CLASSIFICATION OF SOLUTIONS TO GENERAL TODA SYSTEMS WITH SINGULAR SOURCES

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ABSTRACT. We classify all the solutions to the elliptic Toda system associated to a general simple Lie algebra with singular sources at the origin and with finite energy. The solution space is shown to be parametrized by a subgroup of the corresponding complex Lie group. We also show the quantization result for the finite integrals. This work generalizes the previous works in [LWY12] and [Nie16] for Toda systems of types $A$ and $B, C$. However, a more Lie-theoretic method is needed here for the general case, and the method relies heavily on the structure theories of the local solutions and of the $W$-invariants for the Toda system. This work will have applications to nonabelian Chern-Simons-Higgs gauge theory and to the mean field equations of Toda type.

1. Introduction

In this paper, we consider the following Toda systems on the plane. Let $\mathfrak{g}$ be a complex simple Lie algebra of rank $n$, and let $(a_{ij})$ be its Cartan matrix (see [Hel78, Kna02, FH91] and the Appendix for basic Lie theory). The Toda system associated to $\mathfrak{g}$ with singular sources at the origin and with finite energy is the following system of semilinear elliptic PDEs

$$
\begin{align*}
\Delta u_i + 4 \sum_{j=1}^{n} a_{ij} e^{u_j} &= 4\pi \gamma_i \delta_0 \quad \text{on } \mathbb{R}^2, \quad \gamma_i > -1, \\
\int_{\mathbb{R}^2} e^{u_i} \, dx &< \infty, \quad 1 \leq i \leq n,
\end{align*}
$$

where $\delta_0$ is the Dirac delta function at the origin. Here the solutions $u_i$ are required to be real and well-defined on the whole $\mathbb{R}^2$ minus the origin.

When the Lie algebra $\mathfrak{g} = A_1 = \mathfrak{sl}_2$ whose Cartan matrix is (2), the Toda system becomes the Liouville equation

$$
\Delta u + 8 e^u = 4\pi \gamma \delta_0 \quad \text{on } \mathbb{R}^2, \quad \gamma > -1,
$$

where $\delta_0$ is the Dirac delta function at the origin. Here the solutions $u_i$ are required to be real and well-defined on the whole $\mathbb{R}^2$ minus the origin.

The Toda system (1.1) and the Liouville equation (1.2) arise in many physical and geometric problems. For example, in the Chern-Simons theory, the Liouville equation is related to the abelian gauge field theory, while the Toda system is related to nonabelian gauges (see [Yan01, Tar08]). On the geometric side, the Liouville equation is related to conformal metrics on $S^2$ with conical singularities whose Gaussian curvature is a constant. The Toda systems are related to holomorphic curves in projective spaces [Dol97] and the Plücker formulas [GH78], and the periodic Toda systems are related to harmonic maps [Gue97].

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From the analytic point of view, one would like to study the following mean field equation on a compact surface $M$ with a Riemannian metric $g$

$$\Delta_g u + \rho \left( \frac{he^u}{\int_M he^u} - \frac{1}{|M|} \right) = 4\pi N \sum_{j=1}^N \gamma_j \left( \delta_{p_j} - \frac{1}{|M|} \right) \quad \text{on } M,$$

where $h$ is a positive smooth function on $M$ and $|M|$ is the volume of $M$ with respect to $g$. This equation again arises from both conformal change of metrics [KW74, Tro91] with prescribed Gaussian curvature and the Chern-Simons-Higgs theory on the compact surface $M$. There are intense interests and extensive literature on (1.3) concerning solvability, blow-up analysis and topological degrees [Lin14, Mal14, Tar10, CL03].

In general, we are interested in the following mean field equations of Toda type

$$\Delta_g u_i + \rho \sum_{j=1}^n a_{ij} \rho_j \left( \frac{h_j e^{u_j}}{\int_M h_j e^{u_j}} - \frac{1}{|M|} \right) = 4\pi N \sum_{j=1}^N \gamma_{ij} \left( \delta_{p_j} - \frac{1}{|M|} \right) \quad \text{on } M,$$

Such systems have been studied in [JW01, JLW06] for Lie algebras of type $A$. When carrying out the analysis of such systems, there often appears a sequence of bubbling solutions near blow-up points. For that purpose, the fundamental question is to completely classify all entire solutions of the Toda system with finite energy and with singular sources at the origin as in (1.1).

The classification problem for the solutions to the Toda systems has a long history. For the Liouville equation (1.2), Chen and Li [CL91] classified their solutions without the singular source, and Prajapat and Tarantello [PT01] completed the classification with the singular source. For general $A_n = sl_{n+1}$ Toda systems, Jost and Wang [JW02] classified the solutions without singular sources, and Ye and two of the authors [LWY12] completed the classification with singular sources. This later work also invented the method of characterizing the solutions by a complex ODE involving the $W$-invariants of the Toda system. The work [LWY12] has also established the corresponding quantization result for the integrals and the non-degeneracy result for the corresponding linearized systems. The case of $G_2$ Toda system was treated in [Ale15]. In [Nie16], one of us generalized the classification to Toda systems of types $B$ and $C$ by treating them as reductions of type $A$ with symmetries and by applying the results from [Nie12].

In this paper, we complete the classification of solutions to Toda systems for all types of simple Lie algebras, and we also establish the quantization result for the corresponding integrals. We note that the remaining types of Toda systems can not be treated as reductions of type $A$, and a genuinely new method is needed for our purpose. We are able to achieve our goal by systematically applying and further developing the structure theories of local solutions to Toda systems similarly to [LS92, GL14] and of the $W$-invariants as in [Nie14]. We furthermore note that it is the finite energy condition and the strength of the singularities that combine to greatly restrict the form of the solutions. The current work will lay the foundation for future applications to the Chern-Simons-Higgs theory and to the mean field equations.

Our approach of solving (1.1) will heavily use the complex coordinates and holomorphic functions. Let $x = (x_1, x_2)$ be the coordinates on $\mathbb{R}^2$, and we introduce the complex coordinates $z = x_1 + i x_2$ and $\bar{z} = x_1 - i x_2$. Therefore we identify $\mathbb{R}^2$ with
the complex plane $\mathbb{C}$. For simplicity, we write $\bar{\partial}_z = \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y})$, and similarly $\bar{\partial}_\bar{z} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y})$. The Laplace operator is then $\Delta = \bar{\partial}^2 + \bar{\partial}^2 = 4\bar{\partial}_z \partial_{\bar{z}}$.

The coefficient 4 here is responsible for the slightly unconventional coefficient 4 on the left of (1.1), and this coefficient can be easily dealt with (see [Nie16, Remark 1.12]).

Furthermore, our results and proofs are more conveniently presented in a different set of dependent variables. Let $u_i = \sum_{j=1}^n a_{ij} U_j$. Then the $U_i$ satisfy

\begin{equation}
\begin{cases}
U_{i,z\bar{z}} + \exp\left(\sum_{j=1}^n a_{ij} U_j\right) = \pi \gamma^i \delta_0 & \text{on } \mathbb{R}^2, \\
\int_{\mathbb{R}^2} e^{\sum_{j=1}^n a_{ij} U_j} \, dx < \infty,
\end{cases}
\end{equation}

where $\gamma^i = \sum_{j=1}^n a^{ij} \gamma_j$ and $(a^{ij})$ is the inverse matrix of $(a_{ij})$. The first equation can also be written as

\begin{equation}
\Delta U_i + 4e^{h_i} = 4\pi \gamma^i \delta_0 \quad \text{on } \mathbb{R}^2.
\end{equation}

(The $\gamma^i$ were denoted by $\alpha_i$ in [LWY12,Nie16], but we will use $\alpha_i$ to denote the $i$th simple root of the Lie algebra $\mathfrak{g}$ in this paper.)

Throughout the paper, we use the Lie-theoretic setup detailed in the Appendix. Although for the Lie algebras of classical types $A,B,C,D$, the setup can be made fairly concrete, we have chosen to present our result in a Lie-theoretic and intrinsic way, which automatically covers the Lie algebras of exceptional types $G_2, F_4, E_6, E_7$ and $E_8$. Here is our main theorem.

**Theorem 1.6.** Let $G$ be a connected complex Lie group whose Lie algebra is $\mathfrak{g}$ with the Cartan matrix $(a_{ij})$. Let $G = KAN$ be the Iwasawa decomposition of $G$ (see Eq. (A.7)) with $K$ maximally compact, $A$ abelian, and $N$ nilpotent. Let $N_\Gamma$ be the subgroup of $N$ (see Definition 6.2) determined by the $\gamma_i$.

Let $\Phi: \mathbb{C}\backslash \mathbb{R}_{\leq 0} \to N \subset G$ be the unique solution of

\begin{equation}
\begin{cases}
\Phi^{-1} \Phi_z = \sum_{i=1}^n z^{\gamma_i} e_{-\alpha_i} & \text{on } \mathbb{C}\backslash \mathbb{R}_{\leq 0}, \\
\lim_{z \to 0} \Phi(z) = \text{Id},
\end{cases}
\end{equation}

where $\text{Id} \in G$ is the identity element, the limit exists because $\gamma_i > -1$, and the root vectors $e_{-\alpha_i} \in \mathfrak{g}_{-\alpha_i}$ are normalized as in (A.3) and (A.5).

Then all the solutions to (1.4) are

\begin{equation}
u_i = -\log(|i|\Phi^* C^* \Lambda \Phi^* |i|) + 2\gamma_i \log |z|, \quad 1 \leq i \leq n,
\end{equation}

where $C \in N_\Gamma$ and $\Lambda \in A$. Here for $g \in G$, $g^* = (g^\theta)^{-1}$ and $\theta$ is the Cartan involution of $G$, and $|i| \cdot |i|$ is the highest matrix coefficient for the $i$th fundamental representation (see the Appendix).

Consequently, all the solutions to (1.1) are

\begin{equation}
\begin{cases}
u_i = -\sum_{j=1}^n a_{ij} \log(|j|\Phi^* C^* \Lambda \Phi^* |j|) + 2\gamma_i \log |z|, \quad 1 \leq i \leq n,
\end{cases}
\end{equation}
and they satisfy the following quantization result for the integrals (see (7.22))

\[
\sum_{j=1}^{n} \alpha_{ij} \int_{\mathbb{R}^2} e^{u_j} \, dx = \pi(2 + \gamma_i - \kappa \gamma_i), \quad 1 \leq i \leq n, 
\]

where \(\kappa\) is the longest element in the Weyl group and if \(-\kappa \alpha_i = \alpha_k\), then \(-\kappa \gamma_i := \gamma_k\).

Sections 2 to 7 are devoted to the proof of this main theorem, and the approach can be summarized as follows. In Sections 2 and 3, we develop the structure theories of the local solutions and of the \(W\)-invariants for the Toda system, and we relate them. In Sections 4 and 5, we use the finite energy condition and the strength of the singularities to greatly restrict the forms of the \(W\)-invariants and hence of the solutions. In Section 6, we take up the monodromy consideration for the solutions to be well-defined on the punctured plane. Finally in Section 7, we study the quantization result for the finite integrals by establishing the close relationship of our current work with that of Kostant [Kos79] on Toda ODE systems.

Consistently with the results in [LWY12, Nie16] for the Lie algebras of types \(A, B\) and \(C\), the solution space to the Toda system is parametrized by the subgroup \(A_N\Gamma\) of a corresponding complex Lie group \(G\). When all the \(\gamma_i\) are integers, \(A_N\Gamma = A_N\) (see Definition 6.2), and the solution space has the maximal dimension. Since \(N\) is a complex group, the real dimension of \(A_N\) is the same as the real dimension of a real group corresponding to the real Lie algebra \(g_0\) (A.3).

For an element \(g\) in a classical Lie group \(G\), we have that \(g^* = \bar{g}^t\) is the conjugate transpose. The abelian subgroup \(A\) can be chosen to consist of diagonal matrices with positive real entries, and the nilpotent subgroup \(N\) can be chosen to consist of unipotent lower-triangular matrices with complex entries. The fundamental representations are contained in the wedge products of the standard representations together with the spin representations for the \(B\) and \(D\) cases. In Section 8 of the paper, we will relate the above general theorem 1.6 to the previous results [LWY12, Nie16] in the \(A, B\) and \(C\) cases, and we will spell out more details for our general theorem in the \(D\) case.

The nondegeneracy of the linearized system in this general setting of the Toda system (1.1) for a simple Lie algebra will be pursued in a future work.

