Proceedings of the Royal Society of Edinburgh, **131A**, 1457–1480, *2001*

Uniqueness and critical spectrum of boundary spike solutions

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(MS received 21 February 2000; accepted 24 October 2000)

We study the properties of single boundary spike solutions for the following singularly perturbed problem

$$\left.\begin{aligned} \epsilon^2\Delta u - u + u^p = 0 \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\}$$

It is known that at a non-degenerate critical point of the mean curvature function $H(P)$, there exists a single boundary spike solution. In this paper, we show that the single boundary spike solution is unique and moreover it has exactly $(N-1)$ small eigenvalues. We obtain the exact asymptotics of the small eigenvalues in terms of $H(P)$.

1. Introduction

Of concern are properties of boundary spike solutions of the following singularly perturbed Neumann problem,

$$\left.\begin{aligned} \epsilon^2\Delta u - u + u^p = 0 \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $\epsilon > 0$ is a small parameter, Ω is a smooth bounded domain in R^N, $\nu(x)$ is the outer normal at $x \in \partial\Omega$ and $1 < p < (N+2)/(N-2)$ for $N \geqslant 3$ and $1 < p < +\infty$ for $N = 1, 2$.

Problem (1.1) is a typical singular perturbation problem, which arises in many branches of applied science. It can be considered as the stationary equation to the shadow system of the Gierer–Meinhardt system (see [11]), which models biological pattern formation. It is also known as the steady-state problem for a chemotactic aggregation model with logarithmic sensitivity by Keller-Segel [16] (see, for example, [20]).

Lin *et al*. first in [20] established the existence of least-energy solutions and Ni and Takagi in [22] and [23] showed that, for ϵ sufficiently small, the least-energy solution has only one local maximum point P_ϵ and $P_\epsilon \in \partial\Omega$ (therefore, the least-energy solutions have a boundary spike layer). Moreover, $H(P_\epsilon) \to \max_{P\in\partial\Omega} H(P)$ as $\epsilon \to 0$, where $H(P)$ is the mean curvature of P at $\partial\Omega$. In [24], Ni and Takagi constructed multiple spike solutions with spikes on the boundary in an axially symmetric domains. In [25], the author constructed boundary spike solutions in general domains. To state the results, we need to introduce some notation.

It is known that the solution of the following problem,

$$\left.\begin{aligned} \Delta w - w + w^p = 0 \quad \text{in} R^N, \\ w > 0, \quad w(z) \to 0 \quad \text{as } |z| \to \infty, \\ w(0) = \max_{z \in R^N} w(z), \end{aligned}\right\} \tag{1.2}$$

is radial [10] and unique [17]. We denote this solution as w.

Let

$$J_\epsilon(u) = \frac{\epsilon^2}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2} \int_\Omega u^2 - \frac{1}{p+1} \int_\Omega u^{p+1} \tag{1.3}$$

be the energy of $u \in H^1(\Omega)$ and

$$J(w) = \frac{1}{2} \int_{R^N} (|\nabla w|^2 + w^2) - \frac{1}{p+1} \int_{R^N} w^{p+1} \tag{1.4}$$

be the energy of w.

A family of solutions u_ϵ of (1.1) is called *level-*$\frac{1}{2}$ if $\lim_{\epsilon \to 0} \epsilon^{-N} J_\epsilon(u_\epsilon) = \frac{1}{2} J(w)$. Certainly, least-energy solutions are *level-*$\frac{1}{2}$.

In [25], the following result was proved.

Theorem A. *If u_ϵ is a solution of (1.1) and $\lim_{\epsilon \to 0} \epsilon^{-N} J_\epsilon(u_\epsilon) = \frac{1}{2} J(w)$, then, for ϵ sufficiently small, u_ϵ has only one local (hence global) maximum point P_ϵ and $P_\epsilon \in \partial\Omega$. Moreover, $\nabla_{\tau_{P_\epsilon}} H(P_\epsilon) \to 0$ as $\epsilon \to 0$, where $\nabla_{\tau_{P_\epsilon}}$ is the tangential derivative at P_ϵ.*

On the other hand, let $P_0 \in \partial\Omega$ be a non-degenerate critical point of $H(P)$. Then, for ϵ sufficiently small, there exists a solution u_ϵ to (1.1) such that $\epsilon^{-N} J_\epsilon(u_\epsilon) \to \frac{1}{2} J(w)$ and u_ϵ has only one local maximum point P_ϵ and $P_\epsilon \in \partial\Omega$. Moreover, $P_\epsilon \to P_0$.

Remark 1.1. In [8] and [30], multiple boundary spike solutions are constructed at multiple non-degenerate critical points of $H(P)$. On the other hand, single and multiple boundary spike solutions with degenerate critical points of $H(P)$ are studied in [9, 12, 14, 18, 24, 29].

In this paper, we study the properties of u_ϵ constructed in theorem 1. In particular, we establish the uniqueness of u_ϵ and the small eigenvalue estimates of the associated linearized operator. These properties are essential in the study of existence and stability of boundary spike solutions for the corresponding reaction-diffusion system (such as the Gierer–Meinhardt and Keller–Segel systems). See [21, 28] for progress in this direction.

From now on, we fix the point $P_0 \in \partial\Omega$ to be a non-degenerate critical point of $H(P)$.

Our first result concerns the uniqueness of u_ϵ.

Theorem 1.2. *For ϵ sufficiently small, if there are two families of level-$\frac{1}{2}$ solutions $u_{\epsilon,1}$ and $u_{\epsilon,2}$ of (1.1) such that $P_\epsilon^1 \to P_0$, $P_\epsilon^2 \to P_0$, where*

$$u_{\epsilon,1}(P_\epsilon^1) = \max_{P \in \bar{\Omega}} u_{\epsilon,1}(P), \qquad u_{\epsilon,2}(P_\epsilon^2) = \max_{P \in \bar{\Omega}} u_{\epsilon,2}(P),$$

then we have

$$P_\epsilon^1 = P_\epsilon^2, \qquad u_{\epsilon,1} = u_{\epsilon,2}. \tag{1.5}$$

The second result concerns the eigenvalue estimates associated with the linearized operator at u_ϵ,

$$L_\epsilon := \epsilon^2 \Delta - 1 + p u_\epsilon^{p-1}. \tag{1.6}$$

We first note the following result.

LEMMA 1.3. *The following eigenvalue problem*

$$\left.\begin{aligned} \Delta\phi - \phi + p w^{p-1}\phi = \mu\phi \quad &\text{in } R_+^N, \\ \phi \in H^1(R_+^N), \quad \frac{\partial\phi}{\partial y_N} = 0 \quad &\text{on } \partial R_+^N, \end{aligned}\right\} \tag{1.7}$$

where $R_+^N = \{y = (y_1, \dots, y_N) \in R^N \mid y_N > 0\}$ *admits the following set of eigenvalues*

$$\mu_1 > 0, \quad \mu_2 = \cdots = \mu_N = 0, \quad \mu_{N+1} < 0, \dots. \tag{1.8}$$

Moreover, the space of eigenfunctions corresponding to the eigenvalue 0 *is spanned by* $\partial w/\partial y_j$, $j = 1, \dots, N-1$.

Proof. We extend ϕ to R^N by reflection

$$\phi(y_1, \dots, y_{N-1}, y_N) = \phi(y_1, \dots, y_{N-1}, -y_N).$$

Then this lemma follows from theorem 2.12 of [19] and lemma C of [23]. □

Next we introduce an important definition. For each $P \in \bar{\Omega}$, we set $w_{\epsilon,P}$ to be the unique solution of

$$\left.\begin{aligned} \epsilon^2 \Delta u - u + w^p\left(\frac{x-P}{\epsilon}\right) = 0 \quad &\text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad &\text{on } \partial\Omega. \end{aligned}\right\} \tag{1.9}$$

For each $P \in \partial\Omega$, we may assume that $\partial/\partial\tau_j(P)$, $j = 1, \dots, N-1$, are the $(N-1)$-tangential derivatives. (To avoid clumsy notations, we use ∂_j to denote $\partial/\partial\tau_j(P)$. Sometimes we use ∂ to denote any tangential derivative.)

Then we have the following result.

THEOREM 1.4. *For* ϵ *sufficiently small, the following eigenvalue problem,*

$$\left.\begin{aligned} \epsilon^2 \Delta\phi_\epsilon - \phi_\epsilon + p u_\epsilon^{p-1}\phi_\epsilon = \tau_\epsilon\phi_\epsilon \quad &\text{in } \Omega, \\ \frac{\partial\phi_\epsilon}{\partial\nu} = 0 \quad &\text{on } \partial\Omega, \end{aligned}\right\} \tag{1.10}$$

admits exactly $(N-1)$ *eigenvalues* $\tau_\epsilon^1 \leqslant \tau_\epsilon^2 \cdots \leqslant \tau_\epsilon^{N-1}$ *in the interval* $[\frac{1}{2}\mu_{N+1}, \frac{1}{2}\mu_1]$, *where* μ_1 *and* μ_{N+1} *are given by lemma 1.3.*

Moreover, we have the following asymptotic behaviour of τ_ϵ^j,

$$\frac{\tau_\epsilon^j}{\epsilon^2} \to \eta_0 \lambda_j, \quad j = 1, \dots, N-1, \tag{1.11}$$

where $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{N-1}$ *are the eigenvalues of the matrix*

$$G(P_0) := (\partial_i \partial_j H(P_0)),$$

and

$$\eta_0 = \frac{N-1}{N+1} \frac{\int_{R_+^N} (w'(|z|))^2 z_N \, \mathrm{d}z}{\int_{R_+^N} (\partial w / \partial z_1)^2 \, \mathrm{d}z} > 0. \tag{1.12}$$

(Here, $w'(|z|)$ *denotes the radial derivative of* w *with respect to* $|z|$*.)*

Furthermore, the eigenfunction corresponding to τ_ϵ^j*,* $j = 1, \ldots, N-1$*, is given by the following,*

$$\phi_j^\epsilon = \sum_{i=1}^{N-1} (a_{ij} + o(1)) \frac{\partial w_{\epsilon, P_\epsilon}}{\partial \tau_i (P_\epsilon)}, \tag{1.13}$$

where P_ϵ *is the local maximum point of* u_ϵ*,* $\boldsymbol{a}_j = (a_{1j}, \ldots, a_{(N-1)j})^{\mathrm{T}}$ *is the eigenvector corresponding to* λ_j*, namely,*

$$G(P_0) \boldsymbol{a}_j = \lambda_j \boldsymbol{a}_j, \quad j = 1, \ldots, N-1. \tag{1.14}$$

Remark 1.5. The small eigenvalues are usually called *critical* eigenvalues. We mention that critical eigenvalues for Cahn–Hilliard equations and other elliptic problems with small parameters have been studied in [1–7] and the references therein.

