would be of course very interesting in itself but also very difficult. We shall discuss some uniqueness results in Section 5 and study the continuity of \( A_\beta^\lambda \) in Section 6. We compute the precise values of \( \beta_i(\lambda) \), \( i = 1, 2 \) for \( N = 1 \) in Section 7.

### 2 A progress on the Sirakov’s open problem

#### 2.1 Proof of Theorem 1.5: the case of \( \bar{\beta}_1(\lambda) \)

In this subsection, we shall study the Sirakov’s open problem for the small \( \beta \) case. More precisely, we shall find \( \bar{\beta}_1(\lambda) \). Our result will cover the case \( \min\{\mu_1, \mu_2\} \leq \min\{\beta_1(\lambda), \beta_2(\lambda)\} < \sqrt{\mu_1 \mu_2} \). Define

\[
D_\beta(u, v) := \mu_1 |u|^4 \cdot \mu_2 |v|^4 - (\beta |uv|^2)^2, \quad (u, v) \in \mathcal{H}.
\]

and

\[
\begin{align*}
\mathcal{M}_{\lambda}^{+} &:= \{(u, v) \in \mathcal{M}_{\lambda}^{\beta}, D_\beta(u, v) > 0\}, \\
\mathcal{M}_{\lambda}^{0} &:= \{(u, v) \in \mathcal{M}_{\lambda}^{\beta}, D_\beta(u, v) = 0\}, \\
\mathcal{M}_{\lambda}^{-} &:= \{(u, v) \in \mathcal{M}_{\lambda}^{\beta}, D_\beta(u, v) < 0\},
\end{align*}
\]

**Lemma 2.1.** For any \((u, v) \in \mathcal{M}_{\lambda}^{\beta} \), we have

\[
\Phi_\lambda^\beta(u, v) = \max_{(s, t) \in [0, \infty)^2} \Phi_\lambda^\beta(\sqrt{su}, \sqrt{tv}),
\]

and

\[
\Phi_\lambda^\beta(\sqrt{su}, \sqrt{tv}) < \Phi_\lambda^\beta(u, v) \text{ for all } (s, t) \neq (1, 1).
\]

**Proof.** If \(|uv|_2 = 0\), the result is trivial. Next, we assume that \(|uv|_2 > 0\). Let

\[
f(s, t) := \Phi_\lambda^\beta(\sqrt{su}, \sqrt{tv}) = \frac{1}{2} \left( |\nabla u|^2 + \lambda |u|^2 \right) s + \frac{1}{2} \left( |\nabla v|^2 + |v|^2 \right) t
\]

\[
- \frac{1}{4} \left( \mu_1 |u|^4 s^2 + \mu_2 |v|^4 t^2 + 2\beta |uv|^2 st \right).
\]
By $\mathcal{D}_{u,v} \neq 0$, we see that \[ \begin{cases} \frac{\partial}{\partial s} f(s,t) = 0 \\ \frac{\partial}{\partial t} f(s,t) = 0 \end{cases} \] has a unique solution $(s,t) = (1,1)$. We can view $f(s,t)$ as a function defined in $\mathbb{R}^2$. Then we see that $(1,1)$ is the unique critical point of $f(s,t)$ in $\mathbb{R}^2$. Recall the Hessian matrix

\[ H := \begin{pmatrix} \frac{\partial^2 f}{\partial s^2}(1,1) & \frac{\partial^2 f}{\partial s \partial t}(1,1) \\ \frac{\partial^2 f}{\partial s \partial t}(1,1) & \frac{\partial^2 f}{\partial t^2}(1,1) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \mu_1 |u|^4 & -\frac{1}{2} \beta |uv|^2 \\ -\frac{1}{2} \beta |uv|^2 & -\frac{1}{2} \mu_2 |v|^4 \end{pmatrix}, \]

we have that

\[ \text{det}(H) = \frac{1}{4} \mathcal{D}_\beta(u,v) > 0. \]

Hence, $(1,1)$ is a local maxima of $f(s,t)$. We extend $f(s,t)$ to be defined in $\mathbb{R}^2$, then by $\lim_{s^2 + t^2 \to \infty} f(s,t) = -\infty$ due to $\mathcal{D}_\beta(u,v) > 0$, we see that

\[ \max_{(s,t) \in \mathbb{R}^2} f(s,t) \text{ is attained by some inner point } (s^*, t^*) \in \mathbb{R}^2. \]

So $\frac{\partial f}{\partial s}(s^*, t^*) = 0$, $\frac{\partial f}{\partial t}(s^*, t^*) = 0$. By $(u,v) \in \mathcal{M}_{\lambda}^2$ again, we see that $(s^*, t^*) = (1,1)$. Hence,

\[ f(1,1) = \max_{(s,t) \in \mathbb{R}^2} f(s,t) \]

and thus

\[ \Phi_\lambda^\beta(u,v) = \max_{(s,t) \in [0,\infty)^2} \Phi_\lambda^\beta(\sqrt{s}u, \sqrt{t}v). \]

\[ \square \]

**Lemma 2.2.** Assume $u \neq 0, v \neq 0$. If $\mathcal{D}_\beta(u,v) = 0$, then

\[ \sup_{(s,t) \in [0,\infty)^2} \Phi_\lambda^\beta(\sqrt{s}u, \sqrt{t}v) = \max \left\{ \sup_{s>0} \Phi_\lambda^\beta(\sqrt{s}u, 0), \sup_{t>0} \Phi_\lambda^\beta(0, \sqrt{t}v) \right\} \quad (2.5) \]

and there exist some $s > 0, t > 0$ such that $(\sqrt{s}u, \sqrt{t}v) \in \mathcal{M}_{\lambda}^{0,0}$ if and only if

\[ \sup_{s>0} \Phi_\lambda^\beta(\sqrt{s}u, 0) = \sup_{t>0} \Phi_\lambda^\beta(0, \sqrt{t}v). \]

**Proof.** For $u \neq 0, v \neq 0$ with $\mathcal{D}_\beta(u,v) = 0$, we have that $\beta |uv|^2 \neq 0$ and exists some $k > 0$ such that

\[ \frac{\mu_1 |u|^4}{\beta |uv|^2} = \frac{\beta |uv|^2}{\mu_2 |v|^4} = k. \]

Let $(s^*, t^*) \in [0,\infty)^2$ be such that

\[ \Phi_\lambda^\beta(\sqrt{s^*}u, \sqrt{t^*}v) = \max_{(s,t) \in [0,\infty)^2} \Phi_\lambda^\beta(\sqrt{s}u, \sqrt{t}v), \]

then at least one of the following holds:

(i) $s^* > 0, t^* > 0$;
\( s^* = 0, t^* = \frac{|\nabla u|_2^2 + |u|_2^2}{\mu_2|v|_2^2}; \)

\( s^* = \frac{|\nabla u|_2^2 + \lambda|u|_2^2}{\mu_1|u|_2^2}, t^* = 0. \)

In fact, if (i) fails, then it is easy to see that (ii) or (iii) holds. Next, we shall focus on the case (i). If (i) holds, we have that \( (\sqrt{s^2}u, \sqrt{t^2}v) \in \mathcal{M}_{\lambda}^{\beta,0} \) and the following equation satisfied:

\[
\begin{aligned}
\mu_1|u|_4^4 s^* + 1/2|uv|^2 t^* &= |\nabla u|_2^2 + \lambda|u|_2^2, \\
\beta|uv|^2 s^* + |v|^2 t^* &= |\nabla v|_2^2 + |v|_2^2.
\end{aligned}
\]  

(2.6)

Then we also have that

\[
\frac{|\nabla u|_2^2 + \lambda|u|_2^2}{|\nabla u|_2^2 + |v|_2^2} = k.
\]

Therefore, \( (\sqrt{s^2}u, \sqrt{t^2}v) \in \mathcal{M}_{\lambda}^{\beta,0} \) if and only if

\[
\mu_1|u|_4^4 s^2 + 1/2|uv|^2 st = (|\nabla u|_2^2 + \lambda|u|_2^2)s, \quad (s, t) \in (0, \infty)^2.
\]

That is,

\[
\mu_1|u|_4^2 s + \beta|uv|^2 t = |\nabla u|_2^2 + \lambda|u|_2^2, \quad s > 0, t > 0,
\]

which equivalents to that

\[
0 < t < \frac{1}{\beta|uv|^2} \frac{|\nabla u|_2^2 + \lambda|u|_2^2}{\mu_1|u|_4^2} \text{ and } s = \frac{|\nabla u|_2^2 + \lambda|u|_2^2}{\mu_1|u|_4^2} - \frac{\beta|uv|^2}{\mu_1|u|_4^2}t.
\]  

(7.7)

Conversely, it is easy to see that for any \( (s', t') \) satisfying (7.7), then

\[
\Phi^\beta_{\lambda}(\sqrt{s}u, \sqrt{t}v) = \max_{(s, t) \in (0, \infty)^2} \Phi^\beta_{\lambda}(\sqrt{s}u, \sqrt{t}v).
\]

So by taking a limit, we have that

\[
\sup_{(s, t) \in [0, \infty)^2} \Phi^\beta_{\lambda}(\sqrt{s}u, \sqrt{t}v) = \Phi^\beta_{\lambda}\left(\sqrt{\frac{|\nabla u|_2^2 + \lambda|u|_2^2}{\mu_1|u|_4^2}}, 0\right)
\]

\[
= \sup_{s > 0} \Phi^\beta_{\lambda}(\sqrt{s}u, 0) = \Phi^\beta_{\lambda}\left(0, \sqrt{\frac{|\nabla u|_2^2 + \lambda|u|_2^2}{\beta|uv|^2}} v\right).
\]  

(2.8)

Noting that

\[
\frac{|\nabla u|_2^2 + \lambda|u|_2^2}{\beta|uv|^2} = 1 + k = \frac{|\nabla v|_2^2 + |v|^2}{\mu_2|v|_4^2},
\]

we also have that

\[
\Phi^\beta_{\lambda}\left(0, \sqrt{\frac{|\nabla u|_2^2 + \lambda|u|_2^2}{\beta|uv|^2}} v\right) = \sup_{t > 0} \Phi_{\lambda}(0, \sqrt{t}v).
\]
Hence, if (i) holds, then
\[
\sup_{(s,t) \in [0,\infty)^2} \Phi^\beta_{\lambda}(\sqrt{su}, \sqrt{tv}) = \sup_{s > 0} \Phi^\beta_{\lambda}(\sqrt{su}, 0) = \sup_{t > 0} \Phi^\beta_{\lambda}(0, \sqrt{tv}). \tag{2.9}
\]
Combing (i)-(iii) together, we prove this Lemma.