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2. LOCAL SOLUTIONS FROM HOLOMORPHIC FUNCTIONS

In this section, we show that the solutions to the Toda system (1.4) for a general simple Lie algebra locally all come from holomorphic data, which are generalizations of the developing map in the Liouville case. Our approach follows [LS79], [LS92 §3.1, §4.1], and [GL14, Appendix]. Since we only consider real-valued solutions to the Toda systems, we only need the holomorphic data of [LS79] and the compact real form (unitary structure) comes into play. In [GL14], similar results were obtained for the periodic Toda systems and the loop groups.
Therefore this section generalizes the well-known fact that locally the solutions to the Liouville equation
\begin{equation}
U_{z\bar{z}} = -e^{2U} \quad \text{on an open set } D \subset \mathbb{C}
\end{equation}
are
\begin{equation}
U(z) = \log \frac{|f'|}{1 + |f|^2},
\end{equation}
where \( f \) is a holomorphic function on \( D \) whose derivative is nowhere vanishing.

For simplicity, we introduce the notation \( \mathbb{C}^* = \mathbb{C}\setminus\{0\} \) and also recall that \( g_{-1} = \bigoplus_{i=1}^n g_{-\alpha_i} \), as in [A.14]. We use the terminology that a domain is a connected open set in \( \mathbb{C} \).

**Theorem 2.3.** Let \( \{U_i\} \) be a set of solutions to the Toda system (1.4). Then there exists a domain \( D \subset \mathbb{C}\setminus\mathbb{R}_{\leq 0} \) containing 1 and a holomorphic map
\begin{equation}
\eta : D \to g_{-1}; \quad \eta(z) = \sum_{i=1}^n f_i(z) e^{-\alpha_i},
\end{equation}
where the \( f_i \) are holomorphic and nowhere zero on \( D \), such that
\begin{equation}
U_i = -\log \langle i|L^*L|i \rangle + \sum_{j=1}^n a_{ij} \log |f_j|^2,
\end{equation}
where \( L : D \to N \subset G \) satisfies
\begin{equation}
L^{-1}L_z = \eta, \quad L(1) = \text{Id}.
\end{equation}
Here \( L^* = (L^0)^{-1} \), and \( \langle i| : |i \rangle \) is the highest matrix coefficient between the \( i \)th fundamental representation (see the Appendix).

On the other hand, a holomorphic map \( \eta = \sum_{i=1}^n f_i e^{-\alpha_i} : D \to g_{-1} \) as in (2.4) on a simply connected domain \( D \) where the \( f_i \) are nowhere zero gives rise to a set of solutions \( \{U_i\} \) to the Toda system on \( D \).

**Proof.** Using the notation and the setup in the Appendix, Eq. (1.4) has the following zero-curvature equation on \( \mathbb{C}^* \)
\begin{equation}
[\partial_z + A, \partial_{\bar{z}} + A^\theta] = 0, \quad \text{that is},
\end{equation}
\begin{equation}
-A_{\bar{z}} + (A^\theta)_z + [A, A^\theta] = 0, \quad \text{where}
\end{equation}
\begin{equation}
A = -\sum_{i=1}^n \frac{1}{2} U_{i,z} h_{\alpha_i} + \sum_{i=1}^n \exp \left( \frac{1}{2} \sum_{j=1}^n a_{ij} U_j \right) e^{-\alpha_i},
\end{equation}
\begin{equation}
A^\theta = \sum_{i=1}^n \frac{1}{2} U_{i,\bar{z}} h_{\alpha_i} - \sum_{i=1}^n \exp \left( \frac{1}{2} \sum_{j=1}^n a_{ij} U_j \right) e_{\alpha_i}.
\end{equation}

The zero-curvature equation can also be written as the Maurer-Cartan equation
\[ d\omega + \frac{1}{2}[\omega, \omega] = 0 \]
for the following Lie algebra valued differential form
\[ \omega = A dz + A^\theta d\bar{z} \in \Omega^1(\mathbb{C}^*, g). \]
With \( dz = dx_1 + idx_2 \) and \( d\bar{z} = dx_1 - idx_2 \), it is also
\[ \omega = (A + A^\theta)dx_1 + i(A - A^\theta)dx_2. \]
Since the Cartan involution \( \theta \) on \( \mathfrak{g} \) is conjugate linear, we see that \( \omega \) takes value in the fixed subalgebra \( \mathfrak{g}^\theta = \mathfrak{k} \) (see the Appendix Subsection A.3). Therefore by [Sha97, Theorems 6.1 and 7.14], there exists a map on the simply connected domain \( F : \mathbb{C}\backslash \mathbb{R}_{\leq 0} \to K \subset G \) to the compact subgroup \( K = G^\theta \) such that

\[
\begin{aligned}
F^{-1}dF &= \omega \\
F(1) &= \text{Id}.
\end{aligned}
\]

Therefore,

\[
F^{-1}F_z = A, \quad F^{-1}F\bar{z} = A^\theta.
\]

The \( F \) has the following Gauss decomposition (A.2) in a domain \( 1 \in D \subset \mathbb{C}\backslash \mathbb{R}_{\leq 0} \)

\[
F = LM \exp(H)
\]

where \( L : D \to N = N_- \) takes value in the negative nilpotent subgroup, and \( M : D \to N_+ \) takes value in the positive nilpotent subgroup. Furthermore \( H = \sum_{i=1}^n b_i h_{\alpha_i} : D \to \mathfrak{h} \) takes value in the Cartan subalgebra, and \( \exp : \mathfrak{h} \to \mathcal{K} \) is the exponential map to the Cartan subgroup. From \( F(1) = \text{Id} \) in (2.10), we see clearly that \( L(1) = \text{Id} \).

Now we show that \( L \) is holomorphic on \( D \). By the second equation in (2.11), we have

\[
\exp(-H)M^{-1}(L^{-1}L_z)M \exp(H) + \exp(-H)M^{-1}M_z \exp(H) + H_z = A^\theta.
\]

In view of (2.9), the components in \( \mathfrak{n}_- = \oplus_{\alpha \in \Delta} \mathfrak{g}_{-\alpha} \) (see (A.1)) of the above equation give

\[
L^{-1}L_z = 0,
\]

and the components in \( \mathfrak{h} \) give

\[
b_{i, \bar{z}} = \frac{1}{2} U_{i, \bar{z}}, \quad 1 \leq i \leq n.
\]

Thus we see that \( b_{i, \bar{z}} = \frac{1}{2} U_{i, \bar{z}}. \) Taking the conjugate, we also have \( \bar{b}_{i, z\bar{z}} = \frac{1}{2} U_{i, z\bar{z}} \) since \( U_i \) is real. Therefore,

\[
(b_i + \bar{b}_i)z\bar{z} = U_{i, z\bar{z}}.
\]

Hence we have, for \( 1 \leq i \leq n \),

\[
b_i + \bar{b}_i = U_i - p_i
\]

for some real-valued harmonic function \( p_i \) on \( D \).

By the first equation in (2.11), we have

\[
\exp(-H)M^{-1}(L^{-1}L_z)M \exp(H) + \exp(-H)M^{-1}M_z \exp(H) + H_z = A.
\]

Since \( A \in \mathfrak{g}_{-1} \oplus \mathfrak{h} \) by (2.8), we see that \( L^{-1}L_z \in \mathfrak{g}_{-1} \). We denote it by \( \eta \) and write it out in terms of the basis

\[
L^{-1}L_z = \eta = \sum_{i=1}^n f_i(z)e_{-\alpha_i}.
\]

Then the \( f_i \) are holomorphic by (2.13). Furthermore, by (2.8) the component of \( A \) in \( \mathfrak{g}_{-1} \) is \( \sum_{i=1}^n \exp \left( \frac{1}{2} \sum_{j=1}^n a_{ij} U_j \right) e_{-\alpha_i} \) where all the coordinates are nowhere zero, so the \( f_i \) are nowhere zero on \( D \). Thus we have shown (2.6) and (2.4).
Now following the physicists, we denote by $|i\rangle$ a highest weight vector in the $i$th fundamental representation of $G$, and $\langle i|a lowest weight vector in its dual right representation such that $\langle i|Id|i\rangle = 1$ (see the Appendix Subsection A.8).

From (2.12), we have

$$L = F \exp(-H)M^{-1}.$$  

Therefore using the $\ast$ operation from the Appendix Subsection A.4, we have

$$\langle i|L^*L|i\rangle = \langle i|(M^{-1})^* \exp(-H)F^*F \exp(-H)M^{-1}|i\rangle$$

where we have used the following facts. First, by $F \in K$ we have $F^*F = Id$. Secondly, since $M^{-1} \in N_+$ and $|i\rangle$ is a highest weight vector, we have $M^{-1}|i\rangle = |i\rangle$. Similarly, $(M^{-1})^* \in N_-$ and $\langle i|\langle i|(M^{-1})^* = \langle i|$. Finally, we have $h_{\alpha_i}|i\rangle = \delta_{\alpha_i}|i\rangle$ (see Eq. (A.16)). Eq. (2.16) actually shows that $\langle i|L^*L|i\rangle$ is real, and this also follows from the Appendix Subsection A.9. Therefore by (2.14),

$$U_i = -\log \langle i|L^*L|i\rangle + p_i.$$  

Now we show that for the above $U_i$ to satisfy (1.4) with $L$ in (2.15), we must have

$$p_i = \sum_{j=1}^{n} a^j \log |f_j(z)|^2.$$  

This follows from [LS92, §4.1.2] using the Jacobi identity from [LS92, §1.6.4], which is more rigorously proved in [Nie15]. The identity says that for a general element $g \in G^*$, the simply connected Lie group with Lie algebra $\mathfrak{g}$ (see the Appendix Subsection A.8), we have

$$\langle i|g|i\rangle \langle i|g^{-1}e^{-\alpha_i}|i\rangle - \langle i|g^{-1}e^{-\alpha_i}|i\rangle \langle i|e_{\alpha_i}g|i\rangle = \prod_{j \neq i} \langle j|g|j\rangle^{-a_{ij}}.$$  

From (2.17) and that $p_i$ is harmonic, we have

$$U_{i,z\bar{z}} = \frac{-\langle i|L^*L|i\rangle \langle i|L^*L|i\rangle_{z\bar{z}} - \langle i|L^*L|i\rangle_{\bar{z}} \langle i|L^*L|i\rangle_{z}}{\langle i|L^*L|i\rangle^2}.$$  

Now by (2.13) and (2.15), we have

$$\langle i|L^*L|i\rangle_{z\bar{z}} = \langle i|L^*L\eta|i\rangle = f_i(z) \langle i|L^*Le^{-\alpha_i}|i\rangle,$$

where we have used that for the $i$th fundamental representation we have $e_{-\alpha_i}|i\rangle = 0$ for $j \neq i$ (see Eq. (A.16)). Taking the $\ast$ operation and noting (A.6), (2.15) also gives

$$(L^*)_{\bar{z}}(L^*)^{-1} = \eta^* = \sum_{i=1}^{n} f_i(z)e_{\alpha_i}.$$  

Therefore, similarly we have

$$\langle i|L^*L|i\rangle_{z\bar{z}} = \langle i|\eta^*L^*L|i\rangle = \overline{f_i(z)} \langle i|e_{\alpha_i}L^*L|i\rangle.$$  

Furthermore, we have

$$\langle i|L^*L|i\rangle_{z\bar{z}} = |f_i|^2 \langle i|e_{\alpha_i}L^*Le^{-\alpha_i}|i\rangle.$$
Now applying the Jacobi identity \((2.18)\) to \((2.19)\) with \(g = L^* L\) gives

\[
U_{i,z\bar{z}} = -|f_i|^2 \prod_{j=1}^{n} \langle j | L^* L | j \rangle^{-a_{ij}}
\]

by \(a_{ii} = 2\). By \((2.17)\), this is

\[
U_{i,z\bar{z}} = -|f_i|^2 \exp \left( \sum_{j=1}^{n} a_{ij} U_j - \sum_{j=1}^{n} a_{ij} p_j \right)
\]

\[
= -\exp \left( \log |f_i|^2 - \sum_{j=1}^{n} a_{ij} p_j \right) \exp \left( \sum_{j=1}^{n} a_{ij} U_j \right).
\]

Therefore for the \(U_i\) to satisfy \((1.4)\), we need \(\log |f_i|^2 - \sum_{j=1}^{n} a_{ij} p_j = 0\) which gives

\[
p_i = \sum_{j=1}^{n} a_{ij} \log |f_j|^2.
\]

This proves the formula \((2.5)\).