From theorem 1.2, u_ϵ is an isolated critical point of J_ϵ. From theorem 1.4, L_ϵ is invertible. Hence u_ϵ is non-degenerate. We thus obtain the following result.

Corollary 1.6. u_ϵ *is an isolated non-degenerate critical point of* J_ϵ*.*

In the rest of this section, we briefly outline the proofs of theorems 1.2 and 1.4 and the organization of the paper. The main idea is to reduce the problem in $H^2(\Omega)$ into a finite-dimensional problem on the space of spikes and then compute the number of critical points for a finite-dimensional problem. To this end, we use the classical Liapunov–Schmidt reduction method, which has been developed in [29] and [14] for boundary spikes (a similar method has been used in [13, 26]).

We divide the proofs into the following steps.

Step A. Choose good approximate solutions.

Fix $P \in \partial\Omega$. We choose $w_{\epsilon,P}$ defined at (1.9) to be the approximate solution. Note that $w_{\epsilon,P}$ satisfies the Neumann boundary condition and $w_{\epsilon,P}$ is a C^2 function in P. Moreover, by differentiating the equation (1.9) with respect to P, we see that $\partial_j w_{\epsilon,P}$ also satisfies the Neumann boundary condition.

The difference between $w_{\epsilon,P}$ and w plays a very important role in the proof of theorem 1. We collect the properties of $w_{\epsilon,P}$ in § 2.

Step B. The idea now is to write any single boundary spike solution u_ϵ in the following form

$$u_\epsilon = w_{\epsilon,P} + \Phi, \tag{1.15}$$

where P is suitably chosen (near P_0) and Φ is small. In this way, the problem (1.1) can be reduced to looking for (P, Φ).

We first introduce some notation.

Set

$$f(u) := u^p, \Omega_{\epsilon,P} := \{y \mid \epsilon y + P \in \Omega\}. \tag{1.16}$$

Let $H^2_\nu(\Omega_{\epsilon,P})$ be the Hilbert space defined by

$$H^2_\nu(\Omega_{\epsilon,P}) := \left\{ u \in H^2(\Omega_{\epsilon,P}) \;\middle|\; \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega_{\epsilon,P} \right\}. \tag{1.17}$$

Define

$$S_\epsilon(u) := \Delta u - u + f(u) \tag{1.18}$$

for $u \in H^2_\nu(\Omega_{\epsilon,P})$.

Then substituting (1.15) into (1.1), we see that (1.1) is equivalent to

$$S_\epsilon(w_{\epsilon,P} + \Phi) = 0, \qquad \Phi \in H^2_\nu(\Omega_{\epsilon,P}), \tag{1.19}$$

where (P, Φ) are suitably chosen.

STEP C. Fixing any $P \in \partial\Omega$, we solve Φ first.

It is natural to solve Φ by using contraction mapping theorem. The problem is the linearized operator

$$L_{\epsilon,P} := S'_\epsilon(w_{\epsilon,P}) = \Delta - 1 + f'(w_{\epsilon,P}) \tag{1.20}$$

is not uniformly invertible from $H^2_\nu(\Omega_{\epsilon,P})$ to $L^2(\Omega_{\epsilon,P})$. To this end, we solve the Φ module approximate kernel. This procedure has been done in [29] and [14].

Set

$$\mathcal{K}_{\epsilon,P} := \text{span}\{\partial_j w_{\epsilon,P} \mid j = 1, \dots, N-1\} \subset H^2_\nu(\Omega_{\epsilon,P}) \tag{1.21}$$

to be the approximate kernel and

$$\mathcal{C}_{\epsilon,P} := \text{span}\{\partial_j w_{\epsilon,P} \mid j = 1, \dots, N-1\} \subset L^2(\Omega_{\epsilon,P}) \tag{1.22}$$

to be the approximate cokernel.

Let

$$\Lambda := \{P \in \partial\Omega \mid |P - P_0| < \delta\}, \tag{1.23}$$

where δ is so small such that P_0 is the only critical point of $H(P)$ in Λ.

By using the standard Liapunov–Schmidt reduction procedure, for each $P \in \Lambda$, we can find a unique $\Phi = \Phi_{\epsilon,P} \in \mathcal{K}^\perp_{\epsilon,P}$ such that

$$S_\epsilon(w_{\epsilon,P} + \Phi_{\epsilon,P}) \in \mathcal{C}_{\epsilon,P}, \qquad \Phi_{\epsilon,P} \in \mathcal{K}^\perp_{\epsilon,P}. \tag{1.24}$$

We recall this procedure in § 3.

STEP D. Reduction to a finite-dimensional problem.

We then define

$$K_\epsilon(P) := J_\epsilon(w_{\epsilon,P} + \Phi_{\epsilon,P}) : \Lambda \to R. \tag{1.25}$$

A very important observation is the following fact.

LEMMA 3.6. *$u_\epsilon = w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}$ is a critical point of J_ϵ if and only if Q_ϵ is a critical point of K_ϵ in Λ.*

By the above lemma, the uniqueness problem is reduced to counting the number of critical points of K_ϵ in Λ.

Step E. Counting the number of critical points of $K_\epsilon(P)$.

To count the number of critical points of $K_\epsilon(P)$, we need to compute $\partial K_\epsilon(P)$ and $\partial^2 K_\epsilon(P)$.

The following is the basic technical estimate in this paper.

Proposition 4.1. *Let*

$$\tilde{K}_\epsilon(P) = \epsilon^N\{\tfrac{1}{2}J(w) - \epsilon BH(P)\} \tag{1.26}$$

be the reduced energy, where

$$B := \frac{N-1}{N+1}\int_{R^N_+}(w'(|z|))^2 z_N \, \mathrm{d}z. \tag{1.27}$$

Then $K_\epsilon(P)$ is of C^2 in Λ and for ϵ sufficiently small, we have

(1) $K_\epsilon(P) - \tilde{K}_\epsilon(P) = o(\epsilon^{N+1})$;

(2) $\partial K_\epsilon(P) - \partial\tilde{K}_\epsilon(P) = o(\epsilon^{N+1})$ *uniformly for* $P \in \Lambda$;

(3) *if* $Q_\epsilon \in \Lambda$ *is a critical point of* $K_\epsilon(P)$, *then*

$$\partial^2 K_\epsilon(Q_\epsilon) - \partial^2\tilde{K}_\epsilon(Q_\epsilon) = o(\epsilon^{N+1}).$$

We will prove (1) and (2) of proposition 4.1 in § 4. Part (3) of proposition 4.1 will be left to § 6 and the appendix.

We remark that proposition 4.1 is intuitively true. However, even *formal* asymptotic analysis is not so easy. The main reason is that when we expand $\partial_i\partial_j K_\epsilon(P)$, we need to deal with terms of four different orders: $O(\epsilon^{-2})$, $O(\epsilon^{-1})$, $O(1)$, $O(\epsilon)$. Since the error term $\Phi_{\epsilon,P}$ is just of the order $O(\epsilon)$, we need to expand $\Phi_{\epsilon,P}$ up to order $O(\epsilon^2)$ and $\partial_j\Phi_{\epsilon,P}$ up to order $O(\epsilon)$. The order $O(\epsilon^2)$ term of $\Phi_{\epsilon,P}$ is difficult to estimate. (On the other hand, for the study of the corresponding interior peak case, $\Phi_{\epsilon,P}$ can be neglected in all the computations. See [28].) As far as the author knows, our results here are the first rigorous justifications for the asymptotical expansions of critical spectra of boundary spike solutions. It turns out that the order $O(\epsilon)$ term of $\Phi_{\epsilon,P}$ is good enough. We use two important observations to prove (3) of proposition 4.1. The first one is the following simple fact:

$$\frac{\partial w((x-P)/\epsilon)}{\partial x_j} + \frac{\partial w((x-P)/\epsilon)}{\partial P_j} = 0, \quad j = 1,\dots,N. \tag{1.28}$$

This fact eliminates many unnecessary computations. The second one is the so-called Pohozaev identity.

Step F. The proof of theorem 1.2 follows from steps D and E and is done in § 4.

Step G. Estimating the eigenvalues and the proof of theorem 1.4.

