MONOTONICITY FORMULA AND CLASSIFICATION OF STABLE SOLUTIONS TO POLYHARMONIC LANE-EMDEN EQUATIONS

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Abstract. In this paper, we consider polyharmonic Lane-Emden equations

$$(-\Delta)^m u = |u|^{p-1}u, \quad \text{in} \quad \mathbb{R}^n,$$

where $m \geq 3$. We classify the stable or stable outside a compact set solutions when $m = 3$ or 4 for any dimensions and when $m \geq 5$ for large dimensions. In the process, we exhibit the general Joseph-Lundgren exponent (including both local and nonlocal cases) in a concise form and prove related properties. The key ingredient of the proof of the classification is a monotonicity formula for general polyharmonic equations, which may have application in regularity theory for higher order elliptic equations.

Keywords: Polyharmonic equations, Monotonicity formula, General Joseph-Lundgren exponent, Classification of stable solutions

1. Introduction and Statement of Main Results

The polyharmonic Lane-Emden equations

$$(-\Delta)^m u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^n \tag{1.1}$$

have attracted lots of interests in the past few decades as an important semi-linear elliptic equation with many applications in differential geometry and applied physics. Here $n \geq 2, m \geq 1$ and $p \geq 1$.

When $m = 1$, a celebrated result of Gidas and Spruck [21] asserts that the equation has no positive classical solutions if

$$1 < p < p_S := \begin{cases} +\infty & n \leq 2, \\ \frac{n+2}{n-2} & n \geq 3. \end{cases}$$

If $p = p_S$, the equation admits radially symmetric solutions. Indeed, all positive solutions are radially symmetric around some point by Caffarelli, Gidas and Spruck [10]. For non-radial sign-changing solutions we refer to del Pino, Musso, Pacard and Pistoia [13, 14], Musso and Wei [33] and the references therein.

If $p > p_S$, there are very few classification results. A seminal work by Farina [20], using Moser iteration arguments pioneered by Crandall and Rabinowitz [12], proves that there are no nontrivial stable solutions if p is below the Joseph-Lundgren exponent ([26])

$$1 < p < p_{JL} := \begin{cases} \infty & 3 \leq n \leq 10, \\ 1 + \frac{4}{n-4-2\sqrt{n-1}} & n \leq 11. \end{cases}$$

For second order semilinear equations with general nonlinearity, we refer to Dupaigne and Farina [17] and the monograph by Dupaigne [18]. For stable solutions in the bounded domain, we refer to Cabre [5, 7], Cabre and Ros-Oton [6], and the references therein. For recent results on stable solutions of second order elliptic equations, we refer to recent striking results by Cabre and Poggesi [8] and Cabre, Figalli, Ros-Oton and Serra [9] in which they solved a longstanding conjecture on the regularity of the stable solutions in bounded domain with general nonlinearity with optimal dimension.

When $m \geq 2$ and $p \leq \frac{n+2m}{n-2m}$, the classification of positive solutions to (1.1) has been given by Wei and Xu [35]. When $p > \frac{n+2m}{n-2m}$, the classification of stable solutions requires new set of ideas as the Moser iteration techniques as in Farina [20] do not work when $m \geq 2$. In Davila, Dupaigne, Wang and Wei [15], a new scheme of proof via the monotonicity formula is developed. In this paper, we aim to extend this scheme to arbitrary order polyharmonic equations.

First we explain the scheme used in [15]. It consists of the following three essential steps: in Step 1, some basic energy estimates are derived. In Step 2, one derives a monotonicity formula and uses the method of blown-down anaysis to reduce the classification problem to classifying homogeneous stable solutions. Finally, in Step 3, using the fact that p is below the Joseph-Lungren exponent, one excludes the existence of homogeneous stable solutions. For polyharmonic equations, Step 1 is relatively easy. In our recent work [28], a complete classification of homogeneous stable solution to any polyharmonic Emden-Fowler equations is given, which yields Step 3. (See Section 2 for precise statements.) Thus it remains only to consider Step 2–the monotonicity formula. This is the main result of this paper. To establish the monotonicity formula for any order polyharmonic Lane-Emden equations, we develop many methods and techniques as in Section 2 and 3, these methods and techniques have potential application in other polyharmonic problems.