**Remark 2.3.** If \( \beta \leq \sqrt{\mu_1 \mu_2} \), we see that \( M^\beta_{\lambda} = \emptyset \) and \( M^\beta_{\lambda} = M^\beta_{\lambda,+} \cup M^\beta_{\lambda,0} \). In particular, if \( \beta < \sqrt{\mu_1 \mu_2} \), \( M^\beta_{\lambda} = M^\beta_{\lambda,+} \).

**Corollary 2.4.** For any \((u,v) \in M^\beta_{\lambda,+} \cup M^\beta_{\lambda,0}\), we have that
\[
\Phi^\beta_{\lambda}(u,v) = \max_{(s,t) \in [0,\infty)^2} \Phi^\beta_{\lambda}(\sqrt{su}, \sqrt{tv}).
\]

**Proof.** For \((u,v) \in M^\beta_{\lambda}\), we have \( u \neq 0, v \neq 0 \). If \( D^\beta_{\lambda}(u,v) > 0 \), the result follows by Lemma 2.1. And if \( D^\beta_{\lambda}(u,v) = 0 \), the conclusion is also clear by the proof of Lemma 2.2.

**Lemma 2.5.** Assume that \( \sqrt{\mu_1 \mu_2} > \beta_i(\lambda) \) for some \( i \in \{1, 2\} \), then
\[
A^\beta_{\lambda} = \begin{cases} m_{\lambda, \mu_1} & \text{if } i = 1, \\ m_{1, \mu_2} & \text{if } i = 2 \end{cases}
\]
for any \( \beta \in [\beta_i(\lambda), \sqrt{\mu_1 \mu_2}] \)

and
\[
\begin{align*}
A^\beta_{\lambda} &= \begin{cases} m_{\lambda, \mu_1} & \text{if } i = 1, \\ m_{1, \mu_2} & \text{if } i = 2 \end{cases}
\end{align*}
\]
for \( \beta = \beta_i(\lambda) = \min\{\beta_1(\lambda), \beta_2(\lambda)\} \).

**Proof.** We refer to [9, Lemma 2.4]. We note that under the assumptions of [9], \( \sqrt{\mu_1 \mu_2} \in (\beta_2(\lambda), \beta_1(\lambda)) \) and the authors proved the result above for \( i = 2 \). Indeed, their arguments are still valid here.

**Lemma 2.6.** We have the following relations:
\[
\beta_1(\lambda) \leq \mu_1 \left( \frac{N}{4} + \frac{4 - N}{4\lambda} \right), \tag{2.10}
\]
\[
\beta_2(\lambda) \leq \mu_2 \left( \frac{N}{4} + \frac{\lambda(4-N)}{4} \right) \tag{2.11}
\]
and
\[
\begin{align*}
\mu_1 \lambda^{\frac{4-N}{4}} < \beta_1(\lambda) < \mu_1 & \text{ for } \lambda > 1, \\
\mu_1 \lambda^{\frac{N-4}{4}} < \beta_1(\lambda) < \mu_1 \lambda^{-1} & \text{ for } \lambda < 1,
\end{align*}
\]
and
\[
\begin{align*}
\mu_2 \lambda^{\frac{4-N}{4}} < \beta_2(\lambda) < \mu_2 & \text{ for } \lambda > 1, \\
\mu_2 \lambda^{\frac{N-4}{4}} < \beta_2(\lambda) < \mu_2 & \text{ for } \lambda < 1,
\end{align*}
\]
Proof. Recalling that
\[ |\nabla U|_2^2 = \frac{N}{4} S^2, \quad |U|_2^2 = \frac{4-N}{4} S^2, \quad |U|_4^4 = S^2, \]
we have that
\[ \tau(s) \leq \frac{\int_{\mathbb{R}^N} (|\nabla U|^2 + sU^2)}{\int_{\mathbb{R}^N} U^4} = \frac{N}{4} + \frac{4-N}{4} s. \]
Hence, we have
\[ \beta_1(\lambda) = \mu_1 \tau\left(\frac{1}{\lambda}\right) \leq \mu_1 \left(\frac{N}{4} + \frac{4-N}{4\lambda}\right) \]
and
\[ \beta_2(\lambda) = \mu_2 \tau(\lambda) \leq \mu_2 \left(\frac{N}{4} + \frac{\lambda(4-N)}{4}\right). \]
By the Hölder inequality,
\[ |U\varphi|_2^2 \leq |U|_2^2 |\varphi|_4^2 = S|\varphi|_4^2, \]
we have
\[ \tau(\lambda) \geq \inf_{\varphi \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{|\nabla \varphi|_2^2 + \lambda |\varphi|_4^2}{S|\varphi|_4^2} = \frac{1}{\lambda} S\lambda = \lambda^{\frac{4-N}{N}}. \]
So for \( \lambda > 1 \), we have that
\[ \lambda^{\frac{4-N}{N}} < \tau(\lambda) < \lambda \tau(1) = \lambda, \]
and for \( \lambda < 1 \), we have
\[ \lambda^{\frac{4-N}{N}} < \tau(\lambda) < \tau(1) = 1. \]
Then by the definition of \( \beta_i(\lambda), i = 1, 2 \), see (1.16), we obtain (2.12) and (2.13).

Proof of Theorem 1.3: By Lemma 2.6 above, we have that
\[ \min\{\mu_1 \lambda^{\frac{N-4}{4}}, \mu_2 \lambda^{\frac{1-N}{4}}\} < \min\{\beta_1(\lambda), \beta_2(\lambda)\}, \lambda < 1. \]
On the other hand, we have
\[ \min\{\mu_1 \lambda^{\frac{N-4}{4}}, \mu_2 \lambda^{\frac{4-N}{4}}\} \leq \sqrt{\mu_1 \lambda^{\frac{N-4}{4}}} \cdot \mu_2 \lambda^{\frac{4-N}{4}} = \sqrt{\mu_1 \mu_2}. \]
Hence, by Theorem C we obtain the final result.

Remark 2.7. Lemma 2.6 above is an improvement of [1, formula (3)]. The ranges in Theorem B are not the best, see [14, Remark 4]. Theorem 1.3 is also an improvement of Theorem B. Indeed, in Theorem B,
\[ \nu_0 = \frac{\mu_1}{2} \lambda^{-1} + \frac{\mu_2}{2} \lambda^{1-\frac{N}{2}} - \sqrt{\left(\frac{\mu_1}{2} \lambda^{-1} + \frac{\mu_2}{2} \lambda^{1-\frac{N}{2}}\right)^2 - \mu_1 \mu_2 \lambda^{-\frac{N}{2}}} \]
If \( \lambda < 1 \), by a direct computation, one can check that both \( \nu_0 < \mu_1 \lambda^{\frac{k}{k-1}} \) and \( \nu_0 < \mu_2 \lambda^{1-\frac{N}{2}} \).
Corollary 2.8. We have
\[ \beta^* \geq \sqrt{\mu_1 \mu_2}. \]

Proof. If \( \beta^* < \sqrt{\mu_1 \mu_2} \), by Lemma 2.5, we have that for \( \lambda = \lambda^* \), \( \beta = \beta^* \), there holds \( \mathcal{A}^\beta = \mathcal{A}^{\beta, \text{rad}} = m_{\lambda^* \mu_1} = m_{1, \mu_2} \). Formula (1.13) implies that \( \lambda^* = \left( \frac{\mu_1}{\mu_2} \right)^{-\frac{N}{4}} \). If \( \mu_1 = \mu_2 = \mu \), we have known that \( \lambda^* = 1 \) and \( \beta^* = \mu = \sqrt{\mu_1 \mu_2} \). If \( \mu_1 \neq \mu_2 \), we may assume that \( \mu_1 > \mu_2 \) without loss of generality. Then we have that \( \lambda^* > 1 \). So by Lemma 2.6, we have that
\[ \beta^* = \beta_1(\lambda^*) > \mu_1(\lambda^*)^{\frac{N-4}{4}} = \mu_1 \left( \frac{\mu_1}{\mu_2} \right)^{-\frac{1}{2}} = \sqrt{\mu_1 \mu_2}, \]
a contradiction. \(\square\)

Remark 2.9. Assume that \( \mu_2 < \mu_1 \), then \( \lambda^* > 1 \). By Lemma 2.6, we have \( \beta^* > \mu_1 \lambda^* \frac{N-4}{4} \) and \( \beta^* > \mu_2 \lambda^* \frac{4-N}{4} \). Hence, one can also prove
\[ \beta^* > \sqrt{\mu_1 \lambda^* \frac{N-4}{4} \cdot \mu_2 \lambda^* \frac{4-N}{4}} = \sqrt{\mu_1 \mu_2} \]
without Lemma 2.5. However, Lemma 2.5 will be used in our following study.

Corollary 2.10. Assume that \( \mu_1 \geq \mu_2 \), let \( \lambda^* \) be defined by (1.24), then we have
\[ \frac{1}{4 - N} \left( 4 \sqrt{\frac{\mu_1}{\mu_2} - N} \right) \leq \lambda^* \begin{cases} \leq (4 - N) \left( 4 \sqrt{\frac{\mu_1}{\mu_2} - N} \right)^{-1} & \text{if } \frac{\mu_2}{\mu_1} > \frac{N^2}{16}, \\ < +\infty & \text{if } \frac{\mu_2}{\mu_1} \leq \frac{N^2}{16}. \end{cases} \]

Proof. Recalling the definition, we have \( \beta_1(\lambda^*) = \beta_2(\lambda^*) = \beta^* \geq \sqrt{\mu_1 \mu_2} \). Then by the formulas (2.10) and (2.11), we obtain the inequality above. \(\square\)

Proof of Theorem 1.5: Without loss of generality, we may assume that \( \beta_2(\lambda) < \beta_1(\lambda) \). Then by Lemma 2.5, we have that \( \mathcal{A}^\beta = m_{1, \mu_2} \) provided \( \beta = \beta_2(\lambda) \). We only need to prove that \( \mathcal{A}^\beta \) is not attained provided \( \beta = \beta_2(\lambda) \).