To prove theorem 1.4, we decompose the eigenfunction of (1.10) as follows:

$$\phi_\epsilon = \sum_{j=1}^{N-1} \epsilon a_{\epsilon,j} \partial_j w_{\epsilon,Q_\epsilon} + \bar{\phi}_\epsilon, \quad \bar{\phi}_\epsilon \in \mathcal{K}^{\perp}_{\epsilon,Q_\epsilon}.$$

Then we use the results in steps C and E to show that $\bar{\phi}_\epsilon$ is very small.

This is done in § 5.

Finally, we remark that all the theorems above are true if we replace u^p in (1.1) by general nonlinearities $f(u)$ that satisfy assumptions (f1)–(f3) in [14].

Throughout this paper we denote various generic constants by C. We use $O(A)$, $o(A)$ to mean $|O(A)| \leqslant C|A|$, $o(A)/|A| \to 0$ as $|A| \to 0$, respectively. Whenever we have repeated indices, we mean summation of that index from 1 to $N-1$ unless otherwise specified.

2. Preliminaries and analysis of projections

In this section, we study the properties of the approximate function $w_{\epsilon,P}$ defined at (1.9). In particular, we derive the asymptotic expansion of the function as well as its tangential derivatives. Most of the materials in this section are taken from [25] and [29].

We first transform the boundary.

Let $P \in \partial\Omega$. Since $\partial\Omega$ is smooth, we can find an $R_0 > 0$ and $\rho^P : B'(R_0) \to R$ a smooth function such that $\rho^P(0) = 0$, $\nabla_{x'}\rho^P(0) = 0$ and

$$\Omega_1 = \Omega \cap B(R_0) = \{(x', x_N) \in B(R_0) \mid x_N - P_N > \rho^P(x' - P')\},$$
$$\omega_1 = \partial\Omega \cap B(R_0) = \{(x', x_N) \in B(R_0) \mid x_N - P_N = \rho^P(x' - P')\},$$

where $x' = (x_1, \dots, x_{N-1})$ and

$$B(R_0) = \{x \in R^N \mid |x| < R_0\}, \qquad B'(R_0) = \{x' \in R^{N-1} \mid |x'| < R_0\}.$$

Note that

$$H(P) = \frac{1}{N-1} \sum_{k=1}^{N-1} \rho^P_{kk}(0).$$

By Taylor's expansion, we can assume that

$$\rho^P(a) = \tfrac{1}{2}\rho^P_{ij}(0)a_i a_j + \tfrac{1}{6}\rho^P_{ijk}(0)a_i a_j a_k + O(|a|^4)$$

for $a \in R^{N-1}$ small, where

$$\rho^P_i = \frac{\partial\rho^P}{\partial x_i}, \quad \rho^P_{ij} = \frac{\partial^2\rho^2}{\partial x_i x_j}, \quad \text{etc.}$$

To simplify our notation, we denote ρ^P as ρ if there is no confusion.

For $x \in \partial\Omega$, let $\nu(x)$ denote the unit outward normal at x, $\partial/\partial\nu$ denote the normal derivative, $(\tau_1(x), \dots, \tau_{N-1}(x))$ denote the $(N-1)$ linearly independent tangent vectors and $(\partial/\partial\tau_1(x), .., \partial/\partial\tau_{N-1}(x))$ the corresponding $(N-1)$ tangential derivatives at x.

For $x \in \omega_1$, we have

$$\nu(x) = \frac{1}{\sqrt{1+|\nabla_{x'}\rho|^2}}(\nabla_{x'}\rho(x'-P'), -1),$$

$$\frac{\partial}{\partial \nu} = \frac{1}{\sqrt{1+|\nabla_{x'}\rho|^2}}\left\{\rho_j \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_N}\right\}\bigg|_{x_N-P_N=\rho(x'-P')},$$

$$\tau_i(x) = \left(0,\dots,1,..,0,\frac{\partial \rho}{\partial x_i}(x'-P')\right), \quad i=1,\dots,N-1,$$

$$\frac{\partial}{\partial \tau_i(x)} = \frac{\partial}{\partial x_i} + \rho_i(x'-P')\frac{\partial}{\partial x_N}\bigg|_{x_N-P_N=\rho(x'-P')}, \quad i=1,\dots,N-1.$$

For each $u, v \in H^1(\Omega)$, we define

$$\langle u, v\rangle_\epsilon = \epsilon^{-N}\int_\Omega (\epsilon^2 \nabla u\cdot\nabla v + uv). \tag{2.1}$$

We denote $\langle u, u\rangle_\epsilon$ as $\|u\|_\epsilon^2$.

We now analyse $w_{\epsilon,P}$. By the maximum principle, $w_{\epsilon,P} > 0$ in Ω. Let w be the unique solution of (1.2) and

$$h_{\epsilon,P}(x) = w\left(\frac{x-P}{\epsilon}\right) - w_{\epsilon,P}. \tag{2.2}$$

To expand $h_{\epsilon,P}$ in terms of ϵ, we recall the following limit functions defined in [25]. Let $v_{ij}(y)$ be the unique solution of

$$\left.\begin{aligned} \Delta v - v = 0 \quad \text{in } R_+^N, \quad v\in H^1(R_+^N),\\ \frac{\partial v}{\partial y_N} = -\frac{1}{2}\frac{w'}{|y|}y_i y_j \quad \text{on } \partial R_+^N. \end{aligned}\right\} \tag{2.3}$$

Set

$$v_1(x) := \sum_{ij}\rho_{ij}^P(0) v_{ij}\left(\frac{x-P}{\epsilon}\right). \tag{2.4}$$

Let $\chi(a)$ be a function such that $\chi(a) = 1$ for $a \in B_{R_0/2}(P)$, $\chi(a) = 0$ for $a \in (B_{R_0}(P))^c$, where $B_r(P) = \{Q \mid |Q-P| < r\}$.

Set

$$\epsilon y' = x' - P', \qquad \epsilon y_N = x_N - P_N - \rho(x'-P')$$

and

$$h_{\epsilon,P}(x) = \epsilon\sum_{ij}\rho_{ij}^P(0)v_{ij}\left(\frac{x-P}{\epsilon}\right)\chi(x-P) + \epsilon^2\Psi_1^\epsilon(x). \tag{2.5}$$

Then, by proposition 2.1 of [25], we have the following result.

Lemma 2.1. $\|\Psi_1^\epsilon\|_\epsilon \leqslant C$.

We next analyse $\partial_j w_{\epsilon,P}(x)$. By the coordinate system we choose, we can assume that

$$\frac{\partial}{\partial \tau_j(P)} = \frac{\partial}{\partial P_j}.$$

Then $\partial/\partial P_j h_{\epsilon,P}(x)$ satisfies

$$\left.\begin{aligned}\epsilon^2\Delta v - v = 0 \quad \text{in } \Omega,\\ \frac{\partial v}{\partial \nu} = \frac{\partial}{\partial \nu}\frac{\partial}{\partial P_j} w\left(\frac{x-P}{\epsilon}\right) \quad \text{on } \partial\Omega.\end{aligned}\right\}$$

Let w_{kl}^j be the unique solution of

$$\left.\begin{aligned}\Delta v - v = 0 \quad \text{on } R_+^N, \quad v \in H^1(R_+^N),\\ \frac{\partial v}{\partial y_N} = \frac{1}{2}\left(\frac{w''}{|y|^2} - \frac{w'}{|y|^3}\right) y_j y_k y_l \quad \text{on } \partial R_+^N.\end{aligned}\right\} \tag{2.6}$$

Then we have (see proposition 2.2 of [25]) the following result.

PROPOSITION 2.2.

$$\frac{\partial w}{\partial \tau_j(P)} - \frac{\partial w_{\epsilon,P}}{\partial \tau_j(P)} = \left[\sum_{kl} \rho_{jkl}(0) w_{kl}^j\left(\frac{x-P}{\epsilon}\right) - 2\sum_k \rho_{jk}(0) v_{jk}\left(\frac{x-P}{\epsilon}\right)\right]\chi(x-P) + \epsilon \Psi_2^\epsilon(x),$$

where w_{kl}^j, v_{jk} *are defined above and*

$$\|\Psi_2^\epsilon\|_\epsilon \leqslant C.$$

3. Liapunov–Schmidt reduction

In this section, we solve problem (1.1) in an appropriate kernel and cokernel and then reduce our problem to a finite-dimensional one. Since the procedure has been used in many papers (see [8,13,14,18,26,29,30] and the references therein), we will omit most of the details. We refer to § 3 of [14] for further detailed proofs.

We first recall some notation in § 1. Let $f(u)$, $\Omega_{\epsilon,P}$, $H_\nu^2(\Omega_{\epsilon,P})$, S_ϵ, $L_{\epsilon,P}$, $\mathcal{K}_{\epsilon,P}$, $\mathcal{C}_{\epsilon,P}$ and Λ be defined as in § 1.

Fix $P \in \Lambda$. Let $\pi_{\epsilon,P}$ and $\pi_{\epsilon,P}^\perp$ denote the projection of $L^2(\Omega_{\epsilon,P})$ onto $\mathcal{C}_{\epsilon,P}$ and $\mathcal{C}_{\epsilon,P}^\perp$, respectively.

Our goal in this section is to show that the equation

$$\pi_{\epsilon,P}^\perp \circ S_\epsilon(w_{\epsilon,P} + \Phi_{\epsilon,P}) = 0, \qquad \Phi_{\epsilon,P} \in \mathcal{K}_{\epsilon,P}^\perp$$

has a unique solution $\Phi_{\epsilon,P}$ if ϵ is small enough.

We recall the following three results in [14].