We first introduce the higher dimensional Joseph-Lundgren exponent (denoted by $p_{JL}(n,m)$). For $m \geq 3$, it is well-known that the homogeneous singular radial solution of (1.1) is stable under the following condition

$$p\frac{\Gamma(\frac{n}{2}-\frac{m}{p-1})\Gamma(m+\frac{m}{p-1})}{\Gamma(\frac{m}{p-1})\Gamma(\frac{n-2m}{2}-\frac{m}{p-1})} \leq \frac{\Gamma^2(\frac{n+2m}{4})}{\Gamma^2(\frac{n-2m}{4})}, \tag{1.2}$$

where $0 < m < \frac{n}{2}, n \in \mathbb{N}^+$. (See [27].) In [27] it is shown that there exists a unique exponent $p_{JL}(n,m) > \frac{n+2m}{n-2m}$ such that (1.2) is equivalent to

$$p \geq p_{JL}(n,m). \tag{1.3}$$

For $m = 1,2,3,4$, one can get the explicit expression of $p_{JL}(n,m)$. See $p_{JL}(n,m), m = 3,4$ in Appendix 1. For $m \geq 5$, there are no explicit formula for $p_{JL}(n,m)$. However, for $m \geq 5$, we have a convenient explicit expression of $p_{JL}(n,m)$, which plays a role in the proof of Theorem 1.2 below. The following Proposition 1.1 improves the bound in our previous work [27]. The proof of the sharp estimate that $a_{n,m} < 1$ in Proposition 1.1 and the estimate on the borderline dimension are put in Appendix 2.

Proposition 1.1. *For $m \geq 3$, the Joseph-Lundgren exponent can be written as*

$$p_{JL}(n,m) := \begin{cases} +\infty & \text{if } n \leq n_{JL}(m), \\ 1 + \frac{4m}{n-2m-2-2a_{n,m}\sqrt{n}} & \text{if } n > n_{JL}(m). \end{cases} \tag{1.4}$$

Here the borderline dimension $n_{JL}(m)$ is determined as the unique solution of the following equation with parameter m

$$\frac{\Gamma(\frac{n}{2})\Gamma(m+1)}{\Gamma(1)\Gamma(\frac{n-2m}{2})} = \frac{\Gamma^2(\frac{n+2m}{4})}{\Gamma^2(\frac{n-2m}{4})} \tag{1.5}$$

and the implicit parameter $a_{n,m}$ satisfies

$$a_{n,m} < 1, \quad \lim_{n\to\infty} a_{n,m} = 1, \forall\, m \geq 3. \tag{1.6}$$

The borderline dimension $n_{JL}(m)$(view m as a real number here) satisfies

$$n_{JL}(m) > 2m+4; \quad \frac{d}{dm}(n_{JL}(m) - 2m) > 0. \tag{1.7}$$

Remark 1.1. *There is a conjugate exponent $p_1(n,m) < \frac{n+2m}{n-2m}$ found in [27] which is defined by*

$$p_1(n,m) := 1 + \frac{4m}{n-2m-2+2a_{n,m}\sqrt{n}}.$$

$p_1(n,m)$ and $p_{JL}(n,m)$ are connected through

$$\frac{1}{p_1(n,m)} + \frac{1}{p_{JL}(n,m)} = \frac{n-2m-2}{2m}, n > n_{JL}(m).$$

All the arguments in Proposition 1.1 hold for any positive fraction s, i.e., replacing $m_{\in\mathbb{Z}}$ by $s_{>0}$. The threshold $p_{JL}(n,s)$ located in Proposition 1.1 is the critical exponent in classifying the stable solutions. See Theorem 1.1. In comparison, the $p_1(n,s)$ located in Proposition 1.1 is the critical exponent in analyzing the weak solutions with prescribed singularities in fractional Lane-Emden type equations (or the so-called Mazzeo-Pacard program). See original work of Mazzeo and Pacard [31, 32] *and recent works* [1, 2, 3, 22].