Indeed, if \( \mathcal{A}^\beta \) is attained by some \( (u,v) \in \mathcal{M}^\beta_\lambda \), then \( u \neq 0, v \neq 0 \). By Corollary 2.4, we have
\[ m_{1, \mu_2} = \Phi^\beta_\lambda(u,v) = \max_{(s,t) \in [0,\infty)^2} \Phi^\beta_\lambda(\sqrt{su}, \sqrt{tv}) \]
\[ \geq \max_{t \geq 0} \Phi^\beta_\lambda(0, \sqrt{tv}) \]
\[ \geq \min_{\varphi \in H^1(\mathbb{R}^N \setminus \{0\})} \max_{t \geq 0} \Phi^\beta_\lambda(0, \sqrt{tv}) \]
\[ = \Phi^\beta_\lambda(0, U_{1, \mu_2}) = m_{1, \mu_2}. \]

Hence, we obtain \( v = C_1 U_{1, \mu_2} \) for some constant \( C_1 > 0 \). Then by
\[ -\Delta U_{1, \mu_2} + U_{1, \mu_2} = \mu_2 C_1^2 U_{1, \mu_2}^3 + \beta_2(\lambda) u^2 U_{1, \mu_2}, \]
we have \( u = C_2 U_{1,\mu_2} \) with \( \mu_2 C_2^2 + \beta_2(\lambda) C_2^2 = \mu_2 \). Since \((u, v) \in M_\lambda^\beta\), we have that \( C_2 \neq 0 \), and then by \( -\Delta u + \mu_1 u^3 + \beta_2(\lambda) u v^2 \), we obtain that

\[
-\Delta U_{1,\mu_2} + \lambda U_{1,\mu_2} = \left( \mu_1 C_2^2 + \beta_2(\lambda) C_2^2 \right) U_{1,\mu_2}^3.
\]

Hence, the uniqueness result of Kwong[12] implies that \( \lambda = 1 \) and \( \mu_1 C_2^2 + \beta_2(\lambda) C_2^2 = \mu_2 \), a contradiction.

For the asymptotic behavior of the least energy solutions as \( \beta \uparrow \min\{\beta_1(\lambda), \beta_2(\lambda)\} \), we also only prove the case of \( \beta_2(\lambda) < \beta_1(\lambda) \), i.e., \( \lambda < \lambda^* \). By Lemma 2.5, \( A_\lambda^\beta = m_{1,\mu_2} \) provided \( \beta = \beta_2(\lambda) \). Let \( \beta_n \uparrow \beta_2(\lambda) \) as \( n \to \infty \). It is standard to see that \( \{(u_{\lambda_n,\beta_n}, v_{\lambda_n,\beta_n})\} \) is a bounded sequence in \( H_{rad}(\mathbb{R}^N) \). Hence, by a standard argument, up to a subsequence, \((u_{\lambda_n,\beta_n}, v_{\lambda_n,\beta_n}) \to (u, v) \in H_{rad}(\mathbb{R}^N) \) strongly. Since we have proved that \( A_\lambda^\beta \) is not obtained. Hence, \( u = 0 \) or \( v = 0 \). That is, \((u, v) = (0, U_{1,\mu_2})\) or \((u, v) = (U_{\lambda_1,\mu_1}, 0)\). If \( m_{\lambda_1,\mu_1} \neq m_{1,\mu_2} \), then it is easy to see that \((u, v) = (0, U_{1,\mu_2})\). If \( m_{\lambda_1,\mu_1} = m_{1,\mu_2} \), then inequality (1.14) states out that

\[
2\sqrt{m_{1,\mu_2}} \left( \int_{\mathbb{R}^N} \mu_2 v_{\lambda_n,\beta_n}^4 \right)^{\frac{1}{2}} \leq \int_{\mathbb{R}^N} \left( |\nabla v_{\lambda_n,\beta_n}|^2 + v_{\lambda_n,\beta_n}^2 \right)
\]

\[
= \int_{\mathbb{R}^N} \mu_2 v_{\lambda_n,\beta_n}^4 + \int_{\mathbb{R}^N} \beta_n u_{\lambda_n,\beta_n}^2 v_{\lambda_n,\beta_n}^2
\]

\[
\leq \int_{\mathbb{R}^N} \mu_2 v_{\lambda_n,\beta_n}^4 + \frac{\beta_n}{\sqrt{\mu_1 \mu_2}} \left( \int_{\mathbb{R}^N} \mu_1 u_{\lambda_n,\beta_n}^4 \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \mu_2 v_{\lambda_n,\beta_n}^4 \right)^{\frac{1}{2}}.
\]

which implies

\[
2\sqrt{m_{1,\mu_2}} \leq \left( \int_{\mathbb{R}^N} \mu_2 v_{\lambda_n,\beta_n}^4 \right)^{\frac{1}{2}} + \frac{\beta_n}{\sqrt{\mu_1 \mu_2}} \left( \int_{\mathbb{R}^N} \mu_1 u_{\lambda_n,\beta_n}^4 \right)^{\frac{1}{2}}.
\]

Let \( n \to \infty \) and we have

\[
2\sqrt{m_{1,\mu_2}} \leq \left( \int_{\mathbb{R}^N} \mu_2 v^4 \right)^{\frac{1}{2}} + \frac{\beta_2(\lambda)}{\sqrt{\mu_1 \mu_2}} \left( \int_{\mathbb{R}^N} \mu_1 u^4 \right)^{\frac{1}{2}}.
\]

If \((u, v) = (U_{\lambda_1,\mu_1}, 0)\), then we have

\[
2\sqrt{m_{1,\mu_2}} \leq \frac{\beta_2(\lambda)}{\sqrt{\mu_1 \mu_2}} \sqrt{m_{\lambda_1,\mu_1}} = \frac{\beta_2(\lambda)}{\sqrt{\mu_1 \mu_2}} \sqrt{m_{1,\mu_2}}.
\]

Hence, \( \beta_2(\lambda) \geq \sqrt{\mu_1 \mu_2} \), a contradiction to the assumption that

\[
\beta_2(\lambda) = \min\{\beta_1(\lambda), \beta_2(\lambda)\} < \sqrt{\mu_1 \mu_2}.
\]
2.2 Proof of Theorem 1.6: the case of $\hat{\beta}_2(\lambda)$

In this subsection, we consider the following Nehari manifold $\tilde{M}_\lambda^\beta$ defined by (1.17) and define

$$c_\lambda^\beta := \inf_{(u,v) \in \tilde{M}_\lambda^\beta} \Phi^\beta(u,v).$$  \hfill (2.14)

We note that $c_\lambda^\beta = A_\lambda^\beta$ provided $\beta > \max\{\beta_1(\lambda), \beta_2(\lambda)\}$ due to Theorem A-(i).

We also define $\tilde{M}_{\lambda,rad}^\beta := \tilde{M}_\lambda^\beta \cap H_{rad}$ and

$$c_{\lambda,rad}^\beta := \inf_{(u,v) \in \tilde{M}_{\lambda,rad}^\beta} \Phi^\beta(u,v).$$

By the Schwarz symmetrization, it is easy to prove that $c_\lambda^\beta \geq c_{\lambda,rad}^\beta$. Noting that both $(U_{\lambda,\mu_1}, 0)$ and $(0, U_{1,\mu_2})$ belong to $\tilde{M}_\lambda^\beta$, we have

$$c_\lambda^\beta \leq \min\{\Phi_\lambda^\beta(U_{\lambda,\mu_1}, 0), \Phi_\lambda^\beta(0, U_{1,\mu_2})\} = \min\{m_{\lambda,\mu_1}, m_{1,\mu_2}\}. \hfill (2.15)$$

**Lemma 2.11.** For any given $(\lambda, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+$, $c_\lambda^\beta$ is always achieved by some $(u, v) \neq (0, 0), u \geq 0, v \geq 0$.

**Proof.** Recalling that $c_\lambda^\beta = c_{\lambda,rad}^\beta$, let $\{(u_n, v_n)\} \subset \tilde{M}_{\lambda,rad}^\beta$ be a minimizing sequence, then it is standard to prove that $(u_n, v_n) \rightarrow (u, v) \in H_{rad}$ up to a subsequence. It is not difficult to prove that $\tilde{M}_\lambda^\beta$ is closed and bounded away from $(0, 0)$. Hence, $(u, v) \neq (0, 0)$ and $(u, v) \in \tilde{M}_\lambda^\beta$ satisfying

$$\Phi_\lambda^\beta(u, v) = \lim_{n \rightarrow \infty} \Phi_\lambda^\beta(u_n, v_n) = c_\lambda^\beta.$$

Since the functional $\Phi_\lambda^\beta(u, v) = \Phi_\lambda^\beta(\langle u, v \rangle)$ and $(\langle u, v \rangle) \in \tilde{M}_\lambda^\beta$, we may assume that $u \geq 0, v \geq 0$. \hfill \Box

We note that the minimizer $(u, v)$ solves the equation

$$\begin{cases}
-\Delta u + \lambda u = \mu_1 u^3 + \beta uv^2, & \text{in } \mathbb{R}^N, \\
-\Delta v + v = \mu_2 v^3 + \beta u^2 v, & \text{in } \mathbb{R}^N,
\end{cases} \hfill (2.16)$$

since the manifold $\tilde{M}_\lambda$ is of co-dimension 1. However, it may be $u = 0$ or $v = 0$. In such case, we call $(u, 0)$ and $(0, v)$ are semi-trivial solutions. While $u \neq 0, v \neq 0$, we call $(u, v)$ are nontrivial solutions.