PROPOSITION 3.1 (cf. proposition 3.1 of [14]). *Let* $\tilde{L}_{\epsilon,P} := \pi_{\epsilon,P}^\perp \circ L_{\epsilon,P}$. *There exist positive constants* $\bar{\epsilon}, C$ *such that, for all* $\epsilon \in (0, \bar{\epsilon})$ *and* $P \in \Lambda$,

$$\|\Phi\|_{H^2(\Omega_{\epsilon,P})} \leqslant C\|\tilde{L}_{\epsilon,P}\Phi\|_{L^2(\Omega_{\epsilon,P})} \tag{3.1}$$

for all $\Phi \in \mathcal{K}_{\epsilon,P}^\perp$.

PROPOSITION 3.2 (cf. proposition 3.2 of [14]). *For all $\epsilon \in (0,\bar{\epsilon})$ and $P \in \Lambda$, the map*

$$\tilde{L}_{\epsilon,P} = \pi^{\perp}_{\epsilon,P} \circ L_{\epsilon,P} : \mathcal{K}^{\perp}_{\epsilon,P} \to \mathcal{C}^{\perp}_{\epsilon,P}$$

is surjective.

PROPOSITION 3.3 (cf. lemma 3.4 of [14]). *There exists $\bar{\epsilon} > 0$ such that, for any $0 < \epsilon < \bar{\epsilon}$ and $P \in \Lambda$, there exists a unique $\Phi = \Phi_{\epsilon,P} \in \mathcal{K}^{\perp}_{\epsilon,P}$ satisfying $S_\epsilon(w_{\epsilon,P} + \Phi_{\epsilon,P}) \in \mathcal{C}_{\epsilon,P}$ and*

$$\|\Phi_{\epsilon,P}\|_{H^2(\Omega_{\epsilon,P})} \leqslant C\epsilon. \tag{3.2}$$

In fact, $\Phi_{\epsilon,P}$ is actually smooth in P.

LEMMA 3.4 (cf. lemma 3.6 of [14]). *Let $\Phi_{\epsilon,P}$ be defined by proposition 3.3. Then $\Phi_{\epsilon,P}$ is C^1 in P.*

Next, we have the following asymptotic expansion for $\Phi_{\epsilon,P}$. For the proof, please see proposition 3.4 of [29].

PROPOSITION 3.5.

$$\Phi_{\epsilon,P}(x) = \epsilon\biggl(\Phi_0\biggl(\frac{x-P}{\epsilon}\biggr)\chi(x-P)\biggr) + \epsilon^2\Psi_{\epsilon,P}(x), \tag{3.3}$$

where

$$\|\Psi_{\epsilon,P}\|_\epsilon \leqslant C$$

and Φ_0 is the unique solution of

$$\left.\begin{aligned} \Delta\Phi_0 - \Phi_0 + f'(w)\Phi_0 - f'(w)v_1 = 0 \quad \text{in } R^N_+,\\ \frac{\partial\Phi_0}{\partial y_N} = 0 \quad \text{on } \partial R^N_+,\\ \Phi_0 \text{ is orthogonal to the kernel of } L_0, \end{aligned}\right\} \tag{3.4}$$

where v_1 is defined at (2.4) and

$$L_0 := \Delta - 1 + f'(w) : H^2_\nu(R^N_+) \to L^2(R^N_+), \tag{3.5}$$

with

$$H^2_\nu(R^N_+) := \biggl\{u \in H^2(R^N_+) \biggm| \frac{\partial u}{\partial y_N} = 0 \text{ on } \partial R^N_+\biggr\}.$$

REMARK 3.6. Note that the operator $L_0 = \Delta - 1 + f'(w)$ has a non-trivial kernel $\ker(L_0) = \operatorname{span}\{\partial w/\partial y_j \mid j = 1,\dots,N-1\}$ in $H^2_\nu(R^N_+)$. (This follows from lemma 1.2.) However, the function $f'(w)v_1$ is orthogonal to $\ker(L_0)$ (since $v_1(-y_1,\dots,-y_{N-1},y_N) = v_1(y_1,\dots,y_{N-1},y_N)$) and thus there is a unique solution Φ_0 to (3.4).

REMARK 3.7. Formally, as $\epsilon \to 0, x = \epsilon y + P$, $L_{\epsilon,P} \to L_0$, $\Omega_{\epsilon,P} \to R^N_+$ and $H^2_\nu(\Omega_{\epsilon,P}) \to H^2_\nu(R^N_+)$.

Finally, let us recall that $K_\epsilon(P)$ is defined in (1.25). We then have the following result.

LEMMA 3.8. *$Q_\epsilon \in \Lambda$ is a critical point of K_ϵ if and only if $u_\epsilon = w_{\epsilon,Q_\epsilon} + v_{\epsilon,Q_\epsilon}$ is a critical point of J_ϵ.*

Proof. The proof follows from the proofs in § 5 of [14]. For the sake of completeness, we include a proof here.

By proposition 3.3, there exists an $\bar{\epsilon}$ such that, for $0 < \epsilon < \bar{\epsilon}$, we have a C^1 map which, to any $P \in \Lambda$, associates $\Phi_{\epsilon,P} \in \mathcal{K}_{\epsilon,P}^{\perp}$ such that

$$S_\epsilon(w_{\epsilon,P} + \Phi_{\epsilon,P}) = \sum_{l=1}^{N-1} \gamma_l(P)\partial_l w_{\epsilon,P}$$

for some constants $\gamma_l(P)$, $l = 1, \dots, N-1$.

If $u_\epsilon = w_{\epsilon,P} + \Phi_{\epsilon,P}$ is a critical point of J_ϵ, then

$$S_\epsilon(w_{\epsilon,P} + \Phi_{\epsilon,P}) = 0.$$

Hence

$$\int_\Omega (\epsilon^2 \nabla u_\epsilon \nabla \partial_j(w_{\epsilon,P} + \Phi_{\epsilon,P}) + u_\epsilon \partial_j(w_{\epsilon,P} + \Phi_{\epsilon,P})) - \int_\Omega f(u_\epsilon)\partial_j(w_{\epsilon,P} + \Phi_{\epsilon,P}) = -\int_\Omega S_\epsilon(u_\epsilon)\partial_j(w_{\epsilon,P} + \Phi_{\epsilon,P}) = 0,$$

i.e.

$$\partial_j K_\epsilon(P) = 0, \quad j = 1, \dots, N-1,$$

which means that P is a critical point of K_ϵ.

On the other hand, let $Q_\epsilon \in \Lambda$ be a critical point of K_ϵ. Let $u_\epsilon = w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}$. Then we have

$$\partial_j K_\epsilon(Q_\epsilon) = 0, \quad j = 1, \dots, N-1,$$

which implies that

$$\int_\Omega (\epsilon^2 \nabla u_\epsilon \nabla \partial_j(w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}) + u_\epsilon \partial_j(w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon})) - \int_\Omega f(u_\epsilon)\partial_j(w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}) = -\int_\Omega S_\epsilon(u_\epsilon)\partial_j(w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}) = 0$$

for $j = 1, \dots, N-1$.

Therefore, we have

$$\sum_{l=1}^{N-1} \gamma_l(Q_\epsilon) \int_\Omega \partial_l w_{\epsilon,Q_\epsilon} \partial_j(w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}) = 0. \tag{3.6}$$

By proposition 3.3 and the fact that $\Phi_{\epsilon,Q_\epsilon} \in \mathcal{K}_{\epsilon,Q_\epsilon}^{\perp}$,

$$\int_\Omega \partial_l w_{\epsilon,Q_\epsilon} \partial_j \Phi_{\epsilon,Q_\epsilon} = -\int_\Omega \Phi_{\epsilon,Q_\epsilon} \partial_l \partial_l w_{\epsilon,Q_\epsilon} = O(\epsilon^{N-1}). \tag{3.7}$$

On the other hand, by proposition 2.2,

$$\int_\Omega \partial_l w_{\epsilon,Q_\epsilon} \partial_j w_{\epsilon,Q_\epsilon} = \begin{cases} \epsilon^{N-2}(A_0 + o(1)) & \text{if } l = j, \\ O(\epsilon^{N-1}) & \text{if } l \neq j, \end{cases} \tag{3.8}$$

where

$$A_0 = \int_{R_+^N} \left(\frac{\partial w}{\partial y_1}\right)^2. \tag{3.9}$$

By (3.8) and (3.7), the matrix

$$\left(\int_\Omega \partial_l w_{\epsilon,Q_\epsilon} \partial_j (w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon})\right)$$

is diagonally dominant and thus is non-singular. Equations (3.6) then imply that $\gamma_l(Q_\epsilon) = 0$, $l = 1, \dots, N-1$.

Hence $u_\epsilon = w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}$ is a critical point of J_ϵ. □

4. Uniqueness of u_ϵ

In this section, we prove theorem 1.2.

Let $K_\epsilon(P)$ be defined at (1.25) and $\tilde{K}_\epsilon(P)$ be defined at (1.26).

The crucial estimate to prove theorem 1.2 is the following.

PROPOSITION 4.1. *$K_\epsilon(P)$ is of C^2 in Λ and, for ϵ sufficiently small, we have*

(1) $K_\epsilon(P) - \tilde{K}_\epsilon(P) = o(\epsilon^{N+1})$;

(2) $\partial K_\epsilon(P) - \partial \tilde{K}_\epsilon(P) = o(\epsilon^{N+1})$ *uniformly for* $P \in \Lambda$;

(3) *if* Q_ϵ *is a critical point of* $K_\epsilon(P)$, *then*

$$\partial^2 K_\epsilon(Q_\epsilon) - \partial^2 \tilde{K}_\epsilon(Q_\epsilon) = o(\epsilon^{N+1}). \tag{4.1}$$

The proof of proposition 4.1 will be delayed until the end of this section. Let us now use it to prove the uniqueness of u_ϵ.