Now given a holomorphic map \(\eta = \sum_{i=1}^{n} f_i e^{-\alpha_i} : D \to \mathfrak{g}_{-1}\) as in \((2.4)\) on a simply connected domain \(D\) where the \(f_i\) are nowhere zero, we can construct a \(L : D \to G\), which solves \(L^{-1} L_z = \eta\) as in \((2.6)\) but may not necessarily satisfy the condition \(L(1) = \text{Id}\). Construct the \(U_i\) as in \((2.6)\), and they are checked to satisfy the Toda system \((1.4)\) in the same way as above, where the important point is again the Jacobi identity \((2.18)\).

3. \(W\)-Invariants of the Toda systems

In this section, we first present the algebraic theories of \(W\)-invariants of Toda systems as developed in [FF96, Nie14]. Then we present a result relating the \(W\)-invariants with the local solutions from the last section.

By definition, a \(W\)-invariant (also called a characteristic integral) for the Toda system \((1.4)\) is a polynomial in the \(\partial_k^U U_i\) for \(k \geq 1\) and \(1 \leq i \leq n\) whose derivative with respect to \(\bar{z}\) is zero if the \(U_i\) are solutions.

For example, for the Liouville equation \((2.1)\),

\[
W = U_{zz} - U_z^2
\]

is a \(W\)-invariant since \(W_{\bar{z}} = 0\) for a solution \(U\). Furthermore, plugging in the local solution \((2.2)\), we have that

\[
W = \frac{1}{2} \left( f''' - \frac{3}{2} \left( \frac{f''}{f'} \right)^2 \right),
\]

that is, the \(W\)-invariant of the local solution becomes one half of the Schwarzian derivative of the function \(f\). We aim to generalize such results to general Toda systems in this section.

For a general Toda system associated to a simple Lie algebra of rank \(n\), there are \(n\) basic \(W\)-invariants (see [FF96]) so that the other \(W\)-invariants are differential polynomials in these. One of us in [Nie14] has given a concrete construction and a direct proof of the basic \(W\)-invariants \(W_j\) for \(1 \leq j \leq n\). They are obtained by conjugating one side of the following zero-curvature equation of the Toda system to its Drinfeld-Sokolov gauge [DS84], which is in turn related to a Kostant slice of the corresponding Lie algebra [Kos63].
The Toda system \([1.4]\) on \(\mathbb{C}^*\) is also expressed by the following zero-curvature equation

\[
-\partial_z + \epsilon + \sum_{i=1}^n U_{i,z} h_{\alpha_i} - \sum_{i=1}^n e^{\alpha_i} \epsilon = 0,
\]

where \(\epsilon = \sum_{i=1}^n e_{-\alpha_i} \in \mathfrak{g}_{-1}\), and \(u_i = \sum_{j=1}^n a_{ij} U_j\). (This zero-curvature equation is related to but different from the one in \((2.7)\), and the current one is not invariant under the Cartan involution.) Let \(s\) be a Kostant slice of \(\mathfrak{g}\), that is, a homogeneous subspace \(s\) with respect to the principal grading \((A.13)\) such that \(\mathfrak{g} = [\epsilon, \mathfrak{g}] \oplus s\). Then it is known \([\text{Kos63}]\) that \(\dim s = n\). Let \(\{s_j\}_{j=1}^n\) be a homogeneous basis of \(s\) ordered with nondecreasing gradings by \((A.13)\).

**Theorem 3.3** \([\text{Nie14}]\). There exists a unique \(M_0 \in N_+\), an element in the positive nilpotent subgroup whose coefficients depend on the derivatives \(\partial_z U_i\) for \(1 \leq i \leq n\) and \(k \geq 1\), such that

\[
M_0 \left( -\partial_z + \epsilon + \sum_{i=1}^n U_{i,z} h_{\alpha_i} \right) M_0^{-1} = -\partial_z + \epsilon + \sum_{j=1}^n W_j s_j.
\]

Then the \(W_j\) for \(1 \leq j \leq n\) are the basic \(W\)-invariants of the Toda system.

The uniqueness of \(M_0\) is easily proved by induction on the grading \((A.13)\) and the fact from \([\text{Kos59}]\) that \(\ker \text{ad}_\epsilon \cap \mathfrak{b}_+ = 0\).

The right hand side of \((3.4)\) is said to be in the Drinfeld-Sokolov gauge \([\text{DS84}]\).

Here is the main result in this section which relates the \(W\)-invariants with the holomorphic data \(\eta\) in \((2.4)\) for local solutions. It generalizes the relation \((3.2)\) to a general Lie algebra.

**Theorem 3.6.** For the local solutions \((2.5)\) on a simply connected domain \(D\), there exists a \(P_1 \in B_+\), an element in the Borel subgroup whose coefficients depend on the derivatives \(\partial_z^k f_i\) for \(1 \leq i \leq n\) and \(k \geq 0\), such that

\[
P_1 \left( -\partial_z + \sum_{i=1}^n f_i(z) e_{-\alpha_i} \right) P_1^{-1} = -\partial_z + \epsilon + \sum_{j=1}^n W_j s_j,
\]

where the \(W_j\) are the \(W\)-invariants computed by \((3.4)\) for the local solutions \((2.5)\).

Furthermore \(P_1\) is unique up to the finite center of \(G\) and is holomorphic on \(D\).

**Proof.** First we show that for the local solutions \((2.5)\), the \(W\)-invariants are also computed by

\[
M_1 \left( -\partial_z + \epsilon + \sum_{i=1}^n F_i h_{\alpha_i} \right) M_1^{-1} = -\partial_z + \epsilon + \sum_{j=1}^n W_j s_j,
\]

where \(M_1 \in N_+\), and

\[
F_i = \sum_{j=1}^n a^{ij} \partial_z \log f_j = \sum_{j=1}^n a^{ij} \frac{f'_j}{f_j}
\]

is the part of \(U_{i,z}\) in \((3.4)\) using just the summand \(p_i = \sum_{j=1}^n a^{ij} \log |f_j|^2\) in \((2.5)\). We note that the \(\log\)'s are well-defined since \(D\) is simply connected.
If we do not impose that the solutions \( U_i \) in (2.5) are real-valued, then instead of (2.20) we can solve for \( T : D \to G \) such that
\[
\begin{aligned}
T(1) &= Id, \\
TzT^{-1} &= \sum_{i=1}^{n} g_i(\bar{z})e_{\alpha_i},
\end{aligned}
\]
for \( n \) anti-holomorphic functions \( g_1(\bar{z}), \ldots, g_n(\bar{z}) \). Then in analogy to the local solutions (2.5), all the
\[
U_i^T = -\log\langle i|T(\bar{z})L(z)|i \rangle + \sum_{j=1}^{n} a^{ij} \log(f_j(z)g_j(\bar{z})), \quad 1 \leq i \leq n,
\]
are solutions to the Toda system (1.4) on \( D \). Clearly the \( U_i \) in (2.5) corresponds to \( T = L^* \), that is, \( U_iL^* = U_i \) in (2.5).

Now the \( W \)-invariants \( W_j^T \) of the solutions \( U_i^T \) satisfy \( \partial_z W_j^T = 0 \). Since the \( W_j^T \) are polynomials in the \( \partial^k U_i^T \) for \( k \geq 1 \), we see that when computing them, we can in place of the \( U_i^T \) use
\[
\tilde{U}_i^T = -\log\langle i|T(\bar{z})L(z)|i \rangle + \sum_{j=1}^{n} a^{ij} \log(f_j(z)),
\]
that is, we can discard the anti-holomorphic part \( \sum_{j=1}^{n} a^{ij} \log g_j(\bar{z}) \).

Now since \( \partial_z W_j^T = 0 \), we see that the \( W_j^T \) are independent of the \( T(\bar{z}) \) in (3.10). Therefore the \( W_jL^* = W_j \) in (3.4) is that same as the \( W_jId \) using the simplest \( T(\bar{z}) = Id \). Then
\[
\tilde{U}_i^{Id} = \sum_{j=1}^{n} a^{ij} \log f_j(z)
\]
since \( \langle i|Id \cdot L(z) = \langle i|L(z) = \langle i \rangle \) by \( L(z) \in N_\) and hence \( \langle i|Id \cdot L(z)|i \rangle = 1 \). Replacing the \( U_i \) in (3.4) by the above \( U_i^{Id} \) proves (3.8) since \( U_i^{Id} = F_i \) in (3.9).

Now we show that on the simply connected \( D \) there is a conjugation to transform
\(-\partial_z + \sum_{i=1}^{n} f_i e_{-\alpha_i} \) to \( -\partial_z + \epsilon + \sum_{i=1}^{n} F_i h_{\alpha_i} \). We choose
\[
Q_1 = \exp \left( \sum_{k=1}^{n} \left( \sum_{j=1}^{n} a^{kj} \log f_j \right) h_{\alpha_k} \right) \in G.
\]
Since \( [h_{\alpha_k}, e_{-\alpha_i}] = a_{ik} e_{-\alpha_i} \), by (A.10), we have
\[
\text{Ad}_{Q_1} e_{-\alpha_i} = \exp \left( -\sum_{k=1}^{n} \left( \sum_{j=1}^{n} a^{kj} \log f_j \right) a_{ik} \right) e_{-\alpha_i} = \exp \left( -\log f_i \right) e_{-\alpha_i} = \frac{1}{f_i} e_{-\alpha_i}.
\]
It is also clear that
\(-Q_1 \partial_z Q_1^{-1} = \partial_z Q_1 Q_1^{-1} = \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a^{ij} \partial_z \log f_j \right) h_{\alpha_i} = \sum_{i=1}^{n} F_i h_{\alpha_i}.
\]
Therefore we have
\[
Q_1 \left( -\partial_z + \sum_{i=1}^{n} f_i e_{-\alpha_i} \right) Q_1^{-1} = -\partial_z - Q_1 \partial_z Q_1^{-1} + \sum_{i=1}^{n} f_i \frac{1}{f_i} e_{-\alpha_i},
\]
(3.12)
\[
= -\partial_z + \sum_{i=1}^{n} F_i h_{\alpha_i} + \epsilon.
\]
Then Eq. (3.7) is proved with $P_1 = M_1 Q_1 \in B_+$. Again $P_1$ is unique up to the finite center of $G$ by (3.5). Therefore $P_1$ is holomorphic since both the $f_i$ and the $W_j$ in (3.7) are holomorphic. □

Remark 3.13. Consider a curve $L : D \to N_-$ on a simply connected domain such that $L^{-1} L_z = \eta = \sum_{i=1}^{n} f_i(z) e^{-a_i}$. Such a curve is called an integral curve of the standard differential system on $N_- \subset G/B_+$. The above proof shows that there exists a $P_0$ in $B_+$ unique up to the center of $G$ such that

$$P_0(-\partial_z + \eta) P_0^{-1} = -\partial_z + \sum_{j=1}^{n} I_j s_j,$$

It can be shown by the method of moving frames [MB08] that the $I_j$ are the differential invariants for the curve $L(z)$ under the natural action of $G$ on $N_- \subset G/B_+$.

These are the natural generalizations of the Schwarzian derivative for the group $SL_2 \mathbb{C}$ to other simple Lie groups. See also [DZ13].

4. Use the finite energy condition

In this section, we adapt the analytical estimates from [BM91, LWY12] using the finite energy condition and the strength of the singularities to determine the simple forms of the $W$-invariants, which will be shown to further restrict the solutions in the next section.

For a differential monomial in the $U_i$, we call by its degree the sum of the orders of differentiation multiplied by the algebraic degrees of the corresponding factors. For example the above $W = U_{zz} - U_z^2$ in (3.1) for the Liouville equation has a homogeneous degree 2. It is known from [FF96] and also clear from (3.4) that the $W$-invariants $W_j$ involve the $\partial_k^k U_i$ for $k \geq 1$ and that the homogeneous degree of $W_j$ is the same as the degree $d_j$ of the corresponding primitive adjoint-invariant function of the Lie algebra $g$ [Kos59]. We call such degrees the degrees of the simple Lie algebra and we have listed them in the Appendix Subsection A.11.

Proposition 4.1. The $W$-invariants for the Toda system (1.4) are

$$W_j = \frac{w_j}{z^{d_j}}, \quad z \in \mathbb{C}^*, \quad 1 \leq j \leq n,$$

where the $d_j$ are the degrees of the Lie algebra $g$ and the $w_j$ are constants.