**Proposition 2.12.** For any $(\lambda, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+$,

$$c_\lambda^\beta = \inf_{(u,v) \in H \setminus (0,0)} \sup_{t > 0} \Phi^\beta_\lambda(t u, tv)$$

and we also have

$$c_\lambda^\beta = \inf_{\gamma \in \Gamma_\lambda^\beta} \sup_{t \in [0, 1]} \Phi^\beta_\lambda(\gamma(t)),$$

where $\Gamma_\lambda^\beta$ is the mountain pass set given by

$$\Gamma_\lambda^\beta := \left\{ \gamma \in C([0, 1], H_{rad}) : \gamma(0) = (0, 0), \Phi^\beta_\lambda(\gamma(1)) < 0 \right\}.$$
Proof. (1) For any \((u, v) \neq (0, 0)\), by
\[
\Phi_\lambda^\beta(tu, tv) = \frac{1}{2} \left( |\nabla u|^2 + \lambda |u|^2 + |\nabla v|^2 + |v|^2 \right) t^2 - \frac{1}{4} \left( \mu_1 |u|^4 + \mu_2 |v|^4 + 2 \beta \|uv\|^2 \right) t^4,
\]
we see that
\[
t_{(u, v)} := \sqrt{\frac{\mu_1 |u|^4 + \mu_2 |v|^4 + 2 \beta \|uv\|^2}{|\nabla u|^2 + \lambda |u|^2 + |\nabla v|^2 + |v|^2}} > 0
\]
satisfying
\[
(t_{(u, v)} u, t_{(u, v)} v) \in \tilde{M}_\lambda^\beta
\]
and thus
\[
\max_{t > 0} \Phi_\lambda^\beta(tu, tv) = \Phi_\lambda^\beta(t_{(u, v)} u, t_{(u, v)} v) \geq c_\lambda^\beta.
\]
Then, by the arbitrariness of \((u, v)\), we see that
\[
\inf_{(u, v) \in \mathcal{H} \setminus (0, 0)} \sup_{t > 0} \Phi_\lambda^\beta(tu, tv) \geq c_\lambda^\beta.
\]
Conversely, for any \((u, v) \in \tilde{M}_\lambda^\beta\), we have that
\[
\Phi_\lambda^\beta(u, v) = \max_{t > 0} \Phi_\lambda^\beta(tu, tv) \geq \inf_{(u, v) \in \mathcal{H} \setminus (0, 0)} \sup_{t > 0} \Phi_\lambda^\beta(tu, tv).
\]
Also by the arbitrariness of \((u, v) \in \tilde{M}_\lambda^\beta\), we obtain that
\[
c_\lambda^\beta \geq \inf_{(u, v) \in \mathcal{H} \setminus (0, 0)} \sup_{t > 0} \Phi_\lambda^\beta(tu, tv).
\]
Hence,
\[
c_\lambda^\beta = \inf_{(u, v) \in \mathcal{H} \setminus (0, 0)} \sup_{t > 0} \Phi_\lambda^\beta(tu, tv).
\]

(2) For any \((u, v) \in \tilde{M}_\lambda^\beta, \text{rad}\), there exists some \(T > 0\) such that \(\Phi_\lambda^\beta(Tu, Tv) < 0\).
Define
\[
\gamma \in C([0, 1], \mathcal{H}_{\text{rad}}) \text{ such that } \gamma(t) = (tTu, tTv).
\]
Then we see that \(\gamma \in \Gamma_\lambda^\beta\) and thus
\[
\Phi_\lambda^\beta(u, v) = \max_{t \in [0, 1]} \Phi_\lambda^\beta(\gamma(t)) \geq \inf_{\gamma \in \Gamma_\lambda^\beta} \sup_{t \in [0, 1]} \Phi_\lambda^\beta(\gamma(t)).
\]
By the arbitrariness of \((u, v) \in \tilde{M}_\lambda^\beta, \text{rad}\), we have that
\[
c_\lambda^\beta = c_{\lambda, \text{rad}}^\beta \geq \inf_{\gamma \in \Gamma_\lambda^\beta} \sup_{t \in [0, 1]} \Phi_\lambda^\beta(\gamma(t)).
\]
Conversely, we define
\[
d_\lambda^\beta = \inf_{\gamma \in \Gamma_\lambda^\beta} \sup_{t \in [0, 1]} \Phi_\lambda^\beta(\gamma(t))
\]
Then one can see that $d_\lambda^\beta > 0$ is a critical value of $\Phi_\lambda^\beta$ and is obtained by some $(u, v) \in \mathcal{H}_{rad}, (u, v) \neq (0, 0)$. Then we see that $(u, v) \in \bar{\mathcal{M}}_{\lambda, rad}$ and thus

$$d_\lambda^\beta = \Phi_\lambda^\beta(u, v) \geq \inf_{(u,v) \in \bar{\mathcal{M}}_{\lambda, rad}} \Phi_\lambda^\beta(u, v) = c_{\lambda, rad} = c_\lambda^\beta.$$ 

Hence, we also have

$$c_\lambda^\beta = \inf_{\gamma \in T_\lambda^\beta} \sup_{t \in [0, 1]} \Phi_\lambda^\beta(\gamma(t)).$$

\[ \Box \]

**Corollary 2.13.** $c_\lambda^\beta$ is continuous and increases with respect to $\lambda \in \mathbb{R}^+$ and decreases with respect to $\beta \in \mathbb{R}^+$. \[ \Box \]

**Proof.** Following Proposition 2.12, it is easy to see. \[ \Box \]

**Lemma 2.14.** For any given $(\lambda, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+$, let $(u, v) \neq (0, 0)$ be a solution (may me simi-trivial solution) of the system (2.16) with energy $c = \Phi_\lambda^\beta(u, v)$. Then we have the following relations

\[
\left\{ \begin{array}{ll}
|\nabla u|^2 + |\nabla v|^2 = Nc, \\
\lambda|u|^2 + |v|^2 = (4 - N)c, \\
\mu_1|u|^4 + \mu_2|v|^4 + 2\beta|uv|^2 = 4c. 
\end{array} \right. \tag{2.17}
\]

**Proof.** Firstly we have that

$$c = \Phi_\lambda^\beta(u, v)$$

$$= -\frac{1}{2} \left( |\nabla u|^2 + \lambda|u|^2 + |\nabla v|^2 + |v|^2 \right) - \frac{1}{4} \left( \mu_1|u|^4 + \mu_2|v|^4 + 2\beta|uv|^2 \right). \tag{2.18}$$

Secondly, since $(u, v) \neq (0, 0)$ is a solution, we know that $(u, v) \in \bar{\mathcal{M}}_{\lambda}^\beta$, and thus

$$|\nabla u|^2 + \lambda|u|^2 + |\nabla v|^2 + |v|^2 = \mu_1|u|^4 + \mu_2|v|^4 + 2\beta|uv|^2. \tag{2.19}$$

And by the Pohozaev identity, we have

$$(N - 2) \left( |\nabla u|^2 + |\nabla v|^2 \right)$$

$$= -N \left( \lambda|u|^2 + |v|^2 \right) + \frac{N}{2} \left( \mu_1|u|^4 + \mu_2|v|^4 + 2\beta|uv|^2 \right). \tag{2.20}$$

Then by (2.18),(2.19) and (2.20), we obtain (2.17). \[ \Box \]

**Lemma 2.15.** Let $\lambda > 0$, when $\beta < \beta_2(\lambda), (u, v) = (0, U_{1, \mu_1})$ or $\beta < \beta_1(\lambda), (u, v) = (U_{\lambda, \mu_1}, 0)$, the following linearized problem

\[
\left\{ \begin{array}{ll}
\Delta \phi - \lambda \phi + 3\mu_1 u^2 \phi + \beta v^2 \phi + 2\beta uv \phi = 0, & x \in \mathbb{R}^N, \\
\Delta \phi - \phi + 3\mu_2 v^2 \phi + \beta u^2 \phi + 2\beta uv \phi = 0, & x \in \mathbb{R}^N, \\
\phi, \phi \in H^1_{rad}(\mathbb{R}^N) 
\end{array} \right. \tag{2.21}
\]

has only zero solution.
Proof. Let $\beta < \beta_2(\lambda)$, $(u, v) = (0, U_{1,\mu_2})$, then (2.21) becomes

\[
\begin{cases}
\Delta \phi - \lambda \phi + \beta U_{1,\mu_2}^2 \phi = 0, & x \in \mathbb{R}^N, \\
\Delta \phi - \phi + 3\mu_2 U_{1,\mu_2}^2 \phi = 0, & x \in \mathbb{R}^N, \\
\phi, \bar{\phi} \in H^1_{rad}(\mathbb{R}^N).
\end{cases}
\] (2.22)

Note that the eigenvalues of

\[
\Delta \phi - \phi + \nu \mu_2 U_{1,\mu_2}^2 \phi = 0, \phi \in H^1(\mathbb{R}^N)
\]

are

\[
\nu_1 = 1, \nu_2 = \ldots = \nu_{N+1} = 3, \nu_{N+2} > 3,
\]

where the eigenfunction corresponding to $\nu_2$ are spanned by $\frac{\partial U_{1,\mu_2}}{\partial x_j}$, $j = 1, \ldots, N$.

(See [15, Lemma 4.1].) Hence, $\phi = 0$. If there is a nonzero solution $\phi$ to

\[
\Delta \phi - \lambda \phi + \beta U_{1,\mu_2}^2 \phi = 0, \quad x \in \mathbb{R}^N,
\]

we have that

\[
|\nabla \phi|^2 + \lambda |\phi|^2 = \beta |U_{1,\mu_2} \phi|^2.
\]

We obtain a contradiction by the definition of $\beta_2(\lambda)$ since $\beta < \beta_2(\lambda)$.

Similarly, we can prove the same result for $\beta < \beta_1(\lambda), (u, v) = (U_{\lambda,\mu_1}, 0)$.

\[ \square \]

Lemma 2.16. Let $\lambda > 0$, $\bar{\beta}_2(\lambda)$ be defined by (1.20), then $\bar{\beta}_2(\lambda) > 0$. Furthermore, $c^{\bar{\beta}_2(\lambda)}_\lambda = \min\{m_{\lambda,\mu_1}, m_{1,\mu_2}\}$ and only can be obtained by semi-trivial solution.

Proof. Firstly, we prove that $\bar{\beta}_2(\lambda) > 0$. If not, there exists ground state solution $(u_n, v_n)$ to system (1.2) with $\beta = \beta_n$ and $\beta_n \downarrow 0$. We note that $c^{\beta_n}_\lambda = \min\{m_{\lambda,\mu_1}, m_{1,\mu_2}\}$. Without loss of generality, we may assume that $(u_n, v_n)$ are positive radial function. Then it is standard to prove that, up to a subsequence, $(u_n, v_n) \to (u, v)$, and

\[
\limsup c^{\beta_n}_\lambda \leq \min\{m_{\lambda,\mu_1}, m_{1,\mu_2}\}.
\]

However, by [16, Theorem 4.1], $(u_n, v_n)$ is the unique positive solution of (1.2) for $n$ large enough and further more $(u_n, v_n) \to (U_{\lambda,\mu_1}, U_{1,\mu_2})$. So we have $\limsup c^{\beta_n}_\lambda = m_{\lambda,\mu_1} + m_{1,\mu_2}$, a contradiction. Hence, we prove that $\bar{\beta}_2(\lambda) > 0$.