Proof of theorem 1.2. By lemma 3.8, we just need to prove that $K_\epsilon(P)$ has only one critical point in Λ. We prove it in the following steps.

STEP 1. $\deg(\partial K_\epsilon(P), \Lambda, 0) = (-1)^k$, where k is the number of positive eigenvalues of $G(P_0)$.

This follows from (2) of proposition 4.1 and a continuity argument. In fact, by the definition of $\tilde{K}_\epsilon$, we have

$$\deg(\partial \tilde{K}_\epsilon(P), \Lambda, 0) = (-1)^k.$$

By (2), both $K_\epsilon(P)$ and $\tilde{K}_\epsilon(P)$ have no critical points on $\partial\Lambda$ and a continuous deformation argument shows that $\partial K_\epsilon(P)$ has the same degree as $\partial \tilde{K}_\epsilon(P)$ on Λ.

STEP 2. At each critical point Q_ϵ of $K_\epsilon(P)$, we have

$$\deg(\partial K_\epsilon, B_{\delta_\epsilon}(Q_\epsilon) \cap \Lambda, 0) = (-1)^k$$

for δ_ϵ sufficiently small.

This follows from (3) of proposition 4.1 and the fact that P_0 is non-degenerate (thus the eigenvalues of the matrix $(\partial_i\partial_j H(Q_\epsilon))$ are non-zero when $Q_\epsilon \in \Lambda$).

From step 2, we deduce that $K_\epsilon(P)$ has only a finite number of critical points in Λ, say, k_ϵ. By the properties of the degree, we have

$$\deg(\partial K_\epsilon(P), \Lambda, 0) = k_\epsilon(-1)^k.$$

By step 1, $k_\epsilon = 1$.

Theorem 1.2 is thus proved. □

In the rest of this section, we shall prove proposition 4.1. To this end, we first estimate $\partial_j \Phi_{\epsilon,P}$, where $\Phi_{\epsilon,P}$ is defined by proposition 3.3. Note that, by proposition 3.5, the first expansion of $\Phi_{\epsilon,P}$ is $\epsilon\Phi_0$. We will show that this expansion is valid in C^1 for P. That is, the following result holds.

LEMMA 4.2. *For ϵ sufficiently small, we have*

$$\partial_j \Phi_{\epsilon,P} = -\epsilon \frac{\partial \Phi_0((x-P)/\epsilon)}{\partial x_j} + O(\epsilon), \tag{4.2}$$

where Φ_0 is the function defined in proposition 3.5.

Proof. Let us denote

$$\Phi_{0,j} = -\epsilon \frac{\partial \Phi_0((x-P)/\epsilon)}{\partial x_j}.$$

We prove (4.2) by showing

$$\pi_{\epsilon,P}(\partial_j \Phi_{\epsilon,P} - \Phi_{0,j}) = O(\epsilon), \tag{4.3}$$

$$\pi^{\perp}_{\epsilon,P}(\partial_j \Phi_{\epsilon,P} - \Phi_{0,j}) = O(\epsilon). \tag{4.4}$$

We decompose $\partial_j \Phi_{\epsilon,P}$ as follows:

$$\partial_j \Phi_{\epsilon,P} = \Phi_{\epsilon,P,j} + \sum_{k=1}^{N-1} \beta_{jk} \partial_k w_{\epsilon,P}, \quad \Phi_{\epsilon,P,j} \in \mathcal{K}^{\perp}_{\epsilon,P}. \tag{4.5}$$

To show (4.3), we first estimate β_{jk}. Since $\Phi_{\epsilon,P} \perp \partial_l w_{\epsilon,P}$, $l = 1, \dots, N-1$, we have

$$\begin{aligned}\sum_{k=1}^{N-1} \beta_{jk} \int_{\Omega_{\epsilon,P}} (\partial_l w_{\epsilon,P} \partial_k w_{\epsilon,P}) &= \int_{\Omega_{\epsilon,P}} (\partial_j \Phi_{\epsilon,P} \partial_l w_{\epsilon,P}) \\ &= -\int_{\Omega_{\epsilon,P}} (\Phi_{\epsilon,P} \partial^2_{jl} w_{\epsilon,P}).\end{aligned}$$

Hence, by proposition 3.5,

$$\begin{aligned}\beta_{jk} &= \epsilon \int_{R^N_+} \left(\Phi_0 \frac{\partial^2 w}{\partial y_j \partial y_k} \right) A_0^{-1} + O(\epsilon^2) \\ &= -\epsilon \int_{R^N_+} \left(\frac{\partial \Phi_0}{\partial y_j} \frac{\partial w}{\partial y_k} \right) A_0^{-1} + O(\epsilon^2),\end{aligned}$$

where A_0 is given by (3.9). This proves (4.3).

It remains to prove (4.4).

Note that, by proposition 3.3, we have

$$S_\epsilon(w_{\epsilon,P}+\Phi_{\epsilon,P}) = \sum_{l=1}^{N-1} \gamma_l(P)\partial_l w_{\epsilon,P} \tag{4.6}$$

for some constants $\gamma_l(P)$, $l=1,\dots,N-1$. To estimate $\gamma_l(P)$, we observe that lemma 4.1 of [25] gives that

$$\int_\Omega S_\epsilon(w_{\epsilon,P}+\phi_{\epsilon,P})\partial_l w_{\epsilon,P} = O(\epsilon^{N+1}). \tag{4.7}$$

By (4.6), (3.8) and (4.7), we obtain

$$\gamma_l(P) = O(\epsilon^3), \quad l=1,\dots,N-1. \tag{4.8}$$

Applying ∂_j to (4.6), we have that $\Phi_{\epsilon,P,j}$ satisfies

$$S'_\epsilon(w_{\epsilon,P}+\Phi_{\epsilon,P})(\Phi_{\epsilon,P,j})+S'_\epsilon(w_{\epsilon,P}+\Phi_{\epsilon,P})(\beta_{jk}\partial_k w_{\epsilon,P}+\partial_j w_{\epsilon,P})-\sum_{l=1}^{K}\gamma_l\partial^2_{lj}w_{\epsilon,P}\in\mathcal{C}_{\epsilon,P},$$

which implies that

$$\begin{aligned}\pi^{\perp}_{\epsilon,P}\circ S'_\epsilon(w_{\epsilon,P}+\Phi_{\epsilon,P})(\Phi_{\epsilon,P,j})\\ +\pi^{\perp}_{\epsilon,P}\circ S'_\epsilon(w_{\epsilon,P}+\Phi_{\epsilon,P})(\beta_{jk}\partial_k w_{\epsilon,P}+\partial_j w_{\epsilon,P})-\pi^{\perp}_{\epsilon,P}\circ\left(\sum_{l=1}^{K}\gamma_l\partial^2_{lj}w_{\epsilon,P}\right)=0.\end{aligned} \tag{4.9}$$

By (4.8),

$$\left\|\pi^{\perp}_{\epsilon,P}\circ\left(\sum_{l=1}^{K}\gamma_l\partial^2_{lj}w_{\epsilon,P}\right)\right\|_{L^2(\Omega_{\epsilon,P})} = O(\epsilon). \tag{4.10}$$

Equation (4.4) can be proved by expanding (4.9) up to the first order and then comparing with the equations for $\Phi_{0,j}$. We omit the details. □

Finally, we prove proposition 4.1.

Proof of proposition 4.1. Part (1) follows from lemma 3.5 of [14] (see also proposition 3.2 in [23].)

We now prove (2) of proposition 4.1 as follows,

$$\begin{aligned}\partial_j K_\epsilon(P) &= \epsilon^N\langle w_{\epsilon,P}+\Phi_{\epsilon,P},\partial_j w_{\epsilon,P}\rangle_\epsilon-\int_\Omega f(w_{\epsilon,P}+\Phi_{\epsilon,P})\partial_j w_{\epsilon,P}\\ &\quad+\epsilon^N\langle w_{\epsilon,P}+\Phi_{\epsilon,P},\partial_j\Phi_{\epsilon,P}\rangle_\epsilon-\int_\Omega f(w_{\epsilon,P}+\Phi_{\epsilon,P})\partial_j\Phi_{\epsilon,P}\\ &=\int_\Omega S_\epsilon(w_{\epsilon,P}+\Phi_{\epsilon,P})\partial_j w_{\epsilon,P}+\int_\Omega S_\epsilon(w_{\epsilon,P}+\Phi_{\epsilon,P})\partial_j\Phi_{\epsilon,P}\\ &=I_1+I_2,\end{aligned}$$

where I_1 and I_2 are defined at the last equality.

Note that by lemma 4.1 of [25] and the definition of $\tilde{K}_\epsilon(P)$, we have

$$I_1 = \partial_j \tilde{K}_\epsilon + o(\epsilon^{N+1}). \tag{4.11}$$

By (4.8) and lemma 4.2,

$$I_2 = \int_\Omega \sum_{l=1}^{N-1} \gamma_l(P) \partial_l w_{\epsilon,P} \partial_j \Phi_{\epsilon,P} = O(\epsilon^{N+2}). \tag{4.12}$$

Combining the estimates (4.11) and (4.12), we see that part (2) of proposition 4.1 is thus proved.