Proof. This proof is an adaptation of the proof in [LWY12, Eq. (5.10)]. Following [LWY12, Eq. (5.10)], introduce

$$V_i = U_i - 2\gamma^i \log |z|, \quad 1 \leq i \leq n.$$

Then system (1.4) becomes

$$\begin{cases}
\Delta V_i = -4|z|^{2\gamma_i} \exp \left( \sum_{j=1}^{n} a_{ij} V_j \right), \\
\int_{\mathbb{R}^2} |z|^{2\gamma_i} \exp \left( \sum_{j=1}^{n} a_{ij} V_j \right) dx < \infty.
\end{cases}$$
As $\gamma_i > -1$, applying Brezis-Merle’s argument in [BM91], we have that $V_i \in C^{0,\alpha}$ on $\mathbb{C}$ for some $\alpha \in (0,1)$ and that they are upper bounded over $\mathbb{C}$. Therefore we can express $V_i$ by the integral representation formula, and we have

$$\partial_k^z V_i(z) = O(1 + |z|^{2+2\gamma_i-k}) \text{ near } 0,$$

$$\partial_k^z V_i(z) = O(|z|^{-k}) \text{ near } \infty, \forall k \geq 1. \quad (4.4)$$

Therefore from (4.3), we have

$$\partial_k^z U_i(z) = O(|z|^{-k}) \text{ near } 0, \forall k \geq 1,$$

$$\partial_k^z U_i(z) = O(|z|^{-k}) \text{ near } \infty, \forall k \geq 1. \quad (4.5)$$

By $W_j, \bar{z} = 0$ and that $W_j$ has degree $d_j$, we see from the above estimates that $z^{d_j} W_j$ is holomorphic and bounded on $\mathbb{C}^*$. Therefore $z^{d_j} W_j = w_j$ is a constant by the Liouville theorem, and so (4.2) holds.

**Theorem 4.6.** The $W$-invariants for the Toda systems (1.4) are also computed by

$$P_2 \left( - \partial_z + \sum_{i=1}^{n} z^{\gamma_i} e^{-\alpha_i} \right) P_2^{-1} = - \partial_z + \epsilon + \sum_{j=1}^{n} W_j s_j, \quad (4.7)$$

where $P_2 : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \to B_+$ is holomorphic and unique up to the finite center of $G$.

*Proof.* First we show that the $W$-invariants for the Toda system (1.4) are also computed by

$$M_2 \left( - \partial_z + \epsilon + \sum_{i=1}^{n} \frac{\gamma_i}{z} h_{\alpha_i} \right) M_2^{-1} = - \partial_z + \epsilon + \sum_{j=1}^{n} W_j s_j, \quad (4.8)$$

where $M_2 \in N_+$.

The $W_j$ are polynomials in the $\partial_k^z U_i$ for $k \geq 1$, and $W_j = \frac{w_j}{z^{d_j}}$ by (4.2). Since $\gamma_i > -1$, by (4.4) and (4.5) for the orders at 0, we see that all the terms involving $\partial_k^z V_i$ will not appear in the final form of $W_j$ since their orders of pole are not high enough. Therefore the $W_j$ can also be computed using just the $2\gamma_i \log |z| = \gamma_i \log z + \gamma_i \log \bar{z}$ component of $U_i$ in (4.3). Then the term $U_{i,z}$ in (4.4) can be replaced by $\frac{\gamma_i}{z}$, and we get (4.8).

Next we use again the conjugation in (3.11) in our situation, and let

$$Q_2 = \exp \left( \sum_{k=1}^{n} \log z^{\gamma_i} h_{\alpha_k} \right).$$

Then by (3.12) or by direct computation, we have

$$Q_2 \left( - \partial_z + \sum_{i=1}^{n} z^{\gamma_i} e^{-\alpha_i} \right) Q_2^{-1} = - \partial_z + \epsilon + \sum_{i=1}^{n} \frac{\gamma_i}{z} h_{\alpha_i}. \quad (4.9)$$

Then (4.7) is proved with $P_2 = M_2 Q_2 \in B_+$. Similarly to Theorem 3.6, $P_2$ is unique up to the finite center of $G$ and is holomorphic. \qed
5. The holomorphic functions in the local solutions

The $W$-invariants play essential roles in our approach to classify the solutions. The work [LWY12] classified the solutions to the Toda systems of type $A$ by relating them to an ODE whose coefficients are the $W$-invariants. In this section, we will use the $W$-invariants to largely restrict the holomorphic functions $f_i(z)$ \([2.4]\) in the local solutions \([2.5]\) to be $f_i(z) = z^{\gamma_i}$ as long as we allow some constant group element.

**Theorem 5.1.** The local solutions $U_i$ \([2.5]\) on a simply connected domain $D \subset \mathbb{C}\backslash \mathbb{R}_{\leq 0}$ that are solutions to \([1.4]\) have the following form
\[
U_i = -\log(|\Phi^*g^*g\Phi|) + 2\gamma_i \log|z|, \quad 1 \leq i \leq n,
\]
where $\Phi$ satisfies \([1.7]\) and $g \in G$ is a constant group element.

**Proof.** We introduce the notation
\[
\zeta = \sum_{i=1}^n z^{\gamma_i} e^{-\alpha_i},
\]
and then \([1.7]\) becomes $\Phi^{-1}\Phi z = \zeta$.

Consider $s = \exp(\pi i E_0)$ where $E_0$ is defined in \([A.12]\) such that $\alpha_i(E_0) = 1$ for $1 \leq i \leq n$. Then $\text{Ad}_s e^{-\alpha_i} = -e^{-\alpha_i}$ and also $\text{Ad}_s e^{-\alpha_i} = -e^{-\alpha_i}$. Actually $s^2$ is in the center of the group $G$. We also note that such an element $s$ plays an important role in Kostant’s approach to the Toda lattices [Kos79 Eq (3.5.5)], where it is called $m$.

By Theorem \([3.6]\) and Theorem \([4.6]\) we see that for $P = s^{-1}P_2^{-1}P_1 s : D \to B_+$ we have
\[
P^{-1}(\partial_z + \zeta)P = s^{-1}P_1^{-1}P_2 s(\partial_z + \zeta)s^{-1}P_2^{-1}P_1 s
= s^{-1}P_1^{-1}P_2(\partial_z - \zeta)P_2^{-1}P_1 s
= -s^{-1}P_1^{-1}\left(-\partial_z + \epsilon + \sum_{j=1}^n W_js_j\right)P_1 s
= s^{-1}(\partial_z - \eta)s = \partial_z + \eta.
\]
Therefore
\[
P^{-1}P_z = P^{-1}\zeta P = \eta.
\]

Now consider the product $\Phi P : D \to G$ from the simply connected domain $D$ to the group $G$. We have
\[
(\Phi P)^{-1}(\Phi P)z = P^{-1}\Phi^{-1}\Phi z P + P^{-1}P_z = P^{-1}\zeta P + P^{-1}P_z = \eta.
\]
Comparison with \([2.6]\) gives, on the basis that both $L$ and $\Phi P$ are holomorphic on $D$, that
\[
L = g\Phi P,
\]
where $g \in G$ is a constant element and actually $g = ((\Phi P)(1))^{-1}$ by the requirement that $L(1) = \text{Id}$ in \([2.6]\).

Now let $P = QM$ be its decomposition with $Q \in \mathfrak{g}$ and $M \in N_+$, and write $Q = \exp\left(\sum_{j=1}^n q_j h_{\alpha_j}\right)$. Then considering the components in $\mathfrak{g}_{-1}$ from \([5.4]\), we have
\[
\eta = Q^{-1}\zeta Q.
By \([-\hbar_{\alpha_j}, e_{-\alpha_i}] = a_{ij} e_{-\alpha_i}\) from (A.10) and from the definitions \((2.4), (5.3)\), we see that

\[
f_i = z^{\gamma_i} e^{\sum_{j=1}^n a_{ij} q_j}.
\]

It follows that

\[
\sum_{j=1}^n a^{ij} \log f_j = \log z^{\gamma_i} + q_i,
\]

and

\[
\sum_{j=1}^n a^{ij} \log |f_j|^2 = 2\gamma_i \log |z| + q_i + \bar{q}_i, \quad 1 \leq i \leq n.
\]

Since \(|i\rangle\) is a highest weight vector of the \(i\)th fundamental representation, we see that

\[
P|\rangle = Q M |\rangle = Q |\rangle = e^{\bar{q}_i} |\rangle,
\]

by \(M \in N_+\) and \(h_{\alpha_j} |\rangle = \delta_{ij} |\rangle\). Taking the \(*\) operation (and using (A.17)), we have

\[
\langle i| P = e^{q_i} |i\rangle.
\]

Therefore by (5.5) and (5.6), the \(U_i\) from (2.5) becomes

\[
U_i = -\log \langle i| L^* L |i\rangle + \sum_{j=1}^n a^{ij} \log |f_j|^2
\]

\[
= -\log \langle i| P^* \Phi^* g^* g \Phi P |i\rangle + \sum_{j=1}^n a^{ij} \log |f_j|^2
\]

\[
= -\log \langle i| \Phi^* g^* g \Phi |i\rangle - (q_i + \bar{q}_i) + \sum_{j=1}^n a^{ij} \log |f_j|^2
\]

\[
= -\log \langle i| \Phi^* g^* g \Phi |i\rangle + 2\gamma_i \log |z|.
\]

\[\square\]

6. The final monodromy consideration

In the local solution (5.2), write the Iwasawa decomposition (A.7)

\[g = FAC\]

with \(F \in K, A \in A\) and \(C \in N\). Then (5.2) becomes (1.8) by \(F^* F = Id\) and \(A^* = A\).

The solutions \(U_i\) in (1.8) are well-defined on \(C \setminus \mathbb{R}_{\leq 0}\) after the branch cut for the functions \(z^{\gamma_i}\). Clearly the \(U_i\) satisfy the Toda system (1.4) by the other direction in Theorem 2.3 and have the right strength of singularities at the origin. Therefore we have shown that all local solutions are at least defined on \(C \setminus \mathbb{R}_{\leq 0}\) and are of the form (1.8).

We now want to show that for the \(U_i\) in (1.8) to be well-defined on \(C^*\), \(C\) needs to belong to a suitable subgroup \(N_\Gamma\) of \(N\). This subgroup was introduced in [Nie16] already in the classification result for Toda systems of types \(B\) and \(C\).

With the \(E_j \in \mathfrak{h}_0\) as in (A.11), consider the following element in the Cartan subgroup

\[
(6.1) \quad t_\Gamma := \exp \left(2\pi i \sum_{j=1}^n \gamma_j E_j\right) = \exp \left(2\pi i \sum_{j=1}^n \gamma^j \hbar_{\alpha_j}\right) \in \mathfrak{H}.
\]

Clearly \(t_\Gamma^{-1} = t_\Gamma^*\).
Definition 6.2. The subgroup $N_{t^\Gamma} \subset N$ is the centralizer of $t^\Gamma$ in $N$, that is,

$$N_{t^\Gamma} = \{ C \in N \mid Ct^\Gamma = t^\Gamma C \}.$$ 

Remark 6.3. Here is a more concrete description of $N_{t^\Gamma}$. For a positive root $\alpha \in \Delta^+$, write $\alpha = \sum_{i=1}^n m_i \alpha_i$ in terms of the simple roots $\{\alpha_i\}_{i=1}^n$. Define the number $\alpha(\Gamma) = \sum_{i=1}^n m_i \gamma_i$ where we replace $\alpha_i$ by $\gamma_i$. Also define the subset $\Delta^T$ of $\Delta^+$ as $\Delta^T = \{ \alpha^T \mid \alpha(T) \in \mathbb{Z} \}$, and the Lie subalgebra $n_{t^\Gamma}$ of $n$ as $n_{t^\Gamma} = \bigoplus_{\alpha^T \in \Delta^T} \mathfrak{g}_{-\alpha^T}$.

Then $N_{t^\Gamma}$ is the subgroup of $N$ corresponding to $n_{t^\Gamma}$. The reason is that clearly $\alpha(\Gamma) = \alpha(\sum_j \gamma_j E_j)$ by (6.4), and so $\text{Ad}_{t^\Gamma} e_{\alpha} = \exp(2\pi i \alpha(\Gamma)) e_{\alpha}$. Hence we see that

(6.4) $n_{t^\Gamma} = n^{\text{Ad}_{t^\Gamma}}$ and $N_{t^\Gamma} = N^{\text{Ad}_{t^\Gamma}}$

are the fixed point sets of the adjoint actions by $t^\Gamma$.

Theorem 6.5. The $U_i$ in (1.8) are well-defined on $\mathbb{C}^*$ if and only if

$$C \in N_{t^\Gamma}.$$ 

Proof. For the $U_i$ in (1.8) to be well-defined on $\mathbb{C}^*$, we need that the $U_i$ are invariant under the change of $z \mapsto e^{-2\pi i z}$, that is, when one travels once (clockwise) around the origin.