Next, we shall prove that $c^{\bar{\beta}_2(\lambda)}_\lambda = \min\{m_{\lambda,\mu_1}, m_{1,\mu_2}\}$. If it is not true, by the definition of $c^{\bar{\beta}_2(\lambda)}_\lambda$, we have that

\[
c^{\bar{\beta}_2(\lambda)}_\lambda < \min\{m_{\lambda,\mu_1}, m_{1,\mu_2}\},
\]

and thus it is obtain by some positive radial solution $(u, v)$. Recalling that we have proved $\bar{\beta}_2(\lambda) > 0$, by the continuity and monotonicity, there exists some small $\delta > 0$ such that

\[
c^{\beta}_\lambda < \min\{m_{\lambda,\mu_1}, m_{1,\mu_2}\} \quad \text{for } \forall \beta \in (\bar{\beta}_2(\lambda) - \delta, \bar{\beta}_2(\lambda)).
\]

Then, we see that $c^{\beta}_\lambda$ is achieved by positive ground state solution for all $\beta > \bar{\beta}_2(\lambda) - \delta > 0$, a contradiction to the definition of $\bar{\beta}_2(\lambda)$.
Case 1: If \( m_{\lambda,\mu_1} < m_{1,\mu_2} \), then we have that \( c^{\beta_2(\lambda)}_\lambda = m_{\lambda,\mu_1} \). Let \((u, v) \neq (0, 0)\) be a minimizer. Then we have that \( u \neq 0 \). If not, \((u, v) = (0, U_{1,\mu_2})\) and thus
\[
eq \Phi^{\beta_2(\lambda)}_\lambda (0, U_{1,\mu_2}) = m_{1,\mu_2} > m_{\lambda,\mu_1} = c^{\beta_2(\lambda)}_\lambda,
\]
a contradiction.
Define
\[
\mathcal{G}(u, v) := \frac{|\nabla u|^2 + \lambda |u|^2 + |\nabla v|^2 + |v|^2}{\sqrt{\mu_1 |u|^4 + \mu_2 |v|^4 + \beta_2(\lambda) |uv|^2}}
\]
\((u, v) \in \mathcal{H} \setminus \{(0, 0)\}\). (2.23)
Then we have that
\[
\inf_{(u, v) \in \mathcal{H} \setminus \{(0, 0)\}} \mathcal{G}(u, v) = 2\sqrt{c^{\beta_2(\lambda)}_\lambda} = 2\sqrt{m_{\lambda,\mu_1}}.
\]
(2.24)
For \( t \geq 0 \), we have
\[
\mathcal{G}(tu, v) = \frac{(|\nabla u|^2 + \lambda |u|^2) t^2 + |\nabla v|^2 + |v|^2}{\sqrt{\mu_1 |tu|^4 + \mu_2 |v|^4 + \beta_2(\lambda) |uv|^2}}
\]
then we see that
\[
\lim_{t \to \infty} \mathcal{G}(tu, v) = \mathcal{G}(u, 0).
\]
(2.25)
By
\[
2\sqrt{m_{\lambda,\mu_1}} = \inf_{(u, v) \in \mathcal{H} \setminus \{(0, 0)\}} \mathcal{G}(u, v) = \inf_{(u, v) \in \mathcal{H} \setminus \{(0, 0)\}} \inf_{t > 0} \mathcal{G}(tu, v)
\]
\[
\leq \inf_{(u, v) \in \mathcal{H} \setminus \{(0, 0)\}} \lim_{t \to \infty} \mathcal{G}(tu, v) = \inf_{(u, v) \in \mathcal{H} \setminus \{(0, 0)\}} \mathcal{G}(u, 0)
\]
\[
= \inf_{u \in \mathcal{H}^1(\mathbb{R}^N) \setminus \{0\}} \mathcal{G}(u, 0) = 2\sqrt{m_{\lambda,\mu_1}},
\]
we see that 
\[ \overset{\text{“} = \text{”}}{=} \]
holds if and only if \( u = C_1 U_{\lambda,\mu_1} \) for some constant \( C_1 \geq 0 \) and
\[
\lim_{t \to \infty} \mathcal{G}(tU_{\lambda,\mu_1}, v) = \inf_{t > 0} \mathcal{G}(tU_{\lambda,\mu_1}, v).
\]
Furthermore, since \((u, v)\) is a nonnegative solution to
\[-\Delta u + \lambda u = \mu_1 u^3 + \beta_2(\lambda) u^2 \text{ in } \mathbb{R}^N, u(x), v(x) \to 0 \text{ as } |x| \to \infty,\]
by the uniqueness result of [12], we obtain that \( v = 0 \) (in this case, \( u = U_{\lambda,\mu_1} \)) or \( v = C_2 U_{\lambda,\mu_1} \) with \( C_2 > 0 \) such that
\[
\mu_1 C_1^2 + \beta C_2^2 = \mu_1.
\]
We note that \((u, v)\) also solves
\[-\Delta v + v = \mu_2 v^3 + \beta_2(\lambda) v u^2 \text{ in } \mathbb{R}^N, u(x), v(x) \to 0 \text{ as } |x| \to \infty.\]

By [12] again, if \( v \neq 0 \), then we have that \( \lambda = 1 \) and
\[
\beta C_1^2 + \mu_2 C_2^2 = \mu_2,
\]
i.e., \((u, v) = (C_1 U_{\lambda, \mu_1}, C_2 U_{\lambda, \mu_1})\) with
\[
C_1^2 = \frac{(\beta - \mu_1)\mu_2}{\beta^2 - \mu_1\mu_2}, \quad C_2^2 = \frac{(\beta - \mu_2)\mu_1}{\beta^2 - \mu_1\mu_2}.
\]
Therefore, if \( \lambda \neq 1 \), we obtain a contradiction. And if \( \lambda = 1 \), by [14, Theorem 1], we see that \( \bar{\beta}_2(1) = \max\{\beta_1(1), \bar{\beta}_2(1)\} = \max\{\mu_1, \mu_2\} \). Then we see that one of \( C_1, C_2 \) equals 0, also a contradiction. We obtain that \((u, v) = (U_{\lambda, \mu_1}, 0)\).

**Case 2:** If \( m_{\lambda, \mu_1} > m_{1, \mu_2} \), similar to the case 1, we can prove that \( c_\lambda^2(\lambda) = m_{1, \mu_2} = \min\{m_{\lambda, \mu_1}, m_{1, \mu_2}\} \) and it is obtained only by \((u, v) = (0, U_{1, \mu_2})\).

**Case 3:** \( m_{\lambda, \mu_1} = m_{1, \mu_2} \). Let \((u, v)\) be a minimizer. If \( u = 0 \), we have that \( v \) is a positive solution to
\[
-\Delta v + v = \mu_2 v^3 \text{ in } \mathbb{R}^N, \quad v(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty
\]
and
\[
\frac{1}{2} |\nabla v|^2 + v^2 - \frac{1}{4} \mu_2 |v|^4 = \Phi_\lambda^\beta(0, v) = m_{1, \mu_2}.
\]
So we have \( v = U_{1, \mu_2} \), i.e., \((u, v) = (0, U_{1, \mu_2})\). If \( u \neq 0 \), we apply a similar argument as in Case 1, we can prove that \((u, v) = (U_{\lambda, \mu}, 0)\).

In a word, \( c_\lambda^2(\lambda) = \min\{m_{\lambda, \mu_1}, m_{1, \mu_2}\} \) and only can be obtained by semi-trivial solution.

**Proof of Theorem 1.6:** We note that \( m_{\lambda, \mu_1} > (=, <) m_{1, \mu_2} \) if and only if \( \lambda > (=, <) \left( \frac{\mu_1}{\mu_2} \right)^{\frac{1}{2\pi}} \).

We prove the case of \( \left( \frac{\mu_1}{\mu_2} \right)^{\frac{1}{2\pi}} \leq \lambda^* \), and the case of \( \left( \frac{\mu_1}{\mu_2} \right)^{\frac{1}{2\pi}} > \lambda^* \) can be proved in a similar way.

**Case 1:** \( \lambda \in \left( (\frac{\mu_1}{\mu_2})^{\frac{1}{2\pi}}, \lambda^* \right) \). We have that
\[
c_\lambda^2 \leq \min\{m_{\lambda, \mu_1}, m_{1, \mu_2}\} = m_{1, \mu_2}
\]
and
\[
\beta_2(\lambda) = \min\{\beta_1(\lambda), \beta_2(\lambda)\}.
\]
Then for any \( \beta > \beta_2(\lambda) \), we see that \((0, U_{1, \mu_2})\) is a saddle point of \( \Phi_\lambda^\beta \) on \( \tilde{M}_\lambda^\beta \). Then we have that
\[
c_\lambda^\beta < m_{1, \mu_2} = \min\{m_{\lambda, \mu_1}, m_{1, \mu_2}\},
\]
and thus it can be achieved by a positive ground state solution. Then we have that
\[
\bar{\beta}_2(\lambda) \leq \beta_2(\lambda) = \min\{\beta_1(\lambda), \beta_2(\lambda)\}.
\]
We claim that \( \bar{\beta}_2(\lambda) = \beta_2(\lambda) \). If not, by Lemma 2.16, we have that \( 0 < \bar{\beta}_2(\lambda) < \beta_2(\lambda) \). By the definition of \( \bar{\beta}_2(\lambda) \), we can take a positive radial ground state solutions
Lemma 2.15, we see that \( \bar{\beta}_2(\lambda) \). By Lemma 2.16 again, \((u,v)\) is a semi-trivial solution. Hence, by Lemma 2.15, we see that \( \bar{\beta}_2(\lambda) = \beta_2(\lambda) \). Furthermore, if \( \frac{\mu_1}{\mu_2} \neq \lambda^* \), then for \( \lambda \in ((\frac{\mu_1}{\mu_2})^{\frac{2}{n-2}}, \lambda^*] \), we see that \( m_{\lambda,\mu_1} > m_{1,\mu_2} \), and thus \((u,v) = (0, U_{1,\mu_2})\). And for the case \( \lambda = (\frac{\mu_1}{\mu_2})^{\frac{2}{n-2}} \), \( c_\lambda^\beta \) can also be attained by \((U_{\lambda,\mu_1}, 0)\). However, we still have that the ground state solution sequence \((u_n, v_n, \beta_n) \rightarrow (0, U_{1,\mu_2}, \bar{\beta}_2(\lambda))\). If not, by \( \bar{\beta}_2(\lambda) = \beta_2(\lambda) < \beta_1(\lambda) \) since \( \lambda < \lambda^* \), then by Lemma 2.15 again, we can deduce a contradiction.