The proof of (3) is very complicated and thus is left to § 6. □

5. Eigenvalue estimates

In this section, we shall study eigenvalue estimates for

$$L_\epsilon := \epsilon^2 \Delta - 1 + f'(u_\epsilon)$$

and prove theorem 1.4.

By lemma 3.8, we can write

$$u_\epsilon = w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon},$$

where $Q_\epsilon \in \Lambda$, $Q_\epsilon \to P_0$ and Q_ϵ is a critical point of K_ϵ. Let us denote $\Omega_{\epsilon,Q_\epsilon}$ as Ω_ϵ.

Let $(\tau_\epsilon, \phi_\epsilon)$ be a pair such that

$$\left.\begin{aligned} L_\epsilon \phi_\epsilon = \tau_\epsilon \phi_\epsilon \quad \text{in } \Omega \\ \frac{\partial \phi_\epsilon}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{5.1}$$

Suppose $\tau_\epsilon \in [\frac{1}{2}\mu_{N+1}, \frac{1}{2}\mu_1]$ and $\|\phi_\epsilon\|_\epsilon = 1$. We now look for the first expansion for ϕ_ϵ.

Let us first find the limit of τ_ϵ. If $\tau_\epsilon \to \tau_0 \neq 0$, then after a scaling argument and by taking a subsequence, we have that

$$\tilde{\phi}_\epsilon(y) = \phi_\epsilon(Q_\epsilon + \epsilon y) \to \phi_0,$$

where ϕ_0 is a solution of

$$\left.\begin{aligned} \Delta\phi_0 - \phi_0 + f'(w)\phi_0 = \tau_0 \Phi_0 \quad \text{in } R_+^N, \\ \phi_0 \in H^1(R_+^N), \quad \frac{\partial \phi_0}{\partial y_N} = 0 \quad \text{on } \partial R_+^N. \end{aligned}\right\} \tag{5.2}$$

By lemma 1.3, $\tau_0 = \mu_1$, or $\tau_0 = \mu_{N+1}, \dots$. This is impossible. Thus, for any $\tau_\epsilon \in [\frac{1}{2}\mu_{N+1}, \frac{1}{2}\mu_1]$, we must have $\tau_\epsilon \to 0$ as $\epsilon \to 0$. Then, after a scaling and limiting process as before, we have $\tilde{\phi}_\epsilon(y) = \phi_\epsilon(Q_\epsilon + \epsilon y) \to \phi_0$, where ϕ_0 is a solution of (5.2) with $\tau_0 = 0$. By lemma 1.3, there exists a_j such that

$$\phi_0 = \sum_{j=1}^{N-1} a_j \frac{\partial w}{\partial y_j}.$$

Thus we can now decompose ϕ_ϵ as

$$\phi_\epsilon = \sum_{j=1}^{N-1} \epsilon a_{\epsilon,j} \partial_j w_{\epsilon,Q_\epsilon} + \bar{\phi}_\epsilon,$$

where $\bar{\phi}_\epsilon \perp \mathcal{K}_{\epsilon,Q_\epsilon}$ and $\|\bar{\phi}_\epsilon\|_\epsilon = o(1)$.

We estimate $\bar{\phi}_\epsilon$ first in order to show that it can be neglected. Observe that $\bar{\phi}_\epsilon$ satisfies

$$\begin{aligned} L_{\epsilon,Q_\epsilon}\bar{\phi}_\epsilon + \sum_{j=1}^{N-1} a_{\epsilon,j}[f'(u_\epsilon)\epsilon\partial_j w_{\epsilon,Q_\epsilon} - f'(w)\epsilon\partial_j w] \\ = \tau_\epsilon \sum_{j=1}^{N-1} a_{\epsilon,j}\epsilon\partial_j w_{\epsilon,Q_\epsilon} + \tau_\epsilon\bar{\phi}_\epsilon, \quad \bar{\phi}_\epsilon \in \mathcal{K}^{\perp}_{\epsilon,Q_\epsilon}. \end{aligned} \tag{5.3}$$

By propositions 3.1 and 3.2, we have

$$\begin{aligned} \|\bar{\phi}_\epsilon\|_{H^2(\Omega_\epsilon)} &\leqslant C\Big\| \sum_{j=1}^{N-1} a_{\epsilon,j}[f'(u_\epsilon)\epsilon\partial_j w_{\epsilon,Q_\epsilon} - f'(w)\epsilon\partial_j w]\Big\|_{L^2(\Omega_\epsilon)} \\ &\leqslant C\epsilon \sum_{j=1}^{N-1} |a_{\epsilon,j}|. \end{aligned} \tag{5.4}$$

On the other hand, by proposition 3.3, for $P \in \Lambda$, we have

$$S_\epsilon(w_{\epsilon,P} + \Phi_{\epsilon,P}) = \sum_{l=1}^{N-1} \gamma_l(P)\partial_l w_{\epsilon,P}.$$

Applying ∂_j to the above equation and setting $P = Q_\epsilon$, we have

$$S'_\epsilon(u_\epsilon)\partial_j(w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}) - \gamma_l(Q_\epsilon)\partial_j\partial_l w_{\epsilon,Q_\epsilon} \in \mathcal{C}_{\epsilon,Q_\epsilon}.$$

Since u_ϵ is a solution, $\gamma_l(Q_\epsilon) = 0$, $l = 1, \dots, N-1$. Thus we have

$$S'_\epsilon(u_\epsilon)\partial_j(w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}) \in \mathcal{C}_{\epsilon,Q_\epsilon},$$

which implies that

$$L_\epsilon(\partial_j\Phi_{\epsilon,Q_\epsilon}) + [f'(u_\epsilon)\partial_j w_{\epsilon,Q_\epsilon} - f'(w)\epsilon\partial_j w] \in \mathcal{C}_{\epsilon,Q_\epsilon}. \tag{5.5}$$

Comparing (5.5) and (5.3), we see that

$$\bar{\phi}_\epsilon = \sum_{j=1}^{N-1} a_{\epsilon,j}\epsilon(\partial_j\Phi_{\epsilon,Q_\epsilon}) + O\bigg(|\tau_\epsilon|\epsilon \sum_{j=1}^{N-1} |a_{\epsilon,j}|\bigg).$$

Hence

$$\phi_\epsilon = \sum_{j=1}^{N-1} a_{\epsilon,j}\epsilon(\partial_j(w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon})) + O\bigg(|\tau_\epsilon|\epsilon \sum_{j=1}^{N-1} |a_{\epsilon,j}|\bigg). \tag{5.6}$$

Multiplying (5.3) by $\partial_k(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})$ and using (5.6), we obtain

$$\sum_{j=1}^{N-1} a_{\epsilon,j}\int_\Omega [L_\epsilon(\epsilon(\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}))+O(|\tau_\epsilon|\epsilon\Sigma_{j=1}^{N-1}|a_{\epsilon,j}|))\partial_k(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})] = \sum_{j=1}^{N}\tau_\epsilon\int_\Omega a_{\epsilon,j}[\epsilon(\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}))\partial_k(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})]. \tag{5.7}$$

Note that

$$\begin{aligned}\int_\Omega [L_\epsilon(\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}))\partial_k(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})] &= \int_\Omega [L_\epsilon(\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}))\partial_k(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})] \\ &\quad - \int_\Omega S_\epsilon(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})\partial_k(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}) \\ &\qquad\qquad (\text{since } u_\epsilon = w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon} \text{ is a solution of (1.1)}) \\ &= -\partial_j\partial_k K_\epsilon(Q_\epsilon).\end{aligned}$$

Thus, by (3.7), (3.8) and (5.7), we have

$$-a_{\epsilon,j}\epsilon\partial_j\partial_k K_\epsilon(Q_\epsilon) = a_{\epsilon,k}\frac{\tau_\epsilon}{\epsilon}\epsilon^N(A_0+o(1)) + O\biggl(\epsilon^N|\tau_\epsilon|\sum_{j=1}^{N-1}|a_{\epsilon,j}|\biggr),$$

where A_0 is given by (3.9).

By part (3) of proposition 4.1, we have $\tau_\epsilon/\epsilon^2 \to \eta_0\lambda_0$, where η_0 is given by (1.12) and λ_0 satisfies

$$G(P_0)\boldsymbol{a} = \lambda_0\boldsymbol{a},$$

with $\boldsymbol{a} = (\lim_{\epsilon\to 0} a_{\epsilon,1},\dots,\lim_{\epsilon\to 0} a_{\epsilon,N-1})^{\mathrm{T}}$.

In conclusion, we have proved the following result: let $(\tau_\epsilon,\phi_\epsilon)$ satisfy (5.1) with $\tau_\epsilon\in[\frac{1}{2}\mu_{N+1},\frac{1}{2}\mu_1]$. Then

$$\tau_\epsilon = \epsilon^2\eta_0\lambda_j + o(\epsilon^2), \phi_\epsilon = \sum_{i=1}^{N-1}\epsilon(a_{ij}+o(1))\partial_i w_{\epsilon,Q_\epsilon} + o(1) \tag{5.8}$$

for some $j = 1,\dots,N-1$, where λ_j, $\boldsymbol{a}_j = (a_{1j},\dots,a_{(N-1)j})^{\mathrm{T}}$ satisfy

$$G(P_0)\boldsymbol{a}_j = \lambda_j\boldsymbol{a}_j.$$

Finally, let us now use perturbation analysis to show that there exist exactly $(N-1)$ pairs $(\tau_\epsilon,\phi_\epsilon)$ to (5.1) which satisfy (5.8). (Then theorem 1.4 is proved.) By (5.8) and the fact that $\mathcal{K}_{\epsilon,P}$ has dimension $N-1$, all we need to show is that the space $\mathcal{F}_\epsilon$ of eigenfunctions of (5.1) with small eigenvalues has dimension at least $N-1$. Suppose not, let $\dim(\mathcal{F}_\epsilon) < N-1$. We shall derive a contradiction.