By the definition (6.1), we have

$$\sum_{j=1}^n (ze^{2\pi i})^j e_{-\alpha_j} = \sum_{j=1}^n z^{\gamma_j} e^{-2\pi i \gamma_j} e_{-\alpha_j} = \text{Ad}_{t^\Gamma} \left( \sum_{j=1}^n z^{\gamma_j} e_{-\alpha_j} \right).$$

Hence the corresponding solution to

$$\begin{cases} \tilde{\Phi}^{-1} \Phi = -\sum_{j=1}^n (ze^{2\pi i})^j e_{-\alpha_j} & \text{on } \mathbb{C} \setminus \mathbb{R}_{\leq 0}, \\ \tilde{\Phi}(0) = Id, \end{cases}$$

is

$$\tilde{\Phi}(z) = \text{Ad}_{t^\Gamma} \Phi(z) = t^\Gamma \Phi(z) t^{-1}_{\Gamma}.$$ 

Therefore the corresponding solution (1.8) to the Toda system is

$$\tilde{U}_i = -\log \langle i| \tilde{\Phi}^* C^* \Lambda^2 C \tilde{\Phi} |i \rangle + 2 \gamma^i \log |z|$$

$$= -\log \langle i| \tilde{\Phi}^* t_i^\Gamma C^* \Lambda^2 C t_{\Gamma} \Phi |i \rangle + 2 \gamma^i \log |z|,$$

where we have used that $t_i^\Gamma = t_i^{-1}$, $t_{\Gamma}^{-1} |i \rangle = e^{-2\pi i \gamma^i} |i \rangle$, and $\langle i|t_{\Gamma} = e^{2\pi i \gamma^i} |i \rangle$ from (6.1). Note that $\Lambda t_{\Gamma} = t_{\Gamma} \Lambda$ since both belong to the Cartan subgroup $\mathcal{H}$. From $\gamma_i > -1$, it can be shown that the above $\tilde{U}_i$ is equal to the $U_i$ in (1.8) iff

$$Ct_{\Gamma} = t_{\Gamma} C,$$

that is, $C \in N_{t^\Gamma}$. 

\[\square\]
7. Relation with Kostant’s work and quantization

In this section, we obtain an explicit expression for the $\Phi$ in (1.7) inspired by [Kos79] and this allows us to obtain the quantization result for the integrals of our solutions. This section thus establishes a very concrete relationship of our work with that of Kostant for Toda ODEs (see Proposition 7.14).

Following [LWY12], we denote

$$\mu_i = \gamma_i + 1 > 0, \quad 1 \leq i \leq n.$$  

(7.1)

Inspired by [Kos79], we introduce the following notation, using (A.11) and (5.3),

$$w_0 = \sum_{i=1}^{n} \mu_i E_i \in \mathfrak{h},$$  

(7.2)

$$\xi = z\zeta = \sum_{i=1}^{n} z^{\mu_i} e^{-\alpha_i} \in \mathfrak{g}_{-1}.$$  

(7.3)

(It is easy to see from (6.1) that $t^{-1}_{-1} \exp(2i\pi w_0)$ belongs to the center of $G$.) For the concrete expression for $\Phi$ in (1.7), we introduce the following setup after [Kos79].

Let $S$ be the set of all finite sequences

$$s = (i_1, \ldots, i_k), \quad k \geq 0, \quad 1 \leq i_j \leq n.$$  

(7.4)

We write $|s|$ for the length $k$ of the element $s \in S$, and we also write

$$\varphi(s) = \sum_{j=1}^{s} \alpha_{i_j}, \quad \varphi(s, w_0) = \varphi(s)(w_0) = \langle \varphi(s), w_0 \rangle = \sum_{j=1}^{s} \mu_{i_j},$$  

(7.5)

where $\langle \cdot, \cdot \rangle$ is the natural pairing between $\mathfrak{h}'_0$ and $\mathfrak{h}_0$ (see (A.3)). Note that $\varphi(s)$ is equal to the constant function 0 on $\mathfrak{h}$ if $|s| = 0$.

For $0 \leq j \leq |s| - 1$, let $s_j \in S$ be the sequence obtained from $s$ by “cutting off” the first $j$ terms (different from [Kos79])

$$s_j = (i_{j+1}, \ldots, i_{|s|}),$$  

(7.6)

and define

$$p(s, w_0) = \prod_{j=0}^{s-1} \langle \varphi(s_j), w_0 \rangle.$$  

(7.7)

Note then when $|s| = 0$, we have $p(s, w_0) = 1$. Clearly when $|s| \geq 1$, we have

For $s \in S$ as in (7.3), put

$$e_{-s} = e_{-i_k} \cdots e_{-i_2} e_{-i_1}.$$  

(7.8)

We note that the $\xi$ in (7.2) is

$$\xi = \sum_{i=1}^{n} z^{\varphi(i, w_0)} e_{-i},$$  

(7.9)

where the $s$ are the simplest $(i)$ for $1 \leq i \leq n$. 
Proposition 7.10. In the space $\hat{D}(N)$ (see the Appendix Subsection A.8), we have

\begin{equation}
\Phi = \sum_{s \in S} \frac{z^{\varphi(s,w_0)}e^{-s}}{p(s,w_0)}.
\end{equation}

Proof. We denote the right hand side of (7.11) by $\Upsilon$, and we show that it satisfies (1.7) and hence it is equal to $\Phi$. Clearly $\Upsilon(0) = 1$ corresponding to the empty $s$.

We also have

\begin{equation}
\Upsilon z = \sum_{s \in S} \frac{\varphi(s,w_0)z^{\varphi(s,w_0)-1}e^{-s}}{p(s,w_0)} = \frac{1}{z} \sum_{|s| \geq 1} \frac{z^{\varphi(s,w_0)}e^{-s}}{p(s_1,w_0)}
\end{equation}

by (7.7).

Let $s \in S$ and let $\{t^1, \ldots, t^n\}$ be the set of all $t \in S$ such that $t_1 = s$ according to (7.5). Then by (7.9) and (7.4), we have

\begin{equation}
\frac{z^{\varphi(s,w_0)}e^{-s}}{p(s,w_0)} \xi = \sum_{i=1}^n \frac{z^{\varphi(t^i,w_0)}e^{-t^i}}{p(s,w_0)} = \sum_{i=1}^n \frac{z^{\varphi(t^i,w_0)}e^{-t^i}}{p(t^1,w_0)}.
\end{equation}

Therefore we have

\begin{equation}
\Upsilon \xi = \sum_{|s| \geq 1} \frac{z^{\varphi(s,w_0)}e^{-s}}{p(s_1,w_0)}.
\end{equation}

Combining (7.12) and (7.13), we see $\Upsilon z = \frac{1}{z} \Upsilon \xi = \Upsilon \xi$. Therefore $\Upsilon^{-1} \Upsilon z = \zeta$, and (7.11) is proved.

For its own interest, we have the following algebraic characterization of our $\Phi$ in (1.7), where it was originally defined by an ODE.

Proposition 7.14. The unique map $\Psi : \mathbb{C} \setminus \mathbb{R}_{<0} \rightarrow N = N_-$ such that

\begin{equation}
\begin{cases}
\Psi^{-1} w_0 = w := w_0 - \xi, \\
\Psi(0) = Id
\end{cases}
\end{equation}

is $\Psi = \Phi$ from (1.7).

Proof. Our $\Psi$ is modeled after [Kos79], and its existence and uniqueness (up to the center of $G$) follow from Lemma 3.5.2 there. Note that when $z = 0$, $\xi(0) = 0$ and so $\Psi(0)$ can be chosen to be $Id$. This makes the $\Psi$ unique.

Now we show that $\Psi = \Phi$ by verifying that it satisfies the ODE in (1.7). We note that by (7.2) and (A.11), we have

\begin{equation}
w_z = -\sum_{i=1}^n \mu_i z^{\gamma_i} e_{-\alpha_i} = [w_0, \zeta].
\end{equation}

With $\psi := \Psi^{-1} w_z \in n = n_-$, we differentiate (7.15) to get $[w, \psi] = w_z$, which is then

\begin{equation}
[w_0 - z\zeta, \psi] = [w_0, \zeta].
\end{equation}

We now show that this equation implies that $\psi = \zeta$. We write $\psi = \sum_{i \leq -1} \psi_i$ according to the grading (A.13), and we also note that $\zeta \in g_{-1}$. Furthermore by (7.2) and (7.1), we see that

$\ker \text{ad}_{w_0} \cap n_- = 0$. 

The components in $g^{-1}$ of (7.16) give that $[w_0, \psi_{-1}] = [w_0, \zeta]$ and so $\psi_{-1} = \zeta$. The components in $g^{-2}$ of (7.16) give that $[w_0, \psi_{-2}] - [z\zeta, \psi_{-1}] = 0$, and so by $\psi_{-1} = \zeta$ we have $\psi_{-2} = 0$. A quick induction then shows that all $\psi_i = 0$ for $i \leq -3$. Therefore $\psi = \zeta$, and $\Psi = \Phi$ by (1.7).

Remark 7.17. A proof following [Kos79, Prop. 5.8.2] can also be constructed to show directly that $\Psi = \Upsilon$ for the $\Upsilon$ and $\Psi$ in Propositions 7.10 and 7.14.

Let $\lambda$ be a dominant weight, and let $V^\lambda$ be the corresponding irreducible representation. Let $\kappa$ be the longest Weyl group element which maps positive roots to negative roots. Then $\kappa\lambda$ is the lowest weight of $V^\lambda$ (see the Appendix Subsection A.10). Throughout the paper we denote the action of $\kappa$ on $h'_0$ without parentheses.

In the following calculations, we use the Hermitian metric $\{\cdot, \cdot\}$ on $V^\lambda$ which is invariant under the compact subgroup $K_s$ of $G_s$ (see the Appendix Subsection A.9). Choose vectors $v^\lambda \in V^\lambda$ and $v^\kappa\lambda \in V^\kappa\lambda$ in the one-dimensional highest and lowest weight spaces such that
\[
\{v^\lambda, v^\lambda\} = 1, \quad \{v^\kappa\lambda, v^\kappa\lambda\} = 1.
\]
(One can choose $v^\kappa\lambda$ to be $s_0(\kappa)v^\lambda$ where $s_0(\kappa) \in G_s$ induces the longest Weyl element $\kappa$. See [Kos79, Eq. (5.2.10)].)

With the notation (7.4) and (7.8), the vector $e_{-s}v^\lambda$ is a weight vector with weight $-\varphi(s) + \lambda$. Since different weight spaces are orthogonal, by Proposition 7.10 we have
\[
(7.18) \quad \{\Phi v^\lambda, v^\kappa\lambda\} = \left(\sum_{s \in S^\lambda} \frac{c_{s,\lambda}}{p(s, w_0)}\right) z^{(\lambda - \kappa\lambda, w_0)},
\]
where $S^\lambda = \{s \in S | \varphi(s) = \lambda - \kappa\lambda\}$, and $c_{s,\lambda} = \{e_{-s}v^\lambda, v^\kappa\lambda\}$. It is known that coefficient in the parentheses is nonzero by [Kos79, Prop. 5.9.1] (and can be made positive real with suitable choices).

The following theorem is the main result of this section. It provides the asymptotic behavior of our solutions and proves the quantization result for our integrals.

**Theorem 7.19.** The solutions $U_i$ in (1.8) satisfy that
\[
(7.20) \quad U_i = 2(\gamma_i - (\omega_i - \kappa\omega_i, w_0)) \log |z| + O(1), \quad \text{as } z \to \infty,
\]
where $\omega_i$ is the $i$th fundamental weight. Consequently, the solutions $u_i$ in (1.9) satisfy that
\[
(7.21) \quad u_i = -2(2 - \kappa\gamma_i) \log |z| + O(1), \quad \text{as } z \to \infty,
\]
where $-\kappa\omega_i = \alpha_k$ then $-\kappa\gamma_i = \gamma_k$.

The finite integrals in (1.1) are quantized as
\[
(7.22) \quad \sum_{j=1}^n a_{ij} \int_{\mathbb{R}^2} e^{u_j} \, dx = \pi(\mu_i - \kappa\mu_i),
\]
where $-\kappa\mu_i = (-\kappa\omega_i, w_0)$. Therefore we have
\[
(7.23) \quad \sum_{j=1}^n a_{ij} \int_{\mathbb{R}^2} e^{u_j} \, dx = 2\pi(1 + \gamma_i), \quad 1 \leq i \leq n,
\]
except the following cases:
We know that

\[\text{By (1.8) and (A.18), we have}

\tag{7.24}

\sum_{j=1}^{n} a_{ij} \int \mathbb{R}^2 e^{u_j} \, dx = \pi(2 + \gamma_i + \gamma_{n+1-i}), \quad 1 \leq i \leq n.