**Case 2:** \( \lambda < (\frac{\mu_1}{\mu_2})^{\frac{2}{n-2}} \). We have that

\[
c_\lambda^\beta \leq \min\{m_{\lambda,\mu_1}, m_{1,\mu_2}\} = m_{\lambda,\mu_1} < m_{1,\mu_2}
\]

and

\[
\bar{\beta}_2(\lambda) < \beta_1(\lambda).
\]

By Lemma 2.16, we see that \( c_\lambda^\beta \) can be obtained only by \((U_{\lambda,\mu_1}, 0)\). We claim that \( \beta_2(\lambda) = \beta_1(\lambda) \). If not, by Lemma 2.16, we have that \( 0 < \beta_2(\lambda) < \beta_1(\lambda) \). Then by Lemma 2.15, we can deduce a contradiction.

**Case 3:** \( \lambda > \lambda^* \). We have that

\[
c_\lambda^\beta \leq \min\{m_{\lambda,\mu_1}, m_{1,\mu_2}\} = m_{1,\mu_2} < m_{\lambda,\mu_1}
\]

and

\[
\beta_1(\lambda) < \beta_2(\lambda).
\]

By Lemma 2.16, we see that \( c_\lambda^\beta \) can be obtained only by \((0, U_{1,\mu_2})\). Apply a similar argument as above, by Lemma 2.15 we can prove that \( \bar{\beta}_2(\lambda) = \beta_2(\lambda) \). \( \square \)

### 3 The multiplicity of positive solution

As a byproduct, in this section, we presuppose \( \lambda^* \neq (\frac{\mu_1}{\mu_2})^{\frac{2}{n-2}} \) to establish the multiplicity result.

**Proof of Theorem 1.8.** Without loss of generality, we only prove the case \( \lambda^* > (\frac{\mu_1}{\mu_2})^{\frac{2}{n-2}} \). We have \( D_\lambda = ((\frac{\mu_1}{\mu_2})^{\frac{2}{n-2}}, \lambda^*] \) and \( D_\beta = (\beta_2(\lambda), \beta_1(\lambda)) \). So for any \( \lambda \in D_\lambda \), we have that \( m_{\lambda,\mu_1} \geq m_{1,\mu_2} \). By \( \beta > \beta_2(\lambda) \), we see that \((0, U_{1,\mu_2})\) is a strict saddle point of \( \Phi_\lambda^\beta \) on \( \bar{M}_\lambda^\beta \), and thus

\[
0 < c_\lambda^\beta < m_{1,\mu_2} = \min\{m_{\lambda,\mu_1}, m_{1,\mu_2}\}.
\]

Hence, \( c_\lambda^\beta \) can be obtained by a positive radial ground state solution \((u_1, v_1)\).
On the other hand, by $\beta < \beta_1(\lambda)$, we see that $(U_{\lambda,\mu_1}, 0)$ is a local minimal point of $\Phi_{\lambda}^{\beta_1}$ on $\tilde{M}_{\lambda}^{\beta_1}$ and $\Phi_{\lambda}^{\beta_1}(u_1, v_1) < m_{1,\mu_1} \leq m_{\lambda,\mu_1} = \Phi_{\lambda}^{\beta_1}(U_{\lambda,\mu_1}, 0)$, then we can construct a mountain pass type solution $(u_2, v_2)$, which is positive and $\Phi_{\lambda}^{\beta_1}(u_2, v_2) > m_{\lambda,\mu_1} = \max\{m_{\lambda,\mu_1}, m_{1,\mu_2}\}$. 

4 Nonexistence of positive solution

In this section, we shall give another progress for the case of $\lambda > 1$. Firstly, let us recall Corollary 2.10, we see that $\mu_2 \leq \mu_1$ implies that $\lambda^* \geq \frac{1}{4 - N} \left(4\sqrt{\frac{\mu_1}{\mu_2}} - N\right) \geq \sqrt{\frac{\mu_1}{\mu_2}} \geq 1$.

Combining with Lemma 2.6, we have that $[\lambda \mu_2, \frac{\mu_1}{\lambda}] \subset [\beta_2(\lambda), \beta_1(\lambda)]$, $1 \leq \lambda \leq \sqrt{\frac{\mu_1}{\mu_2}}.$

Proof of Theorem 1.4. Suppose we have a positive solution $(u, v)$ of system (1.2). Then it is well known that $u, v$ are radial functions (up to transform) and strictly radially decreasing if $\beta > 0$. So we have that $\nabla u = u'(r) \frac{r}{|x|}, \nabla v = v'(r) \frac{r}{|x|}$ with $u'(r) < 0, v'(r) < 0$, and thus $\int_{\mathbb{R}^N} \langle \nabla u, \nabla v \rangle > 0.$

We multiply the first equation in (1.2) by $v$, the second equation by $u$, and integrate the resulting equations over $\mathbb{R}^N$. This yields

$$(1 - \lambda) \int_{\mathbb{R}^N} \langle \nabla u, \nabla v \rangle = \int_{\mathbb{R}^N} [(\beta - \lambda \mu_2) v^3 u + (\mu_1 - \lambda \beta) v u^3]. \quad (4.1)$$

By $\lambda > 1$, the left hand side of (4.1) $\leq 0$. However, by $\lambda \mu_2 \leq \beta \leq \frac{\mu_1}{\lambda}$ and $u > 0, v > 0$, the right hand side of (4.1) $\geq 0$, a contradiction. Hence, (1.2) has no positive solution if $(\lambda, \beta) \in \left[1, \frac{\mu_1}{\mu_2}\right] \times \left[\lambda \mu_2, \frac{\mu_1}{\lambda}\right].$ 

5 The uniqueness of ground state solution

For $\lambda = 1$, system (1.2) has no solution in the regime $\beta \in [\min\{\mu_1, \mu_2\}, \max\{\mu_1, \mu_2\}]$, if $\mu_1 \neq \mu_2$. On the other hand, for $\beta \in (0, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, +\infty)$ it is also easy to see that

$$u_\beta(x) = \sqrt{\frac{\beta - \mu_2}{\beta^2 - \mu_1 \mu_2}} U(x), \quad v_\beta(x) = \sqrt{\frac{\beta - \mu_1}{\beta^2 - \mu_1 \mu_2}} U(x)$$

solve (1.2). The solution $(u_\beta, v_\beta)$ is nondegenerate in the space $H^1_{rad}(\mathbb{R}^3, \mathbb{R}^2)$; see [10, Lemma 2.2]. Sirakov [14, Remark 2] conjectured that, up to translations, $(u_\beta, v_\beta)$
is the unique positive solution of (1.2). Wei and Yao [16, Theorem 4.1, Theorem 4.2] proved this conjecture for \( \beta > \max\{\mu_1, \mu_2\} \) and for \( 0 < \beta < \beta_0 \) close to 0. Chen and Zou [9, Theorem 1.1] proved the conjecture in case \( \beta_0' < \beta < \min\{\mu_1, \mu_2\} \) close to \( \min\{\mu_1, \mu_2\} \). The remaining range \( \beta \in [\beta_0, \beta_0'] \) is open up to now.

Consider a weaker problem: what about the uniqueness of the ground state solution? Chen and Zou [8, Theorem 4.2] prove that \( (u_\beta, v_\beta) \) is the unique ground state solution of (1.2). However, if \( \lambda \neq 1 \), the symmetry of (1.2) is broken, and the problem becomes more complicated.

By the implicit function theorem, Wei and Yao [16] obtained the uniqueness of positive solution for \( \beta \) near 0. By the bifurcation method, under the assumption \( (1 - \lambda)(\mu_2 - \mu_1) \leq 0 \), Chen and Zou [9] prove the uniqueness of ground state solution for \( \beta \in (\beta_1(\lambda) - \delta(\lambda), \beta_1(\lambda)) \), here \( \delta(\lambda) \) is a number close to 0 and depends on \( \lambda \).

We can obtain the following result in this direction.

**Theorem 5.1.** (i) Let \( \tilde{\beta}_2(\lambda) \) be defined by (1.21), then there exists \( \delta_1 > 0 \) such that for any

\[
\beta \in \left( \tilde{\beta}_2(\lambda), \beta_2(\lambda) + \delta_1 \right),
\]

the ground state solution of (1.2) is unique up to a translation, which has the energy

\[
c^\beta_\lambda = \inf_{(u,v) \in \mathcal{A}_\lambda^\beta} \Phi^\beta_\lambda(u,v).
\]

(ii) Let \( \beta_1(\lambda), \beta_2(\lambda) \) be defined by (1.16), if \( \min\{\beta_1(\lambda), \beta_2(\lambda)\} < \sqrt{\mu_1 \mu_2} \), then there exists \( \delta_2 > 0 \) such that for any

\[
\beta \in \left( \min\{\beta_1(\lambda), \beta_2(\lambda)\} - \delta_2, \min\{\beta_1(\lambda), \beta_2(\lambda)\} \right),
\]

the ground state solution of (1.2) is unique up to a translation, which has the energy

\[
\mathcal{A}_\lambda^\beta = \inf_{(u,v) \in \mathcal{A}_\lambda^\beta} \Phi^\beta_\lambda(u,v).
\]

**Proof.** (i) We only prove the case of \( 0 < \lambda < \frac{\mu_1}{\mu_2} \), the other case can be proved in a similar way. Then by Theorem 1.6, we have \( \tilde{\beta}_2(\lambda) = \beta_1(\lambda) \) and \( c^\beta_\lambda = c^\beta_\lambda(\lambda) = m_{\lambda, \mu_1} < m_{1, \mu_2} \). By Lemma 2.16, \( c^\beta_\lambda(\lambda) \) is only obtained by \( (U_{\lambda, \mu_1}, 0) \). Suppose there exists \( \delta_1 > 0 \), \( (u_1, v_1, \beta_1), (u_2, v_2, \beta_2) \) are two different positive radial solutions to system (1.2). Up to a subsequence, we have that \( (u_1, v_1, \beta_1) \rightarrow (U_{\lambda, \mu_1}, 0, \beta_1) \) is a bifurcation from \( (U_{\lambda, \mu_1}, 0, \beta_1) \). By the assumption, we also have that \( (u_2, v_2, \beta_2) \rightarrow (U_{\lambda, \mu_1}, 0, \beta_1) \) is another bifurcation from \( (U_{\lambda, \mu_1}, 0, \beta_1) \). However, by Lemma 2.3 in [10], this is a bifurcation from a simple eigenvalue, hence there can not be two different bifurcations (see [6, 7] or [2, Lemma 3.1]), that is, we get a contradiction. Therefore, there exists some \( \delta_1 > 0 \) such that for any \( \beta \in (\beta_2(\lambda), \beta_2(\lambda) + \delta_1) \), the ground state solution of (1.2) is unique up to a translation.