Set

$$a := \min(\tfrac{1}{4}\mu_1, \tfrac{1}{4}|\mu_{N+1}|). \tag{5.9}$$

From the proof of (5.8), it is easy to see that there exists

$$\psi_j^\epsilon = \sum_{i=1}^{N-1} a_{ij}^\epsilon \epsilon \partial_i (w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}), \quad j = 1, \dots, N-1,$$

such that $a_{ij}^\epsilon \to a_{ij}$ and ψ_j^ϵ are orthogonal functions in $L^2(\Omega_\epsilon)$. Moreover, it is easy to see from the previous computations that

$$L_\epsilon \psi_j^\epsilon = \epsilon^2 \eta_0 \lambda_j \psi_j^\epsilon + r_j,$$

where

$$\|r_j\|_{L^2(\Omega_\epsilon)} = o(1).$$

Note that, by (5.8), for ϵ small, $\mathcal{F}_\epsilon$ is also the span of eigenfunctions of L_ϵ with eigenvalues in $I = [-a, a]$. Let $\mathcal{E}_\epsilon$ be the span of ψ_j^ϵ, $j = 1, \dots, N-1$. Let $H := L^2(\Omega_\epsilon)$ and

$$\boldsymbol{d}(\mathcal{E}_\epsilon, \mathcal{F}_\epsilon) = \sup\{d(x, \mathcal{F}_\epsilon) \mid x \in \mathcal{E}_\epsilon,\ \|x\|_H = 1\}.$$

By (5.8), we can write $\mathcal{F}_\epsilon = \operatorname{span}\{\tilde{\psi}_{j_i}^\epsilon \mid i = 1, \dots, \dim(\mathcal{F}_\epsilon)\}$, where

$$\|\tilde{\psi}_{j_i}^\epsilon - \psi_{j_i}^\epsilon\|_{L^2(\Omega_\epsilon)} = o(1).$$

Since $\dim(\mathcal{F}_\epsilon) < \dim(\mathcal{E}_\epsilon)$, there is a $\psi_{i_0}^\epsilon \in \mathcal{E}_\epsilon$ for some i_0 such that

$$\int_{\Omega_{\epsilon,P}} \psi_{i_0}^\epsilon \tilde{\psi}_{j_i}^\epsilon = o(1), \quad i = 1, \dots, \dim(\mathcal{F}_\epsilon),$$

which implies that

$$\boldsymbol{d}(\mathcal{E}_\epsilon, \mathcal{F}_\epsilon) \geqslant 1 - \delta \tag{5.10}$$

for some δ small.

We are now in a position to use the following lemma due to [15].

LEMMA 5.1. *Let A be a self-adjoint operator on a Hilbert space H, I a compact interval in R, and $\{\psi_1, \dots, \psi_N\}$ linearly independent normalized elements in $D(A)$. Assume that the following conditions are true.*

(i) $A\psi_j = \mu_j \psi_j + r_j$, $\|r_j\| < \epsilon'$ *and* $\mu_j \in I$, $j = 1, \dots, N$.

(ii) *There is a number $a > 0$ such that I is a-isolated in the spectrum of A,*

$$(\sigma(A) \setminus I) \cap (I + (-a, a)) = \emptyset.$$

Then

$$\boldsymbol{d}(E, F) = \sup\{d(x, F) \mid x \in E,\ \|x\|_H = 1\} \leqslant \frac{N^{1/2}\epsilon'}{a(\lambda_{\min})^{1/2}},$$

where

$$E = \operatorname{span}\{\psi_1, \dots, \psi_N\},$$
$$F = \textit{closed subspace associated to } \sigma(A) \cap I,$$
$$\lambda_{\min} = \textit{the smallest eigenvalue of the matrix } (\langle \psi_i, \psi_j \rangle).$$

Applying the above lemma to $A = L_\epsilon$, $E = \mathcal{E}_\epsilon$, $F = \mathcal{F}_\epsilon$, $\psi_i = \psi_i^\epsilon$, $I = [-a, a]$, with a defined by (5.9) and ϵ' small, we have

$$\boldsymbol{d}(\mathcal{E}_\epsilon, \mathcal{F}_\epsilon) = O(\epsilon'),$$

which contradicts (5.10).

6. Estimate of $\partial_i \partial_j K_\epsilon$

In this section, we compute $\partial_i \partial_j K_\epsilon$ and therefore finish the proof of part (3) of proposition 4.1.

Let Q_ϵ be a critical point of $K_\epsilon(P)$ in Λ. By part (2) of proposition 4.1,

$$\partial_j H(Q_\epsilon) = \frac{1}{N-1} \rho_{kkj}^{Q_\epsilon}(0) = o(1), \quad j = 1, \dots, N-1. \tag{6.1}$$

Let

$$D_j = \frac{\partial}{\partial \tau_j(Q_\epsilon)} + \frac{\partial}{\partial x_j}.$$

We first recall the following facts.

LEMMA 6.1.

(1) *$D_j g_1(w((x - Q_\epsilon)/\epsilon)) = 0$ and $D_j g_2(w'((x - Q_\epsilon)/\epsilon)) = 0$, where g_1 and g_2 are any C^1 functions.*

(2) *If u_ϵ is a solution of (1.1), then we have*

$$\int_{\partial\Omega} [\tfrac{1}{2}\epsilon^2 |\nabla u_\epsilon|^2 + \tfrac{1}{2} u_\epsilon^2 - F(u_\epsilon)] \nu_j \, \mathrm{d}x = 0, \quad j = 1, \dots, N, \tag{6.2}$$

where

$$F(u) = \int_0^u f(s) \, \mathrm{d}s.$$

Proof. (1) follows by direct computations. (2) follows from (2.14) of lemma 2.3 of [27]. □

Using (1) of lemma 6.1, we can prove the following:

$$\begin{aligned} J := \int_\Omega D_i [\nabla u_\epsilon \cdot \nabla \partial_j (w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}) + u_\epsilon \partial_j (w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}) \\ - f(w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon})(\partial_j (w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}))] \\ = o(\epsilon^{N+1}). \end{aligned} \tag{6.3}$$

The proof of (6.3) requires some work. Please see the appendix.

We can now estimate $\partial_i \partial_j K_\epsilon(Q_\epsilon)$. The main idea is to use D_j to reduce integrating in Ω to integrating on $\partial\Omega$ and then use the Pohozaev identity.

By definition, we have

$$\begin{aligned}
\partial_i\partial_j K_\epsilon(Q_\epsilon) &= \epsilon^N \langle \partial_i(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}), \partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})\rangle_\epsilon \\
&\quad - \int_\Omega f'(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})\partial_i(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}) \\
&\quad + \epsilon^N \langle w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}, \partial^2_{ij}(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})\rangle_\epsilon \\
&\quad - \int_\Omega f(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})\partial^2_{ij}(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}) \\
&= \int_\Omega \partial_i[\nabla u_\epsilon\cdot\nabla\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}) + u_\epsilon\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}) \\
&\qquad\qquad - f(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})(\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}))] \\
&= -\int_\Omega \frac{\partial}{\partial x_i}[\nabla u_\epsilon\cdot\nabla\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}) + u_\epsilon\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}) \\
&\qquad\qquad - f(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})(\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}))] \\
&\quad + \int_\Omega D_i[\nabla u_\epsilon\cdot\nabla\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}) + u_\epsilon\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}) \\
&\qquad\qquad - f(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})(\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}))] \\
&= -\int_{\partial\Omega} \partial_j[\tfrac{1}{2}(\epsilon^2|\nabla u_\epsilon|^2+u_\epsilon^2) - F(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})]\nu_i(x)\,\mathrm{d}x + o(\epsilon^{N+1}) \\
&\qquad\qquad \text{(integrating by parts and using (6.3))} \\
&= \int_{\partial\Omega} \frac{\partial}{\partial x_j}[\tfrac{1}{2}(\epsilon^2|\nabla u_\epsilon|^2+u_\epsilon^2) - F(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})]\nu_i(x)\,\mathrm{d}x + o(\epsilon^{N+1}) \\
&\qquad\qquad \text{(using } D_j \text{ and (1) of lemma 6.1)} \\
&= \int_{\partial\Omega} \frac{\partial}{\partial x_j}[\tfrac{1}{2}(\epsilon^2|\nabla u_\epsilon|^2+u_\epsilon^2) - F(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})\nu_i(x)]\,\mathrm{d}x + o(\epsilon^{N+1}) \\
&\quad - \int_{\partial\Omega} [\tfrac{1}{2}(\epsilon^2|\nabla u_\epsilon|^2+u_\epsilon^2) - F(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})]\nu_{i,j}(x)\,\mathrm{d}x + o(\epsilon^{N+1}) \\
&= I_3 + I_4,
\end{aligned}$$

where I_3 and I_4 are defined at the last equality.