Remark 1.2. *The explicit Joseph-Lundgren exponent for the case $m=1$ and $m=2$ was given in Farina* [20] *and Joseph-Lundgren* [26] *respectively. Namely, in the setting of* (1.4),

$$a_{n,1}=\sqrt{\frac{n-1}{n}} \tag{1.8}$$

and

$$a_{n,2}=\sqrt{\frac{2(n-1)(n^2-2n-2)}{n(n^2+4+n\sqrt{(n-4)^2+4})}}. \tag{1.9}$$

The explicit Joseph-Lundgren exponent for the case $m=3$ is much harder, however, a powerful and elegant form was given by Harrabi and Rahal [23], *they found that*

$$p_{JL}(n,3):=\begin{cases}+\infty & \text{if } n\le 14,\\ \frac{n+4-2\sqrt{\beta_c}}{n-8-2\sqrt{\beta_c}} & \text{if } n\ge 15,\end{cases}$$

where

$$\beta_c=Z_c+\frac{3n^2+12}{12}, Z_c=\frac{\sqrt[3]{\mathbb{K}_0+\mathbb{K}_1}+\sqrt[3]{\mathbb{K}_0-\mathbb{K}_1}}{12},$$

and

$$2\mathbb{K}_0=-27n^6+324n^5-756n^4-2592n^3+25776n^2+5184n-23744,$$
$$2\mathbb{K}_1=\sqrt{(2\mathbb{K}_0)^2-4(192n^2+256)^3}.$$

In the setting of (1.4), *for the case $m=3$(sixth-order Joseph-Lundgren exponent, see Harrabi and Rahal* [23]*)*

$$a_{n,3}=\sqrt{\frac{\beta_c}{n}}. \tag{1.10}$$

An alternative and equivalent form of Joseph-Lundgren exponent for $m=3$ was given independently by the authors in [29], *and the Joseph-Lundgren exponent for $m=4$ was given by the authors in* [30], *see the details in Appendix.*

The estimate in (1.6) *is easy to see for the case $m=1$ and $m=2$, as shown in* (1.8) *and* (1.9) *respectively. It can be also checked directly for the case $m=3$ by the formulas given by the authors* [29] *and Harrabi-Rahal* [23] *independently and $m=4$ by the authors* [30]. *For the general m(integers or fractional numbers), we will prove this estimate* (1.6) *in the Appendix. The estimate* (1.6) *is important in the proof of Theorem 1.2 below.*

Remark 1.3. *The monotonicity of the borderline dimension in* (1.7) *is mainly used when m is a purely fractional number. From* (1.7), *one has*

$$n_{JL}(k)-2k<n_{JL}(s)-2s<n_{JL}(k+1)-2(k+1) \text{ for } k<s<k+1. \tag{1.11}$$

When m is an integer, then the equation (1.5) *becomes an algebraic equation and hence, the borderline dimension $n_{JL}(m)$ can be easily obtained by solving such an algebraic equation, we give a table(1.3) for the $n_{JL}(m)$ for $m=1,2,\cdots 15$*

However, when m is non-integer, it is seems no way to get the elementary explicit formula for borderline dimension $n_{JL}(s)$ since (1.5) *is a transcendental equation. Fortunately, we have the effectively estimate* (1.11) *by* (1.7).

TABLE 1. **Borderline dimension $n_{JL}(m)$ for various integers m.**

$n_{JL}(1) = 10$	$n_{JL}(2) \approx 12.56534446$	$n_{JL}(3) \approx 14.99710770$
$n_{JL}(4) \approx 17.34988211$	$n_{JL}(5) \approx 19.64978469$	$n_{JL}(6) \approx 21.91152553$
$n_{JL}(7) \approx 24.14428125$	$n_{JL}(8) \approx 26.35419459$	$n_{JL}(9) \approx 28.54559575$
$n_{JL}(10) \approx 30.72166083$	$n_{JL}(11) \approx 32.88479373$	$n_{JL}(12) \approx 35.03686103$
$n_{JL}(13) \approx 37.17934306$	$n_{JL}(14) \approx 39.31343498$	$n_{JL}(15) \approx 41.44011654$

To simplify the notations for polyharmonic equations, we introduce the following notations

$$\nabla^j \circ w = \begin{cases} \Delta^{\frac{j}{2}} w, & j \text{ is even,} \\ \nabla \Delta^{\frac{j-1}{2}}, w & j \text{ is odd,} \end{cases} \tag{1.12}$$

$$\nabla_\theta^j \circ w = \begin{cases} \Delta_\theta^{\frac{j}{2}} w, & j \text{ is even,} \\ \nabla_\theta \Delta_\theta^{\frac{j-1}{2}} w, & j \text{ is odd,} \end{cases}$$

where $\theta = \frac{x}{|x|}$, $\Delta_\theta = \Delta_{S^{n-1}}$ and $\nabla_\theta = \nabla_{S^{n-1}}$ denotes the co-variant derivative on S^{n-1}.