(ii) Indeed, under the additional assumption of \( (1 - \lambda)(\mu_2 - \mu_1) \leq 0 \), this conclusion is proved by Chen and Zou (see [9, Theorem 1.2-(iv)]). However, when remove the assumption of \( (1 - \lambda)(\mu_2 - \mu_1) \leq 0 \), by Theorem 1.5, we can also prove that \( \tilde{\beta}_1(\lambda) = \)}
min\{β_1(λ), β_2(λ)\} and \(A_λ^β\) is not attained provided \(β = \min\{β_1(λ), β_2(λ)\}\). The remaining proof is similar as above.

6 The continuity of \(A_λ^β\)

The continuity of \(c_λ^β\) can be seen from the definition easily (see Corollary 2.13). However, if \((λ, β) ∈ \Lambda\) with
\[
\Lambda := \{(λ, β) ∈ R^+ × R^+ : β < \min\{β_1(λ), β_2(λ)\}\},
\]
we see that \(A_λ^β \neq c_λ^β\), and the continuity of \(A_λ^β\) is not trivial. We write \(P_1 : R^+ × R^+ → R^+\) for the projection from \((λ, β)\) onto the \(λ\)-component. Let
\[
A_λ^β := \Lambda ∩ R^+ × \{β\}.
\]
In this section, we show that \(A_λ^β\) (cf. (1.5)) is continuous for \(λ ∈ P_1(Λ^β)\) with \(β < \sqrt{μ_1μ_2}\). We remark that \(A_λ^β > \max\{m_1, m_2\}\) for \(λ ∈ P_1(Λ^β)\) with \(β < \sqrt{μ_1μ_2}\). We also note that \(P_1(Λ^β) \neq \emptyset\) provided \(β < \sqrt{μ_1μ_2}\), since Corollary 2.8 indicates that \(β < β^*\).

Lemma 6.1. For any \(u ≠ 0, v ≠ 0\) satisfying \(D_β(u, v) > 0\), then there exists a unique \((s, t) ∈ R^2\) such that
\[
\begin{cases}
|∇u|^2 + λ|u|^2 = \mu_1|u|^2s + β|uv|^2t, \\
|∇v|^2 + |v|^2 = \mu_2|v|^2t + β|uv|^2s.
\end{cases}
\]
(6.1)

And if \((s, t) ∈ (0, ∞)^2\), then \((\sqrt{su}, \sqrt{tv}) ∈ M_λ^{β^+}\) and
\[
Φ_λ(\sqrt{su}, \sqrt{tv}) = \max_{(s', t') ∈ [0, ∞)^2} Φ_λ(\sqrt{s'}u, \sqrt{t'}v).
\]

Proof. By \(D_β(u, v) > 0\), we see that the matrix
\[
\begin{pmatrix}
\mu_1 |u|^4 + β|uv|^2 \\
β|uv|^2 + μ_2 |v|^4
\end{pmatrix}
\]
is invertible. Hence, there exists a unique \((s, t) ∈ R^2\) such that (6.1) holds. Precisely, we have
\[
\begin{cases}
s = \frac{|∇u|^2 + λ|u|^2 - μ_1|u|^2 - β|uv|^2}{D_β(u, v)}, \\
t = \frac{|∇v|^2 + |v|^2 - μ_2|v|^2 - β|uv|^2}{D_β(u, v)}.
\end{cases}
\]
(6.2)

If \(s > 0, t > 0\), we have \((\sqrt{su}, \sqrt{tv}) ∈ M_λ^{β^+}\) and \(D_β(\sqrt{su}, \sqrt{tv}) = stD_β(u, v) > 0\). Then by Lemma 2.1, we obtain that \(Φ_λ^β(\sqrt{su}, \sqrt{tv}) = \max_{(s', t') ∈ [0, ∞)^2} Φ_λ^β(\sqrt{s'}u, \sqrt{t'}v)\).

□
Remark 6.2. For a fixed $\beta > 0$, the above $(s, t)$ depends on $(u, v)$ and $\lambda$, we may denote it by $s_\beta(\lambda, u, v), t_\beta(\lambda, u, v)$. Furthermore, for such a fixed $(u, v)$, $s_\beta(\lambda, u, v)$ and $t_\beta(\lambda, u, v)$ are continuous by $\lambda$. By $D_\beta(u, v) > 0$, we also have the property that $s_\beta(\lambda, u, v)$ increases with respect to $\lambda$ while $t_\beta(\lambda, u, v)$ decreases. So they are differentiable almost everywhere where respect to $\lambda > 0$.

Theorem 6.3. Assume that $\beta < \sqrt{\mu_1/\mu_2}$ is fixed. Then $A_\lambda^\beta$ (cf. (1.5)) is continuous in $\lambda$ and increasing strictly with respect to all $\lambda \in P_1(\Lambda^\beta)$.

Proof. For $\forall \varepsilon > 0$, there exists some $(u, v) \in M_\lambda^\beta$, such that
\[
A_\lambda^\beta \leq \Phi_\lambda^\beta(u, v) < A_\lambda^\beta + \varepsilon.
\]
By $(u, v) \in M_\lambda^\beta$, we have that $s_\beta(\lambda, u, v) = 1, t_\beta(\lambda, u, v) = 1$. Then there exists some small $\eta_\lambda > 0$ such that $s_\beta(\lambda, u, v) > 0, t_\beta(\lambda, u, v) > 0$ for all $\lambda \in (\lambda - \eta_\lambda, \lambda + \eta_\lambda)$.

Thus, $(\sqrt{s_\beta(\lambda, u, v)u}, \sqrt{t_\beta(\lambda, u, v)v}) \in M_\lambda^\beta$ for $\lambda \in (\lambda - \eta_\lambda, \lambda + \eta_\lambda)$. Then
\[
\Phi_\lambda^\beta(\sqrt{s_\beta(\lambda, u, v)u}, \sqrt{t_\beta(\lambda, u, v)v})
\]

is continuous with respect to $\lambda \in (\lambda - \eta_\lambda, \lambda + \eta_\lambda)$. Now, for any sequence $\{\lambda_n\} \subset (\lambda - \eta_\lambda, \lambda + \eta_\lambda)$ with $\lambda_n \to \lambda$, we have
\[
\Phi_\lambda^\beta(s_\beta(\lambda_n, u, v)u, t_\beta(\lambda_n, u, v)v) \to \Phi_\lambda^\beta(u, v) < A_\lambda^\beta + \varepsilon.
\]

Hence, we have
\[
\limsup_{\lambda_n \to \lambda} A_\lambda^\beta \leq \limsup_{\lambda_n \to \lambda} \Phi_\lambda^\beta(s_\beta(\lambda_n, u, v)u, t_\beta(\lambda_n, u, v)v) < A_\lambda^\beta + \varepsilon. \tag{6.3}
\]

Furthermore, by the arbitrariness of $\varepsilon > 0$, we have
\[
\limsup_{\lambda_n \to \lambda} A_\lambda^\beta \leq A_\lambda^\beta. \tag{6.4}
\]

It is known that $A_\lambda^\beta$ is attained by some solution $(u_\lambda, v_\lambda) \in M_\lambda^\beta$ since $\lambda \in P_1(\Lambda^\beta)$ and $\beta < \sqrt{\mu_1/\mu_2}$. We claim that
\[
A_\lambda^\beta \leq f(\lambda') < f(\lambda) = A_\lambda^\beta, \ \forall \lambda' \in (\lambda - \eta_\lambda, \lambda) \text{ with a suitable small } \eta_\lambda, \tag{6.5}
\]
where
\[
f(\lambda') := \Phi_\lambda^\beta(\sqrt{s_\beta(\lambda', u_\lambda, v_\lambda)u_\lambda}, \sqrt{t_\beta(\lambda', u_\lambda, v_\lambda)v_\lambda})
\]

\[
= \frac{1}{4} \left[ (|\nabla u_\lambda|_2^2 + \lambda'|u_\lambda|_2^2)s_\beta(\lambda', u_\lambda, v_\lambda) + (|\nabla v_\lambda|_2^2 + |v_\lambda|_2^2)t_\beta(\lambda', u_\lambda, v_\lambda) \right]. \tag{6.6}
\]

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To prove this claim, we fix \( \lambda \in P_1(\Lambda^\beta) \), and let \((u_\lambda, v_\lambda) \in \mathcal{M}_\lambda^\beta \) be the least energy solution. Consider \( \lambda' \in (\lambda - \eta_\lambda, \lambda + \eta_\lambda) \) with \( \eta_\lambda \) small enough, by Lemma 6.1, we have

\[
\left( \sqrt{s_\beta'(\lambda', u_\lambda, v_\lambda)} u_\lambda + \sqrt{t_\beta'(\lambda', u_\lambda, v_\lambda)} v_\lambda \right) \in \mathcal{M}_\lambda^\beta, \quad \lambda' \in (\lambda - \eta_\lambda, \lambda + \eta_\lambda) \tag{6.7}
\]

and

\[
(s_\beta(\lambda', u_\lambda, v_\lambda), t_\beta(\lambda', u_\lambda, v_\lambda)) \to (1, 1) \quad \text{as} \quad \lambda' \to \lambda. \tag{6.8}
\]

Noting that \( f(\lambda') \) is continuous, furthermore, it is differentiable for almost every \( \lambda' \). Insert the formula (6.2) into (6.6). By a direct computation, we have that

\[
\frac{df(\lambda')}{d\lambda'}|_{\lambda' = \lambda} = \frac{1}{2} \int_{\mathbb{R}^N} u_\lambda^2 > 0. \tag{6.9}
\]

The above claim implies that \( A_{\lambda_n}^\beta \) increases strictly by \( \lambda \in P_1(\Lambda^\beta) \).