We first compute I_3,

$$\begin{aligned}
I_3 &= \epsilon^{N-1}\int_{R^{N-1}} \frac{\partial}{\epsilon\partial y_j}[(\tfrac{1}{2}[|\nabla w|^2+w^2] - F(w))\nu_i(\epsilon y)]\sqrt{1+|\nabla\rho|^2}\,\mathrm{d}y + o(\epsilon^{N+1}) \\
&= -\epsilon^{N-1}\int_{R^{N-1}} [(\tfrac{1}{2}[|\nabla w|^2+w^2] - F(w))\nu_i(\epsilon y)]\frac{\partial}{\partial x_j}\sqrt{1+|\nabla\rho|^2}\,\mathrm{d}y + o(\epsilon^{N+1}) \\
&= -\epsilon^{N+1}\int_{R^{N-1}} [(\tfrac{1}{2}[|\nabla w|^2+w^2] - F(w))\rho_{it}y_t]\rho_{jm}\rho_{ml}y_l\,\mathrm{d}y + o(\epsilon^{N+1}) \\
&= -\epsilon^{N+1}\rho_{il}\rho_{lm}\rho_{mj}\int_{R^{N-1}} [(\tfrac{1}{2}[|\nabla w|^2+w^2] - F(w))]y_1^2\,\mathrm{d}y + o(\epsilon^{N+1}).
\end{aligned}$$

(Here, the function ρ and its derivatives are computed at Q_ϵ.)

To estimate I_4, we note that

$$\nu_{i,j}(x) = \frac{1}{\sqrt{1+|\nabla\rho|^2}}\biggl(\rho_{ij} - \rho_i \frac{\nabla\rho\nabla\rho_j}{1+|\nabla\rho|^2}\biggr).$$

Since $\nu_N = -1/\sqrt{1+|\nabla\rho|^2}$, we have

$$\nu_{i,j}(x) + \rho_{ij}^{Q_\epsilon}(0)\nu_N = \rho_{ijm}x_m + \tfrac{1}{2}\rho_{ijmk}x_m x_k - \rho_{ik}\rho_{jt}\rho_{tm}x_m x_k + O(|x|^3). \tag{6.4}$$

Applying part (2) of lemma 6.1 with $j = N$, we have

$$\int_{\partial\Omega}[\tfrac{1}{2}\epsilon^2|\nabla u_\epsilon|^2 + \tfrac{1}{2}u_\epsilon^2 - F(u_\epsilon)]\nu_N \,\mathrm{d}x = 0. \tag{6.5}$$

By (6.5), we have

$$\begin{aligned}
I_4 &= -\int_{\partial\Omega}[\tfrac{1}{2}\epsilon^2|\nabla u_\epsilon|^2 + \tfrac{1}{2}u_\epsilon^2 - F(u_\epsilon)]\nu_{i,j}(x)\,\mathrm{d}x + O(e^{-\delta/\epsilon}) \\
&= -\int_{\partial\Omega}[\tfrac{1}{2}\epsilon^2|\nabla u_\epsilon|^2 + \tfrac{1}{2}u_\epsilon^2 - F(u_\epsilon)][\nu_{i,j}(x) + \rho_{ij}^{Q_\epsilon}(0)\nu_N]\,\mathrm{d}x + o(\epsilon^{N-1}) \\
&= -\int_{\partial\Omega}[\tfrac{1}{2}\epsilon^2|\nabla u_\epsilon|^2 + \tfrac{1}{2}u_\epsilon^2 - F(u_\epsilon)]\rho_{ijm}x_m \\
&\quad - \frac{1}{2}\int_{\partial\Omega}[\tfrac{1}{2}\epsilon^2|\nabla u_\epsilon|^2 + \tfrac{1}{2}u_\epsilon^2 - F(u_\epsilon)]\rho_{ijmk}x_m x_k \\
&\quad + \int_{\partial\Omega}[\tfrac{1}{2}\epsilon^2|\nabla u_\epsilon|^2 + \tfrac{1}{2}u_\epsilon^2 - F(u_\epsilon)]\rho_{tm}\rho_{ik}\rho_{jt}x_m x_k + o(\epsilon^{N+1}) \quad \text{(by (6.4))} \\
&= -\epsilon^{N+1}\biggl\{\tfrac{1}{2}\rho_{ijmk}\int_{R^{N-1}}(\tfrac{1}{2}|\nabla w|^2 + \tfrac{1}{2}w^2 - F(w))y_m y_k\biggr\} \\
&\quad + \epsilon^{N+1}\biggl\{\rho_{tm}\rho_{ik}\rho_{jt}\int_{R^{N-1}}(\tfrac{1}{2}|\nabla w|^2 + \tfrac{1}{2}w^2 - F(w))y_m y_k\biggr\} + o(\epsilon^{N+1}).
\end{aligned}$$

Combining the estimates for I_3 and I_4 together, we obtain

$$\begin{aligned}
I_3 + I_4 &= -\tfrac{1}{2}\rho_{ijmk}\epsilon^{N+1}\int_{R^{N-1}}(\tfrac{1}{2}|\nabla w|^2 + \tfrac{1}{2}w^2 - F(w))y_m y_k + o(\epsilon^{N+1}) \\
&= -\tfrac{1}{2}(N-1)\partial_i\partial_j H(Q_\epsilon)\epsilon^{N+1} \\
&\qquad\qquad \times \int_{R^{N-1}}(\tfrac{1}{2}|\nabla w|^2 + \tfrac{1}{2}w^2 - F(w))y_1^2\,\mathrm{d}y + o(\epsilon^{N+1}) \\
&= -\tfrac{1}{2}\partial_i\partial_j H(Q_\epsilon)\epsilon^{N+1}\int_{R^{N-1}}(\tfrac{1}{2}|\nabla w|^2 + \tfrac{1}{2}w^2 - F(w))|y|^2\,\mathrm{d}y + o(\epsilon^{N+1}).
\end{aligned}$$

By lemma 3.3 of [22],

$$\begin{aligned}
B &= \frac{N-1}{N+1}\int_{R_+^N}(w'(|z|))^2 z_N\,\mathrm{d}z \\
&= \tfrac{1}{2}(N-1)\int_{R_+^N}\{\tfrac{1}{2}(|\nabla w|^2 + w^2) - F(w)\}z_N\,\mathrm{d}z.
\end{aligned}$$

Since w is radially symmetric, it is not hard to see that

$$\int_{R_+^N}\{\tfrac{1}{2}(|\nabla w|^2+w^2)-F(w)\}z_N\,\mathrm{d}z=\frac{1}{N-1}\int_{R^{N-1}}\{\tfrac{1}{2}(|\nabla w|^2+w^2)-F(w)\}|z|^2\,\mathrm{d}z.$$

Thus

$$B=\frac{1}{2}\int_{R^{N-1}}(\tfrac{1}{2}|\nabla w|^2+\tfrac{1}{2}w^2-F(w))|y|^2\,\mathrm{d}y$$

and

$$I_3+I_4=-\epsilon^{N+1}B\partial_i\partial_j H(Q_\epsilon)+o(\epsilon^{N+1}).$$

Part (3) of proposition 4.1 is thus proved.

Appendix A. Estimates of J

In this appendix, we give the proof of the estimate for J in § 6. Some of the estimates are long and straightforward. We shall omit most of the details.

Estimate A.

$$D_j w_{\epsilon,Q_\epsilon}=-\epsilon\sum_{ij}\rho_{klj}^{Q_\epsilon}(0)v_{kl}\biggl(\frac{x-Q_\epsilon}{\epsilon}\biggr)+O(\epsilon^2),$$

where v_{kl} is defined by (2.3).

Proof. By direct computations. □

Estimate B.

$$\epsilon^N\langle D_i(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon}),\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})\rangle_\epsilon -\int_\Omega D_i f(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})=o(\epsilon^{N+1}).$$

Proof.

$$\begin{aligned}\text{Left-hand side}&=\int_\Omega[f'(w)\partial_j w D_i(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})\\&\qquad-f'(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})D_i(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})\partial_j(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})]\\&=o(\epsilon^{N+1}),\end{aligned}$$

since the first term in $D_i(w_{\epsilon,Q_\epsilon}+\Phi_{\epsilon,Q_\epsilon})$ is an even function. □

Estimate C.

$$D_j\frac{\partial}{\partial\tau_i(Q_\epsilon)}w_{\epsilon,Q_\epsilon}=\sum_{kl}\rho_{ijkl}^{Q_\epsilon}(0)w_{kl}^j\biggl(\frac{x-Q_\epsilon}{\epsilon}\biggr)-2\sum_k\rho_{kij}^{Q_\epsilon}(0)v_{jk}+O(\epsilon),$$

where w_{kl}^j is defined by (2.6).

Proof. By direct computations. □

ESTIMATE D.

$$\epsilon^N \langle (w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}), D_i\partial_j(w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon})\rangle_\epsilon - \int_\Omega f(w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}) D_i\partial_j(w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}) = o(\epsilon^{N+1}).$$

Proof.

$$\text{Left-hand side} = \int_\Omega [f(w) - f(w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon})] D_i\partial_j(w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon}).$$

Note that the first expansion of $f(w) - f(w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon})$ is an even function, while by estimate C, the first expansion of $D_i\partial_j(w_{\epsilon,Q_\epsilon} + \Phi_{\epsilon,Q_\epsilon})$ consists of an odd function and an even function. For the odd function part, it is of $o(\epsilon^{N+1})$. For the even function part, we obtain $\epsilon^{N+1}\rho^{Q_\epsilon}_{kki}(0) + o(\epsilon^{N+1}) = o(\epsilon^{N+1})$ by (6.1). □

Combining estimates B and D, we obtain the estimate for J.

Acknowledgments

The author thanks the referee for many valuable comments. This research is supported by an Earmarked Grant from RGC of Hong Kong.

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(*Issued 14 December 2001*)