\]

(ii) \(g = D_{2n+1}\) and \(i = 2n\) or \(2n+1\), we have

\[\text{−} \int \mathbb{R}^2 e^{u_{2n-1}} \, dx + 2 \int \mathbb{R}^2 e^{u_{2n}} \, dx = \pi(2 + \gamma_{2n} + \gamma_{2n+1}),

\tag{7.25}

\]

(iii) \(g = E_6\) and using the labeling in the Appendix Subsection A.5, we have

\[\sum_{j=1}^{6} a_{ij} \int \mathbb{R}^2 e^{u_j} \, dx = \pi(2 + \gamma_1 + \gamma_6), \quad i = 1, 6,

\tag{7.26}

\]

\[\sum_{j=1}^{6} a_{ij} \int \mathbb{R}^2 e^{u_j} \, dx = \pi(2 + \gamma_3 + \gamma_5), \quad i = 3, 5,

\]

Proof. To show the asymptotic behavior \(\text{(7.20)}\), we prove

\[e^{-U_i} = |z|^{-2\gamma_i} |z|^{2(\omega_i - \kappa \omega_i, w_0)} (c_i + o(1)), \quad \text{as } z \to \infty.

\tag{7.27}

\]

By (1.8) and (A.18), we have

\[e^{-U_i} = |z|^{-2\gamma_i} \langle i | \Phi^* C^* A^2 C \Phi | i \rangle = |z|^{-2\gamma_i} \{ \Lambda C \Phi v^{\omega_i}, \Lambda C \Phi v^{\omega_i} \}.

\]

We know that

\[\{ \Lambda C \Phi v^{\omega_i}, \Lambda C \Phi v^{\omega_i} \} = \sum_v \left| \{ \Lambda C \Phi v^{\omega_i}, v \} \right|^2,

\]

where the sum ranges over an orthonormal basis of weight vectors of the \(i\)th fundamental representation \(V_i\). By Proposition 7.10 and that \(\mu_i > 0\) from (7.1), we see that the highest power of \(z\) appears in the norm squared of \(\{ \Lambda C \Phi v^{\omega_i}, v^{\omega_i} \}\), that is, the coordinate of \(\Lambda C \Phi v^{\omega_i}\) in the lowest weight vector \(v^{\omega_i}\). Furthermore, by (A.17) we have

\[\{ \Lambda C \Phi v^{\omega_i}, v^{\omega_i} \} = \{ \Phi v^{\omega_i}, C^* \Lambda A v^{\omega_i} \} = \Lambda^{\omega_i} (\{ \Phi v^{\omega_i}, v^{\omega_i} \} + \text{lower order terms}),

\]

where \(\Lambda^{\omega_i} = \exp(\langle \kappa \omega_i, H \rangle)\) if \(\Lambda = \exp(H)\) for \(H \in \mathfrak{h}\). Using (7.18), we see that (7.27) holds with

\[c_i = (\Lambda^{\omega_i})^2 \sum_{s \in S^{\omega_i}} \frac{c_{s,\omega_i}}{p(s, w_0)}^2.

\]

By \(u_i = \sum_{j=1}^{n} a_{ij} U_j\), \(\gamma_i = \sum_{j=1}^{n} a_{ij} \gamma_j\), \(\alpha_i = \sum_{j=1}^{n} a_{ij} \alpha_j\), and \(\langle \alpha_i, w_0 \rangle = \mu_i\), and assuming that \(-\kappa \alpha_i = \alpha_k\) for some \(1 \leq k \leq n\), we see

\[u_i = \sum_{j=1}^{n} a_{ij} U_j = 2(\gamma_i - \langle \alpha_i - \kappa \alpha_i, w_0 \rangle) \log |z| + O(1)

\]

\[= 2(\gamma_i - \mu_i - \mu_k) \log |z| + O(1) = -2(2 + \gamma_k) \log |z| + O(1)

\]

\[= -2(2 - \kappa \gamma_i) \log |z| + O(1),

\]

which is (7.21).
Integrating (1.5) and using (7.20), we have
\[ 4 \int_{\mathbb{R}^2} e^{\mu_i} \, dx = 4\pi \gamma^i - \lim_{R \to \infty} \int_{\partial \mathcal{B}_R} \frac{\partial U_i}{\partial \nu} \, ds \]
\[ = 4\pi \langle \omega_i - \kappa \omega_i, w_0 \rangle. \]

The linear combinations of the above using the Cartan matrix clearly give (7.22),
which implies the more concrete version (1.10) in Theorem 1.6.

Except three cases, \(-\kappa = \text{Id}\) and so (7.23) follows immediately. The three exceptions are for the Lie algebras of type \(A_n, D_{2n+1}\) and \(E_6\), where \(-\kappa\) represents the symmetry of the Dynkin diagram of the Lie algebra. In the \(A_n\) case,
\[ -\kappa \alpha_i = \alpha_{n+1-i}, \quad 1 \leq i \leq n, \]
that is, \(-\kappa\) is the reflection of the Dynkin diagram about its center. Therefore we have (7.24), and this is the quantization result in [LWY12, Theorem 1.3], up to the factor of 4.

In the \(D_{2n+1}\) case, we have
\[ -\kappa \alpha_{2n} = \alpha_{2n+1}, \quad -\kappa \alpha_{2n+1} = \alpha_{2n}, \quad -\kappa \alpha_i = \alpha_i, \quad 1 \leq i \leq 2n - 1, \]
that is, \(-\kappa\) permutes the last two roots at the node and preserves the others. This proves (7.25).

In the \(E_6\) case, using the labeling in the Appendix Subsection A.5 we have
\[ -\kappa \alpha_1 = \alpha_6, \quad -\kappa \alpha_6 = \alpha_1, \]
\[ -\kappa \alpha_3 = \alpha_5, \quad -\kappa \alpha_5 = \alpha_3, \]
\[ -\kappa \alpha_2 = \alpha_2, \quad -\kappa \alpha_4 = \alpha_4. \]
This then proves (7.26).

8. Examples and relations with previous results

For classical groups, we discuss the Lie-theoretic setups in more concrete terms.

To make our result more explicit especially to the analysts, we present the examples of \(A_2\) Toda systems and relate them with the previous results in [LWY12, Nie16]. Then we will present the example of \(D_4\) to illustrate our new result in this paper.

It is known [FH91] that the Lie groups \(SL_{n+1} \mathbb{C}\) and \(Sp_{2n} \mathbb{C}\) are simply connected, but \(\pi_1(SO_m \mathbb{C}) \cong \mathbb{Z}/2\) for \(m \geq 3\) and its universal cover is \(Spin_m \mathbb{C}\). However as discussed in the Appendix Subsection A.8, all the calculations can be done in the simpler \(SO_m \mathbb{C}\).

Then for an element \(g\) in these classical groups, \(g^* = g^t\) is the conjugate transpose. The compact subgroups \(K \subset G\) are characterized by \(g^{-1} = g^*\), and they are
\[ SU(n+1) \subset SL_{n+1} \mathbb{C}, \quad SO(2n+1) \subset SO_{2n+1} \mathbb{C}, \]
\[ Sp(2n) \subset Sp_{2n} \mathbb{C}, \quad SO(2n) \subset SO_{2n} \mathbb{C}. \]

Clearly for \(SL_{n+1} \mathbb{C}\), the nilpotent subgroup \(N\) consists of unipotent lower-triangular matrices with complex entries, and the abelian subgroup \(A\) consists of diagonal matrices with positive entries that multiply to 1.
Choose the symmetric and skew-symmetric bilinear forms $\kappa_m$ and $\Omega_{2n}$ for $SO_m\mathbb{C}$ and $Sp_{2n}\mathbb{C}$ to be
\begin{equation}
\kappa_m = \begin{pmatrix} 1 & \cdot & \cdot \\ \cdot & \ddots & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}_{m \times m}, \quad \Omega_{2n} = \begin{pmatrix} -\kappa_n & \kappa_n \end{pmatrix}.
\end{equation}
Then the nilpotent subgroups $N \subset G$ in these cases consist of unipotent lower-triangular matrices with complex entries, which preserve the corresponding bilinear forms. The abelian subgroups $A \subset G$ consist of diagonal matrices with positive entries whose symmetric pairs with respect to the secondary diagonal multiply to 1.

The first fundamental representations of these classical groups are just their standard representations, that is, the standard actions of $SL_{n+1}\mathbb{C}$ on $\mathbb{C}^{n+1}$, $SO_m\mathbb{C}$ on $\mathbb{C}^m$, and $Sp_{2n}\mathbb{C}$ on $\mathbb{C}^{2n}$. The $i$th fundamental representations are the irreducible representations with the same highest weights in the $i$th wedge products of the standard representations, except that at the end there is 1 spin representation for $B_n$ and two half-spin representations for $D_n$. Therefore the highest matrix coefficients $\langle i | \cdot | i \rangle$ are just the leading principal minors of rank $i$ of the matrices in the standard representations, except for the spin representations where one needs to do more work.

Example 8.2 ($A_2$ Toda system). The group is $G = SL_3\mathbb{C}$, and the solution to
\[
\begin{cases}
\Phi^{-1} \Phi_z = \sum_{i=1}^2 z^{\gamma_i} e_{-\alpha_i} = \begin{pmatrix} 0 & z^{\gamma_1} & 0 \\ z^{\gamma_1} & 0 & z^{\gamma_2} \\ 0 & z^{\gamma_2} & 0 \end{pmatrix} 
\end{cases}
\]
is
\[
\Phi(0) = \text{Id}
\]
\[
\Phi(z) = \begin{pmatrix} 1 & \frac{z^{\mu_1}}{\mu_1} & \frac{z^{\mu_1 + \mu_2}}{\mu_2(\mu_1 + \mu_2)} \\ \frac{z^{\mu_1}}{\mu_1} & 1 & \frac{z^{\mu_2}}{\mu_2} \\ \frac{z^{\mu_1 + \mu_2}}{\mu_2(\mu_1 + \mu_2)} & \frac{z^{\mu_2}}{\mu_2} & 1 \end{pmatrix}.
\]

General elements $\Lambda \in A$ and $C \in N$ are of the form
\[
\Lambda = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ c_{10} \\ c_{20} \\ c_{21} \end{pmatrix},
\]
where $\lambda_i > 0$ and $\lambda_0 \lambda_1 \lambda_2 = 1$. Then
\[
X := \Lambda C \Phi = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} \begin{pmatrix} \frac{z^{\mu_1}}{\mu_1} + c_{10} \\ \frac{z^{\mu_1} + z^{\mu_2}}{\mu_2(\mu_1 + \mu_2)} + c_{21} \frac{z^{\mu_1}}{\mu_1} \\ \frac{z^{\mu_2}}{\mu_2} + c_{20} \frac{z^{\mu_1} + z^{\mu_2}}{\mu_2(\mu_1 + \mu_2)} + c_{21} \frac{z^{\mu_2}}{\mu_2} \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix}.
\]

Therefore by Theorem 1.6 we see that
\[
e^{-U_1} = |z|^{-2y_1}(1|X^*X|1) = |z|^{-2y_1}(X^*X)_{1,1}
\]
\[
= |z|^{-2y_1} \left( \lambda_0^2 + \lambda_1^2 \left| \frac{z^{\mu_1}}{\mu_1} + c_{10} \right|^2 + \lambda_2^2 \left| \frac{z^{\mu_1 + \mu_2}}{\mu_2(\mu_1 + \mu_2)} + c_{21} \frac{z^{\mu_1}}{\mu_1} + c_{20} \right|^2 \right).
\]
Noting that our $\gamma_1$ is their $\alpha_3$, this clearly matches with the main Theorem 1.1 in \cite{LWY12} after some suitable correspondence. In particular, Eq. (1.11) there
matches our condition that $\lambda_0 \lambda_1 \lambda_2 = 1$ by the coefficient 4 in our equation (1.1) and that our power functions now have the coefficients for the $\mu$'s.

Furthermore, the condition that if $\gamma_{j+1} + \cdots + \gamma_i \notin \mathbb{Z}$ for some $j < i$, then $c_{ij} = 0$ exactly matches our current condition that $C \in N_1$.

For the Toda systems of types $A$ (and $C$ and $B$), the $U_1$ determines the other $U_i$'s. So our current result is compatible with [LWY12].