In the following, we shall prove that \( A_{\lambda_n}^\beta \) is continuous in \( P_1(\Lambda^\beta) \). Fix any \( \lambda \in P_1(\Lambda^\beta) \) and let any sequence \( \{\lambda_n\} \subset P_1(\Lambda^\beta) \) with \( \lambda_n \to \lambda \) in \( P_1(\Lambda^\beta) \). We denote

\[
\lambda := \min_{n \in \mathbb{N}} \{\lambda_n\},
\]

then we have \( \lambda \in P_1(\Lambda^\beta) \). The monotonicity implies that

\[
A_{\lambda_n}^\beta \geq A_{\lambda}^\beta > \max\{m_{\Delta, \mu_1}, m_{1, \mu_2}\}. \tag{6.10}
\]

By (6.4) again, we have that \( \{u_{\lambda_n}\}, \{v_{\lambda_n}\} \) are bounded in \( H^1(\mathbb{R}^N) \). Since they are radial, up to a subsequence if necessary, it is standard to prove that there exists some \((u, v) \in \mathcal{H}_{\text{rad}} \) such that

\[
(u_{\lambda_n}, v_{\lambda_n}) \to (u, v) \quad \text{in} \quad \mathcal{H}.
\]

By (6.10), we have that

\[
\Phi_{\lambda_n}^\beta(u, v) = \lim_{\lambda_n \to \lambda} \Phi_{\lambda_n}^\beta(u_{\lambda_n}, v_{\lambda_n}) > \max\{m_{\Delta, \mu_1}, m_{1, \mu_2}\}. \tag{6.11}
\]

So we have \((u, v) \neq (0, 0)\). If \( u = 0 \), we see that \( v \) is a positive solution to

\[-\Delta v + v = \mu_2 v^3 \quad \text{in} \quad \mathbb{R}^N, \quad v(x) \to 0 \quad \text{as} \quad |x| \to \infty.
\]

So we have \((u, v) = (0, U_{1, \mu_2})\), a contradiction to \( \lambda \in P_1(\Lambda^\beta) \) with \( \beta < \sqrt{\mu_1 \mu_2} \).

Similarly, if \( v = 0 \), we have \((u, v) = (U_{\lambda, \mu_1}, 0)\), also a contradiction to \( \lambda \in P_1(\Lambda^\beta) \) with \( \beta < \sqrt{\mu_1 \mu_2} \).

Hence, \( u \neq 0, v \neq 0 \) and thus \((u, v) \in \mathcal{M}_\lambda^\beta \).

\[
\lim_{n \to \infty} A_{\lambda_n}^\beta = \lim_{n \to \infty} \Phi_{\lambda_n}^\beta(u_{\lambda_n}, v_{\lambda_n})
= \lim_{n \to \infty} \frac{1}{4} \left( |\nabla u_{\lambda_n}|^2 + \lambda_n |u_{\lambda_n}|^2 + |\nabla v_{\lambda_n}|^2 + |v_{\lambda_n}|^2 \right)
= \frac{1}{4} \left( |\nabla u|^2 + \lambda |u|^2 + |\nabla v|^2 + |v|^2 \right) = \Phi_{\lambda}^\beta(u, v) \geq A_{\lambda}^\beta
\]
Thus, combining with (6.4) again, we obtain that
\[ \lim_{n \to \infty} A^\beta_{\lambda_n} = A^\beta_{\lambda}. \]
That is, \( A^\beta_{\lambda} \) is continuous with respect to \( \lambda \in P_1(\Lambda^\beta) \) with \( \beta < \sqrt{\mu_1\mu_2} \). \( \square \)

7 Appendix: Computation of \( \beta_i(\lambda) \) for \( N = 1 \)

Consider the eigenvalue problem
\[-\Delta \varphi + s\varphi = \tau(s)U^2 \varphi \text{ in } \mathbb{R}^N, \varphi \in H^1(\mathbb{R}^N).\]
It is standard to prove that \( \tau(s) \) can be achieved by some radial function \( \varphi(x) = \varphi(|x|) \).
Set \( \varphi(x) = U^\gamma(x)F(x) \), for \( N = 1 \), we have
\[ F_{xx} + 2\gamma \frac{U'}{U} F_x + \left[ \gamma \frac{U''}{U} + \gamma(\gamma - 1) \frac{U'^2}{U^2} - s + \tau(s)U^2 \right] F = 0. \]
Noting that
\[ \frac{U''}{U} = 1 - U^2 \frac{U'^2}{U^2} = 1 - \frac{1}{2} U^2, \]
so we take \( \gamma = \sqrt{s} \) and thus \( \varphi = U^\sqrt{s} F \), we have
\[ F_{xx} + 2\sqrt{s} \frac{U'}{U} F_x + \left[ \tau(s) - s + \frac{\sqrt{s}}{2} \right] U^2 F = 0. \quad (6.1)\]
Introduce
\[ z = \frac{1}{2}(1 - \frac{U'}{U}), \]
then
\[ \frac{dz}{dx} = \frac{1}{4} U^4 = \frac{1}{2} (1 - \frac{U'^2}{U^2}) = 2z(1 - z), \]
and
\[ \frac{d^2z}{dx^2} = 2 \frac{dz}{dx} (1 - z) - 2z \frac{dz}{dx} = 2(1 - 2z) \frac{dz}{dx} = 4z(1 - z)(1 - 2z). \]
Hence,
\[ F_x = F_z \frac{dz}{dx} = 2z(1 - z)F_z \]
and
\[ F_{xx} = 4z^2(1 - z)^2 F_{zz} + 4z(1 - z)(1 - 2z)F_z. \]
Noting that
\[ \frac{U'}{U} = 1 - 2z \]
and
\[ U^2 = 2 - 2 \frac{U''}{U^2} = 2 - 2(1 - 2z)^2 = 8z(1 - z), \]

So we have
\[ z(1 - z)F_{zz} + (1 + \sqrt{s})(1 - 2z)F_z + [2\tau(s) - s - \sqrt{s}]F = 0. \quad (6.2) \]

We also note that for \( x \in [0, \infty) \), we have \( z \geq \frac{1}{2} \). For \( x = 0 \), by \( \frac{U''}{U} = 1 - 2z = 0 \), we have \( z = \frac{1}{2} \). And for \( |x| \to \infty \), by \( 0 < U^2 = 8z(1 - z) \to 0 \) and \( z \geq \frac{1}{2} \), we have that \( z \to 1 \). Especially, for \( x \in (0, \infty) \), we have \( z \in (\frac{1}{2}, 1) \).

Hence, \( z = 0 \) is a singular point of the hypergeometric equation (6.2) but not a singular point of solution \( F \). Set
\[
\begin{cases}
-ab = 2\tau(s) - s - \sqrt{s}, \\
a + b + 1 = 2(1 + \sqrt{s}), \\
c = 1 + \sqrt{s}
\end{cases}
\]

then \( a, b \) are the two roots of
\[ t^2 - (1 + 2\sqrt{s})t + [s + \sqrt{s} - 2\tau(s)] = 0. \]

We obtain that
\[
a, b = \frac{1 + 2\sqrt{s} \pm \sqrt{1 + 4\sqrt{s} + 4s - 4(s + \sqrt{s} - 2\tau(s))}}{2}
= \frac{1 + 2\sqrt{s} \pm \sqrt{1 + 8\tau(s)}}{2}.
\]

Equation (6.2) has two linear independent solution
\[ F(x; 0) := _2F_1(a; b; c; z); \quad F(x; 1 - c) := x^{1-c}_2F_1(a - c + 1; b - c + 1; 2 - c; z). \]

By \( 1 - c = -\sqrt{s} \), we see that \( x = 0 \) is a singular point of \( F(x; 1 - c) \). Hence, up to a multiplier,
\[ F(x) = _2F_1(a; b; c; z). \]

By Gauss-Kummer identity and the relation
\[ _2F_1(a; b; c; z) = (1 - z)^{c-a-b} _2F_1(c - a; c - b; c; z) \]

we have that
\[
\lim_{z \to 1} (1 - z)^{a+b-c} _2F_1(a; b; c; z) = _2F_1(c - a; c - b; c; 1) = \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}.
\]

Since \( a + b - c = \tau(s) > 0 \) with \( s > 0 \), so a solution that is regular at both \( z = 0 \) and \( z = 1 \) can only exists if \( \Gamma(t) \) has a pole at \( a \) or \( b \). In other words \( a = 0, -1, -2, \cdots \) or \( b = 0, -1, -2, \cdots \). Hence, we have
\[
\frac{1 + 2\sqrt{s} - \sqrt{1 + 8\tau(s)}}{2} = -\ell, \quad \ell \geq 0.
\]

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Thus,
\[
\tau(s) = \frac{(1 + 2\sqrt{s} + 2\ell)^2 - 1}{8} = \frac{s + \sqrt{s} + \ell^2 + \ell + 2\ell\sqrt{s}}{2}.
\]
Combining with \(\tau(1) = 1\), we see that
\[
\frac{1 + \sqrt{1} + \ell^2 + \ell + 2\ell\sqrt{1}}{2} = 1 \Rightarrow \ell^2 + 3\ell = 0 \Rightarrow \ell = 0 \text{ or } \ell = -3.
\]
Since \(\ell \geq 0\), we finally obtain that \(\ell = 0\) and thus
\[
\tau(s) = \frac{s + \sqrt{s}}{2}.
\]
So we have that
\[
\beta_1(\lambda) = \mu_1\tau\left(\frac{1}{\lambda}\right) = \frac{\mu_1}{2}\left(\frac{1}{\lambda} + \sqrt{\frac{1}{\lambda}}\right)
\]
and
\[
\beta_2(\lambda) = \mu_2\tau(\lambda) = \frac{\mu_2}{2}(\lambda + \sqrt{\lambda}).
\]
So let
\[
\beta_1(\lambda^*) = \beta_2(\lambda^*),
\]
we obtain that
\[
\lambda^* = \left(\frac{\mu_1}{\mu_2}\right)^2.
\]

References