We recall that a solution to (1.1) is called stable if

$$E(u;\varphi) := \int_{\mathbb{R}^n} |\nabla^m \circ \varphi|^2 dx - p \int_{\mathbb{R}^n} |u|^{p-1} \varphi^2 dx \geq 0, \text{ for } \forall \varphi \in C_0^\infty(\mathbb{R}^n). \tag{1.13}$$

More generally we say that a solution to (1.1) is stable outside a compact set if there exists a compact set $\mathcal{K} \subset \mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} |\nabla^m \circ \varphi|^2 dx - p \int_{\mathbb{R}^n} |u|^{p-1} \varphi^2 dx \geq 0, \text{ for } \forall \varphi \in C_0^\infty(\mathbb{R}^n \backslash \mathcal{K}). \tag{1.14}$$

We first state the basic energy estimates.

Theorem A (Harrabi [24, 25])**.** *For the stable solution of the polyharmonic Lane-Emden equation* (1.1)*, there exists constant C independent of R such that*

$$\begin{aligned} \int_{B_R} |u|^{p+1} + |\nabla^m \circ u|^2 &\leq C R^{-2m} \int_{B_{2R} \backslash B_R} u^2, \\ \int_{B_R} |u|^{p+1} + |\nabla^m \circ u|^2 &\leq C R^{n-2m\frac{p+1}{p-1}}. \end{aligned} \tag{1.15}$$

In particular, for the sub Sobolev critical exponent cases, i.e., $1 < p \leq \frac{n+2m}{n-2m}$, the stable solution of the polyaharmonic Lane-Emden equation (1.1) *must be trivial.*

Remark 1.4. *By introducing a new weighted semi-norm and using the iteration skillfully, Harrabi* [25] *obtained much more general estimate(universal estimate) for more general operator and non-linearities.*

The subcritical cases was solved by Harrabi [24, 25] as stated in Theorem A. Therefore, it remains to consider the supercritical cases($p > \frac{n+2m}{n-2m}$). It should be noted that the supercritical cases are essentially hard hence we shall establish the corresponding monotonicity formulas to handle(Theorem 1.2). We have the following almost complete classification results for the supercritical cases.

Theorem 1.1. *Assume that either $m = 3, 4$, $n \geq 2m+1$ or $m \geq 5$ and $n \geq n(m)$, where $n(m)$ is some constant depending on m, to be defined later. Let u be a stable outside a compact set solution of* (1.1)*. If $\frac{n+2m}{n-2m} < p < p_{JL}(n,m)$, then the solution $u \equiv 0$.*

Remark 1.5. *The threshold $p_{JL}(n,m)$ in Theorem 1.1 is sharp by considering the radial solution. If $p \leq \frac{n+2m}{n-2m}$, the results in Theorem 1.1 holds for any $n \geq 2m+1$.*

Remark 1.6. *The constant $n(m) \geq 2m+1$ in Theorem 1.1 is determined in Theorem 1.2 in pages 12-15.*

As we remarked earlier, the core argument in the proof of Theorem 1.1 is the monotonicity formula which we introduce next. Let

$$B_r := \{y \in \mathbb{R}^n : |y| < r\}$$

and

$$u^\lambda(x) := \lambda^{\frac{2m}{p-1}} u(\lambda x), \quad \lambda > 0. \tag{1.16}$$

A key property we will use frequently is the following invariance: if u satisfies (1.1) then u^λ also satisfies (1.1).

We define the refined energy with boundary terms

$$\begin{aligned} \mathcal{E}(u^\lambda; m) := & \int_{B_1} \frac{1}{2}|\nabla^m u^\lambda|^2 - \frac{1}{p+1}|u^\lambda|^{p+1} \\ & + \int_{\partial B_1} \sum_{i+j+2s \leq 2m-1} C_{i,j,s} \nabla_\theta^s \frac{d^i u^\lambda}{d\lambda^i} \nabla_\theta^s \frac{d^j u^\lambda}{d\lambda^j} \end{aligned}$$

where $C_{i,j,s}$ are coefficients to be determined in terms of m, n and p.

The following monotonicity formula is the main result of this paper.