Similarly for Toda systems of type $C$ and $B$, the current results are related to those in [Nie16]. Now let us present the $D_4$ example in some detail, since this illustrates our new result.

Example 8.3 ($D_4$ Toda system). For simplicity, we present this example using concrete numbers. Let $\gamma_1 = -\frac{1}{2}, \gamma_2 = 1, \gamma_3 = 2, \gamma_4 = 3$, and then $\gamma^1 = 3, \gamma^2 = \frac{13}{2}, \gamma^3 = \frac{17}{4}, \gamma^4 = \frac{19}{4}$ by $\gamma^i = \sum_{j=1}^n a^{i,j} \gamma_j$.

The solution to $D_4$ Toda system

\[
\Phi^{-1} \Phi_\mathbb{C} = \sum_{j=1}^4 z^{\gamma_j} e^{-\alpha_j} = \begin{pmatrix}
0 & 0 \\
0 & z & 0 \\
z^2 & 0 & z^3 & 0 & 0 \\
-z^3 & -z^2 & -z & 0 & -\frac{1}{\sqrt{z}} & 0
\end{pmatrix}
\]

is 

\[
\Phi(0) = Id
\]

\[
\Phi(z) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2\sqrt{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & \frac{1}{2} z^2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} z^4 & \frac{1}{\sqrt{2}} z^6 & \frac{1}{\sqrt{2}} z^4 & 0 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} z^4 & \frac{1}{\sqrt{2}} z^6 & \frac{1}{\sqrt{2}} z^8 & \frac{1}{\sqrt{2}} z^{10} & 1 & 0 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} z^2 & -\frac{1}{\sqrt{2}} z^4 & -\frac{1}{\sqrt{2}} z^6 & -\frac{1}{\sqrt{2}} z^8 & \frac{1}{\sqrt{2}} z^{10} & 1 & 0 & 0 & 0 \\
-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} z^2 & -\frac{1}{\sqrt{2}} z^4 & -\frac{1}{\sqrt{2}} z^6 & -\frac{1}{\sqrt{2}} z^8 & -\frac{1}{\sqrt{2}} z^{10} & \frac{1}{\sqrt{2}} z^{12} & -2 & 0 & 0 \\
-64 & -768 & -23 & -384 & -24 & -65 & 4 & 2 & -2 \sqrt{z} & 1
\end{pmatrix}
\]

General elements $\Lambda \in A$ and $\Gamma \in N$ are of the form

$\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_4^{-1}, \lambda_3^{-1}, \lambda_2^{-1}, \lambda_1^{-1}), \quad \lambda_i > 0,$

$\begin{pmatrix}
1 & c_{21} & 1 \\
c_{31} & c_{32} & 1 \\
c_{41} & c_{42} & c_{43} & 1 \\
c_{51} & c_{52} & c_{53} & 0 & 1 \\
c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & 1 \\
c_{71} & c_{72} & c_{73} & c_{74} & c_{75} & c_{76} & 1 \\
c_{81} & c_{82} & c_{83} & c_{84} & c_{85} & c_{86} & c_{87} & 1
\end{pmatrix}$

Solving $C^t \kappa_8 C = \kappa_8$ (see [8.3]), we see that the coordinates are the $c$'s above the secondary diagonal, that is, the $c_{ij}$ for $j < i \leq 8 - j$, $1 \leq j \leq 3$, and the other $c$'s are solved in these.
Moreover, for the current $\gamma$’s, $\alpha(\Gamma)$ in Remark 6.3 is not an integer if $\alpha$ contains $\alpha_1$. Therefore the subgroup $N_\Gamma$ in our case is obtained by letting $c_{i1} = 0$ for $2 \leq i \leq 7$, which are coordinates corresponding to the roots $\alpha_1$, $\alpha_1 + \alpha_2$, $\alpha_1 + \alpha_2 + \alpha_3$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, $\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$.

The solution space in this case has dimension $16 = 4 + 2 \cdot 6$ and is parametrized by the positive numbers $\lambda_1, \ldots, \lambda_4$ and the complex numbers $c_{32}, c_{43}, c_{53}, c_{42}, c_{52}, c_{62}$, which are coordinates corresponding to the roots $\alpha_2, \alpha_3, \alpha_4, \alpha_2 + \alpha_3, \alpha_2 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4$.

Therefore with $X = \Delta C \Phi$, we see that

$$e^{-U_1} = |z|^{-2\gamma^2}(X^*X)_{1,1},$$

$$e^{-U_2} = |z|^{-2\gamma^2}(X^*X)_{[1,2],[1,2]},$$

where $U_2$ involves the leading principal $2 \times 2$ minor.

Now to determine $U_3$ and $U_4$, we need the 3rd and the 4th fundamental representations of $D_4$, which are the two half-spin representations. (There is the triality, the symmetry for $\alpha_1, \alpha_3$ and $\alpha_3$ in the case of $D_4$, but we disregard that since that symmetry is not valid for higher $D_n$.) Then $U_3$ and $U_4$ are expressed in terms of the highest matrix coefficient of $X^*X$ for the spin representations. These can be concretely computed, with the help of a computer algebra system such as Maple, by the Lie algebra spin representations and the exponential map. (See [FH91] for more details on the spin representations).

Appendix A. Background and setup from Lie theory

In this appendix, we provide the details for the Lie-theoretic background and setup needed in this paper. Our basic references are [Kna02][Hel78][FH91].

A.1. Cartan subalgebra and root space decomposition. Let $\mathfrak{g}$ be a complex simple Lie algebra. The Killing form $B(X,Y) = \text{Tr}(\text{ad}_X \text{ad}_Y)$ is a symmetric nondegenerate bilinear form on $\mathfrak{g}$, where $\text{ad}_X : \mathfrak{g} \to \mathfrak{g}; Z \mapsto [X,Z]$ is the adjoint action of $X$. Let $\mathfrak{h}$ be a fixed Cartan subalgebra, whose dimension $n$ is the rank of the Lie algebra.

Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$, where $\Delta$ denotes the set of roots. The roots are linear functions on the Cartan subalgebra $\mathfrak{h}$, and for $X_\alpha \in \mathfrak{g}_\alpha$ and $H \in \mathfrak{h}$, we have $[H, X_\alpha] = \alpha(H)X_\alpha$. It is known that $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = 1$.

Let $\Delta = \Delta^+ \cup \Delta^-$ be a fixed decomposition of the set of roots into the sets of positive and negative roots, and let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the set of positive simple roots.

We furthermore introduce the following standard subalgebras of $\mathfrak{g}$

$$n = n_- = \oplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}, \quad n_+ = \oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad b_+ = \mathfrak{h} \oplus n_+.$$  

A.2. Gauss decomposition. Let $G$ be a connected complex Lie group whose Lie algebra is $\mathfrak{g}$. Let $\mathcal{H}$ be the Cartan subgroup of $G$ corresponding to $\mathfrak{h}$. Denote the subgroups of $G$ corresponding to $n = n_-, n_+$ and $b_+$ in (A.1) by $N = N_-, N_+$ and $B_+$. Here $B_+$ is called a Borel subgroup of $G$. Then by the Gauss decomposition (see [LS92] Eq. (1.5.6)) and [Kos79] Eq. (2.4.4)), there exists an open and dense subset $G_\Gamma$ of $G$, called the regular part, such that

$$G_\Gamma = N_- N_+ \mathcal{H}.$$
We note that $\mathcal{H}N_+ = N_+\mathcal{H}$ by $hn = (hnh^{-1})h$, where $h \in \mathcal{H}$ and $n \in N_+$. Clearly $G_r$ contains the identity element of $G$.

A.3. **Split and compact real forms.** By Theorem 6.6 and Corollary 6.10, the complex Lie algebra $\mathfrak{g}$ has a split real form

(A.3) \[ \mathfrak{g}_0 = \mathfrak{h}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathbb{R}e_\alpha, \]

where $\mathfrak{h}_0 = \{H \in \mathfrak{h} \mid \alpha(H) \in \mathbb{R}, \forall \alpha \in \Delta\}$, and the $e_\alpha \in \mathfrak{g}_\alpha$ form a Cartan-Weyl basis.

There exists a basis $H_\alpha$ for $1 \leq i \leq n$ of $\mathfrak{h}_0$ such that

(A.4) \[ B(H_{\alpha_i}, H) = \alpha_i(H), \quad \forall H \in \mathfrak{h}. \]

The positive definite Killing form $B$ on $\mathfrak{h}_0$ also introduces an inner product on the real dual space $\mathfrak{h}_0^* = \text{Hom}_{\mathbb{R}}(\mathfrak{h}_0, \mathbb{R})$ by $(\alpha_i, \alpha_j) = B(H_{\alpha_i}, H_{\alpha_j})$ for $1 \leq i, j \leq n$.

The split real form (A.3) defines a Cartan decomposition of the Lie algebra into vector subspaces

\[ \mathfrak{g} = \mathfrak{t} \oplus i\mathfrak{t}, \]

\[ \mathfrak{t} = i\mathfrak{h}_0 + \sum_{\alpha \in \Delta^+} \mathbb{R}(e_\alpha - e_{-\alpha}) + \sum_{\alpha \in \Delta^+} \mathbb{R}i(e_\alpha + e_{-\alpha}), \]

\[ i\mathfrak{t} = \mathfrak{h}_0 + \sum_{\alpha \in \Delta^+} \mathbb{R}i(e_\alpha - e_{-\alpha}) + \sum_{\alpha \in \Delta^+} \mathbb{R}(e_\alpha + e_{-\alpha}). \]

Here $\mathfrak{t}$ is subalgebra and it is compact in the sense that the restriction of the Killing form $B|_{\mathfrak{t} \times \mathfrak{t}}$ is negative definite.

The Cartan decomposition (A.3) determines the Cartan involution $\theta$ of $\mathfrak{g}$ which is $+1$ on $\mathfrak{t}$ and $-1$ on $i\mathfrak{t}$. We often write $X^\theta$ for $\theta(X)$ with $X \in \mathfrak{g}$. Define $X^* = -X^\theta$, and we see that

(A.6) \[ H_{\alpha_i}^* = H_{-\alpha_i}, \quad (iH_{\alpha_i})^* = -iH_{\alpha_i}, \quad e_{\alpha_i}^* = e_{-\alpha_i}, \quad \text{and} \quad (ie_{\alpha_i})^* = -ie_{-\alpha_i}. \]

Therefore $*$ and $\theta$ are conjugate linear with respect to the split real form.

A.4. **Iwasawa decomposition.** From the Cartan decomposition (A.3), we also have the following Iwasawa decomposition Proposition 6.43 of the Lie algebra into Lie subalgebras

\[ \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{a} \oplus \mathfrak{n}, \quad \text{where} \quad \mathfrak{a} = \mathfrak{h}_0, \quad \mathfrak{n} = \mathfrak{n}_-. \]

Let $G$ be a connected complex Lie group whose Lie algebra is $\mathfrak{g}$. Then we have the corresponding Iwasawa decomposition on the group level Theorem 6.46

(A.7) \[ G = KAN, \]

where $K, A$ and $N$ are the subgroups in $G$ corresponding to $\mathfrak{t}, \mathfrak{a}$ and $\mathfrak{n}$. The subgroup $K$ is compact, the subgroup $A$ is abelian, and the subgroup $N$ is nilpotent. The groups $A$ and $N$ are simply connected. (The usual version using $\mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ implies our current version using $\mathfrak{n}_-$ after one application of the Cartan involution $\theta$.)

The Cartan involution $\theta$ on $\mathfrak{g}$ also lifts to the group $G$ Theorem 6.31, and we continue to denote it by the same notation. Then $K = G^\theta$ is the subgroup fixed by $\theta$. For $g \in G$, define $g^* = (g^\theta)^{-1}$. Then $(gh)^* = h^*g^*$, and an element $F \in K$ if and only if $F^*F = Id$. Furthermore, from (A.6) we see that $(N_+)^* = N_-$. 
For the purpose of doing representation theory, we let $G^*$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then it is known that $G^*$ is the universal covering of a general $G$ whose Lie algebra is $\mathfrak{g}$. Let $G^* = K^*A^*N^*$ be its Iwasawa decomposition, while $G = KAN$ is the Iwasawa decomposition of $G$. Then $A$ and $N$ are simply connected and hence isomorphic to $A^*$ and $N^*$. Only $K^*$ is a covering of $K$.