Theorem 1.2. *Assume that*

- $m = 3$, $\frac{n+6}{n-6} < p < p_T(n)$, *where*

$$p_T(n) := \begin{cases} +\infty & \text{if} \quad n \leq 30, \\ \frac{5n+30-\sqrt{15n^2-60n+190}}{5n-30-\sqrt{15n^2-60n+190}} & \text{if} \quad n \geq 31; \end{cases} \tag{1.17}$$

- $m = 4$, $\frac{n+8}{n-8} < p < p_{JL}(n, 4)$;
- $m \geq 5$, $\frac{n+2m}{n-2m} < p < p_{JL}(n, m)$ *for* $n \geq n(m)$ *where* $n(m)$ *is a large integer to be given.*

Then there exist positive constant $C(n,p)$ such that

$$\frac{d}{d\lambda}\mathcal{E}(u^\lambda; m) \geq C(n,p) \int_{\partial B_1} \lambda (\frac{du^\lambda}{d\lambda})^2 = C(n,p)\lambda^{\frac{2m}{p-1}-n} \int_{\partial B_\lambda} (\frac{2m}{p-1}u + \lambda \partial_r u)^2.$$

Remark 1.7. *The boundary terms $\int_{\partial B_1} \sum_{i+j+2s \leq 2m-1} C_{i,j,s} \nabla_\theta^s \frac{d^i u^\lambda}{d\lambda^i} \nabla_\theta^s \frac{d^j u^\lambda}{d\lambda^j}$ may be complicated. However by the energy estimates in Section 6 they can be controlled in the blow-down analysis. The existence of such positive constant $C(n,p)$ and real constants $C_{i,j,s}$ for $m \geq 5$ and $m = 4$, are determined in Section 3 and Section 5 respectively. The constant $n(m)$ is determined in Section 3.*

Remark 1.8. *The tri-harmonic $m = 3$ has been announced in [29], and we omit the proof here. Blatt [4] derived independently the monotonicity formula under condition that $\frac{n+6}{n-6} < p < p_B(n)$, where*

$$p_B(n) := \begin{cases} +\infty & \text{if} \quad n \leq 20, \\ \frac{n+28}{n-20} & \text{if} \quad n \geq 21. \end{cases}$$

It is easy to see that $p_T(n) > p_B(n)$. By the monotonicity in [29] and Blatt [4], we [29] and independently Harrabi and Rahal [23] classified the stable solutions for the tri-harmonic Lane-Emden equation (the case $m = 3$ in (1.1)). We remark that the method to analyze the sixth-order Joseph-Lundgren exponent is quite different from ours in [29], and we believe that their method can be applied to obtain an elegant form of eighth-order Joseph-Lundgren exponent, see more on Remark 1.2.

The monotonicity formula in Theorem 1.2 has applications for the study of tri- and quad-harmonic functions

$$\Delta^m u = 0, m = 3, 4. \tag{1.18}$$

Let

$$\mathcal{E}_\infty(u^\lambda;m)=\mathcal{E}(u^\lambda;m)+\int_{B_1}\frac{1}{p+1}|u^\lambda|^{p+1}.$$

As a consequence of Theorem 1.2, we state monotonicity formula for tri- and quad- harmonic map, which has application in regularity theory for polyharmonic functions. For the biharmonic maps we refer to Chang, Wang and Yang [11] and references therein.

Corollary 1.1. *Let u be a weak solution of the higher order harmonic maps (1.18). Assume that one of the following conditions*

- $m=3$, $7\le n\le 30$;
- $m=4$, $9\le n\le 17$.

Then

$$\frac{d}{d\lambda}\mathcal{E}_\infty(u^\lambda;m)\ge C(n)\lambda^{-n}\int_{\partial B_\lambda}\Big(\lambda\partial_r u\Big)^2,$$

where $C(n)>0$ is a constant independent of λ.

The rest of the paper is organized as follows: In Section 2, we present some preliminaries in our consequent analysis. In Section 3, we develop some general differential and integral inequalities to prove Theorem 1.2 in the cases $m\ge 5$. We prove the Theorem 1.2 in the case $m=4$ in Section 4 and Section 5. In Section 6, we state the scheme of the proof of Theorem 1.1 by the results in Sections 3-5.