A.5. Cartan matrices and classification of simple Lie algebras. We further specify the normalization of the $e_{\alpha_i}$ and $e_{-\alpha_i}$ for $1 \leq i \leq n$ in (A.3) by requiring that

\begin{equation}
\alpha_i(h_{\alpha_i}) = 2, \quad \text{where } h_{\alpha_i} = [e_{\alpha_i}, e_{-\alpha_i}].
\end{equation}

By [Kna02, Eq. (2.93)], the relation of the $h_{\alpha_i}$ with the basis (A.4) is that

\begin{equation}
h_{\alpha_i} = \frac{2H_{\alpha_i}}{B(H_{\alpha_i}, H_{\alpha_i})} = \frac{2H_{\alpha_i}}{(\alpha_i, \alpha_i)}.
\end{equation}

The benefit of the choice of the $h_{\alpha_i}$ is that the Lie subalgebra generated by $e_{\alpha_i}, h_{\alpha_i}$, and $e_{-\alpha_i}$ is isomorphic to a copy of $\mathfrak{sl}_2$ with the standard basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ 

Then the Cartan matrix ($a_{ij}$) of $\mathfrak{g}$ is

\begin{equation}
a_{ij} = \alpha_i(h_{\alpha_j}) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}, \quad 1 \leq i, j \leq n.
\end{equation}

There are four infinite series of classical complex simple Lie algebras and five exceptional Lie algebras with the following Cartan matrices

- $A_n = \mathfrak{sl}_{n+1}$
- $B_n = \mathfrak{so}_{2n+1}$
- $C_n = \mathfrak{sp}_{2n}$
- $D_n = \mathfrak{so}_{2n}$
- $G2$:
- $F4$:
- $E6$:
- $E7$:
- $E8$:

In the above, the labelling of the roots for the exceptional Lie algebras follows [Kna02 pp 180-1]. We require that $n \geq 2$ for $B_n$ and $C_n$, and $n \geq 3$ for $D_n$. 
Furthermore we have the following isomorphisms
\[ B_2 \cong C_2, \quad D_3 \cong A_3. \]

### A.6. Principal grading.

There exist \( E_j \in \mathfrak{h}_0 \) such that
\[ \alpha_i(E_j) = \delta_{ij}, \quad \text{for } 1 \leq i, j \leq n. \]

By (A.10), we have \[ E = \sum_{k=1}^{n} a^{kj} h_{ak}, \] where \((a^{kj})\) is the inverse Cartan matrix. Define the so-called principal grading element
\[ \alpha_i(E_j) = \delta_{ij}, \quad \text{for } 1 \leq i, j \leq n. \]

By (A.11), we have \[ E_0 = \sum_{j=1}^{n} E_j \in \mathfrak{h}_0, \quad \text{such that } \alpha_i(E_0) = 1, \quad \text{for } 1 \leq i \leq n. \]

Define \( \mathfrak{g}_j = \{ x \in \mathfrak{g} | [E_0, x] = jx \}. \) Then
\[ \mathfrak{g}_j = \sum_j \mathfrak{g}_j \]
is the principal grading of \( \mathfrak{g} \), and we have
\[ \mathfrak{g}_{-1} = \bigoplus_{i=1}^{n} \mathfrak{g}_{-\alpha_i}. \]

### A.7. Representation spaces.

The weight lattice is \( \Lambda_W = \{ \beta \in \mathfrak{h}_0' \mid (\beta, h_{\alpha_i}) \in \mathbb{Z}, \forall 1 \leq i \leq n \} \). A weight \( \beta \) is called dominant if \((\beta, \alpha_i) \geq 0\) for all \( 1 \leq i \leq n \). The weight lattice is a lattice generated by the fundamental weight \( \omega_i \) for \( 1 \leq i \leq n \) such that
\[ \omega_i(h_{\alpha_j}) = \frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}. \]

An irreducible representation \( \rho \) of \( \mathfrak{g} \) on a finite-dimensional complex vector space \( V \) has the weight space decomposition \( V = \bigoplus V_{\beta} \), where \( \beta \in \Lambda_W \) and \( \rho(H)(v) = \beta(H)v \) for \( H \in \mathfrak{h} \) and \( v \in V_{\beta} \). There exists a unique highest weight \( \lambda \) with a one-dimensional highest weight space \( V_{\lambda} \) such that \( \rho(n_+) V_{\lambda} = 0 \).

The Theorem of the Highest Weight [Kna02, Theorem 5.5] asserts that up to equivalence, the irreducible finite-dimensional complex representations of \( \mathfrak{g} \) stand in one-one correspondence with the dominant weights which sends an irreducible representation to its highest weight. We denote the irreducible representation space corresponding to a dominant weight \( \lambda \) by \( V^\lambda \).

There is a canonical pairing between the dual space \( V^* = \text{Hom}(V, \mathbb{C}) \) and \( V \) denoted by \( \langle w, v \rangle \in \mathbb{C} \) with \( v \in V \) and \( w \in V^* \). \( V^* \) has a right representation \( \rho^* \) of \( \mathfrak{g} \) defined by
\[ \langle w \rho^*(X), v \rangle = \langle w, \rho(X)v \rangle, \quad X \in \mathfrak{g}. \]

The representation corresponding to the \( i \)th fundamental weight \( \omega_i \) is called the \( i \)th fundamental representation \( V_i \) of \( \mathfrak{g} \). We choose a highest weight vector in \( V_i \) and following the physicists [LS92] we called it by \( |i \rangle \). We choose a vector \( |i \rangle \) in the lowest weight space in \( V_i^\omega \) and require that \( \langle i | Id | i \rangle = 1 \) for the identity element \( Id \in G \). For simplicity, we will omit the notation \( \rho \) for the representation.

Therefore we have (see [LS92, Eq. (1.4.19)])
\[ X|i \rangle = 0, \forall X \in \mathfrak{n}_+; \quad h_{\alpha_j}|i \rangle = \delta_{ij}|i \rangle; \quad \text{and } e_{-\alpha_j}|i \rangle = 0, \forall j \neq i. \]

That is, in the \( i \)th fundamental representation, only \( e_{-\alpha_i} \) may bring the highest weight vector down. Similarly we have \( \langle i | Y = 0 \) for \( Y \in \mathfrak{n} = \mathfrak{n}_- \), and \( \langle i | e_{\alpha_j} = 0 \) if \( j \neq i \).
A.8. Lift the representation to the group. Let $G^*$ be a connected and simply connected Lie group whose Lie algebra is $\mathfrak{g}$. Then all the irreducible representations of $\mathfrak{g}$ lift to representations of $G^*$, and in particular the fundamental representations $V_i$ do. For $g \in G^*$, the pairing $\langle i | g | i \rangle$ is called the highest matrix coefficient \cite{LS92} p. 45 because it is the matrix coefficient for the highest weight vector.

It is clear that $g(i) = |i|$ for $g \in N_+$, $\langle i | g | i \rangle = |g \in N_-\rangle$, and $\exp(H) | i \rangle = e^{-\omega(H) | i \rangle}$ for $H \in \mathfrak{h}$ from the corresponding Lie algebra facts in Subsection A.7.

A representation of $\mathfrak{g}$ also leads to a representation for the universal enveloping algebra $U(\mathfrak{g})$ \cite{Kna92}. For $\mu, \nu \in U(\mathfrak{g})$ and $g \in G^*$, $\langle i | g \mu | i \rangle$ denotes the pairing of $\langle i | \mu | i \rangle$ in $V_i$ with $g(\mu | i \rangle)$ in $V_i$.

In our main Theorem \ref{thm:main} we can work with a general Lie group $G$ whose Lie algebra is $\mathfrak{g}$ instead of only the simply connected $G^*$. The reason is that the simply connected compact subgroup $K^*$ of $G^*$ is used only in passing. Our results are expressed using $N$ and $A$, and they are simply connected and the same for a general $G$ and the simply connected $G^*$ (see Subsection A.4).

In Section 7 on asymptotic behaviors and quantization, we also need the following setup from \cite{Kos79} §5.7. Let $\mathbb{C}[G^*]$ be the group algebra of $G^*$ where $G^*$ is regarded as an abstract group. Let $D(G^*) = \mathbb{C}[G^*] \otimes U(\mathfrak{g})$. We endow $D(G^*)$ with the weak * topology, and let $\hat{D}(G^*)$ be its completion.

Also let $D(N) = D(N_-)$ be the subalgebra of $D(G^*)$ generated by $N = N_-$ and $n = n_-$. and let $D(N) \to D(G^*)$ be the closure of $D(N)$ in $\hat{D}(G^*)$. Our Proposition \ref{prop:asymptotics} is stated in this space $\hat{D}(N)$, where an element in $N$ is expressed in $U(n)$.

A.9. The real structure using invariant Hermitian form. There is a more concrete realization of the dual $V_i^*$ in Subsection A.7. By averaging, there is a Hermitian metric $\{\cdot,\cdot\}$ on $V_i$ (conjugate linear in the second position) invariant under the compact group $K^*$ of a simply connected $G^*$. The important property of this Hermitian form is that \cite{Kos79} Eq. (5.11)]

\begin{equation}
\langle gu, v \rangle = \{u, g^* v \}, \quad g \in G^*, \; u, v \in V_i.
\end{equation}

Therefore we have an isomorphism $\hat{V}_i \to V_i^*; v \mapsto \{\cdot, v\}$, where $\hat{V}_i$ is the vector space $V$ with the conjugate scalar multiplication. Furthermore the right representation of $G$ on $V^*_i$ now corresponds to the right action $\hat{V}_i \times G \to \hat{V}_i; (v, g) \mapsto g^* v$ by comparing \eqref{eq:action} and \eqref{eq:inner_product}. Choose $v_\alpha \in \hat{V}_i$ to be a highest weight vector for the $i$th fundamental representation, and we require that $\{v_\alpha, v_\beta\} = 1$. Then the term in \eqref{eq:inner_product} is, with $g = \Lambda C \Phi$,

\begin{equation}
\langle i | g^* g | i \rangle = \{ g^* g v_\alpha, v_\beta \} = \{ g v_\alpha, g v_\beta \} > 0.
\end{equation}

A.10. Weyl group. The Weyl group $W$ of a Lie algebra $\mathfrak{g}$ is the finite group generated by the reflections in the simple roots on $\mathfrak{h}_0' \setminus \{\beta\}$

\begin{equation}
r_i(\beta) = \beta - \frac{2(\beta, \alpha_i)}{\alpha_i, \alpha_i} \alpha_i, \quad 1 \leq i \leq n.
\end{equation}

In the Weyl group, there is a unique element $\kappa \in W$ that is the longest element in the sense that when one writes it as a product of the simple reflections it has the maximal length. (Actually the maximal length is the number of positive roots.) This $\kappa$ on $\mathfrak{h}_0'$ maps all the positives roots to the negative roots and vice versa.
The \( \kappa \) is \(-Id\) except in the following cases: \( A_n \), \( D_{2n+1} \) and \( E_6 \), where \(-\kappa\) represents the outer-automorphism of the corresponding Lie algebra represented by the symmetry of its Dynkin diagram. (See [Dav08, Remark 13.1.8].)

Therefore \(-\kappa \alpha_i\) is still a simple root and it is hence \( \alpha_k \) for some \( 1 \leq k \leq n \). In fact \(-\kappa \alpha_i = \alpha_i\) except the three cases mentioned above.

For the irreducible representation \( V_\lambda \) with a highest weight \( \lambda \), its lowest weight is \( \kappa \lambda \).

### A.11. Degrees of primitive adjoint-invariant functions on \( g \)

The degrees of the primitive homogeneous adjoint-invariant functions of the simple Lie algebras are listed as follows:

<table>
<thead>
<tr>
<th>Lie algebras</th>
<th>degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_n )</td>
<td>2, 3, \ldots, n + 1</td>
</tr>
<tr>
<td>( B_n )</td>
<td>2, 4, \ldots, 2n</td>
</tr>
<tr>
<td>( C_n )</td>
<td>2, 4, \ldots, 2n</td>
</tr>
<tr>
<td>( D_n )</td>
<td>2, 4, \ldots, 2n - 2, n</td>
</tr>
<tr>
<td>( G_2 )</td>
<td>2, 6</td>
</tr>
<tr>
<td>( F_4 )</td>
<td>2, 6, 8, 12</td>
</tr>
<tr>
<td>( E_6 )</td>
<td>2, 5, 6, 8, 9, 12</td>
</tr>
<tr>
<td>( E_7 )</td>
<td>2, 6, 8, 10, 12, 14, 18</td>
</tr>
<tr>
<td>( E_8 )</td>
<td>2, 8, 12, 14, 18, 20, 24, 30</td>
</tr>
</tbody>
</table>

### References


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