2. Some preliminaries

In this section, we present several preliminaries which are of independent interest and may be useful subsequent sections. We first give a new decomposition of the polyharmonic operator by introducing new combinatorial operators, which plays an important role in classifying the homogeneous stable solutions. Next we build the connection between $\frac{\partial^j}{\partial r^j}u^\lambda$ and $\frac{\partial^i}{\partial\lambda^i}u^\lambda$, which then provides a bridge in proving Theorem 1.2 for the cases $m\ge 5$. Thirdly we give an alternative decomposition of the tri-harmonic operator using the derivatives with respect to λ, which is useful in prove Theorem 1.2 in the cases $m=3$, or 4. Finally, we derive some basic differentiation by parts formulas. They are used in the proof of Theorem 1.2 in the cases $m=3$ or 4.

2.1. **Decomposition of polyharmonic operator.** We first introduce some combinatorial operators to decompose the polyharmonic operator Δ^m.

First, let $a(x):=x(n-2-x)$, where n is the dimension. $a(x)$ is naturally related to the radial Laplacian operator $\partial_{rr}+\frac{n-1}{r}\partial_r$. In fact, it is easy to see that $(\partial_{rr}+\frac{n-1}{r}\partial_r)r^{-x}=a(x)r^{-x-2}$.

Next, for $m\ge 1$, we define the symmetric function

$$J_{t,m}(x):=\sum_{0\le i_1<i_2<\cdots<i_{m-t}\le m-1}\Pi_{j=1}^{m-t}a(x+2i_j),\tag{2.1}$$

which is associated with the following symmetric differential operator

$$P_{t,m}:=\sum_{all\ the\ different\ arrangements\ of}(\underbrace{r^{-2},\cdots,r^{-2}}_{t},\underbrace{\partial_{rr}+\frac{n-1}{r}\partial_r,\cdots,\partial_{rr}+\frac{n-1}{r}\partial_r}_{m-t})\tag{2.2}$$

For unifying the notations, it is natural to assume that $P_{t,m}=0$ if $t>m$ or $t<0$. For example, when $t=1,m=4$ we have

$$\begin{aligned}P_{1,4}=&(\partial_{rr}+\frac{n-1}{r}\partial_r)^2(r^{-2}(\partial_{rr}+\frac{n-1}{r}\partial_r))+(\partial_{rr}+\frac{n-1}{r}\partial_r)^3r^{-2}\\&+(\partial_{rr}+\frac{n-1}{r}\partial_r)(r^{-2}(\partial_{rr}+\frac{n-1}{r}\partial_r)^2)+r^{-2}(\partial_{rr}+\frac{n-1}{r}\partial_r)^3.\end{aligned}$$

The differential operator $P_{t,m}$ and the symmetric function $J_{t,m}(x)$ are related by

$$P_{t,m} \circ r^{-x} = (-1)^{m-t} J_{t,m}(x) r^{-x-2(m-t)}. \tag{2.3}$$

By the definition (2.2) it is easy to see that we also have the following recursive relation.

Proposition 2.1.

$$P_{j,m+1} = (\partial_{rr} + \frac{n-1}{r}\partial_r) P_{j,m} + r^{-2} P_{j-1,m}.$$

Now we turn to the spherical decomposition of the polyharmonic operator.

Proposition 2.2. *(Decomposition of polyharmonic operator)*

$$\Delta^m = \sum_{j=0}^{m} \Delta_\theta^j P_{j,m}, \quad m = 1, 2, 3 \cdots. \tag{2.4}$$

Proof. We prove it by induction. For the case $m = 1$, since $P_{0,1} = \partial_{rr} + \frac{n-1}{r}\partial_r, P_{1,1} = r^{-2}$ and $\Delta = \partial_{rr} + \frac{n-1}{r}\partial_r + r^{-2}\Delta_\theta$, then (2.4) holds immediately.

Now we suppose that $\Delta^m = \sum_{j=0}^{m} \Delta_\theta^j P_{j,m}$. Let us consider Δ^{m+1}:

$$\begin{aligned}
\Delta^{m+1} = \Delta\Delta^m &= (\partial_{rr} + \frac{n-1}{r}\partial_r + r^{-2}\Delta_\theta) \sum_{j=0}^{m} \Delta_\theta^j P_{j,m} \\
&= \sum_{j=0}^{m} \Delta_\theta^j (\partial_{rr} + \frac{n-1}{r}\partial_r) P_{j,m} + \sum_{j=0}^{m} \Delta_\theta^{j+1} r^{-2} P_{j,m} \\
&= \sum_{j=0}^{m} \Delta_\theta^j (\partial_{rr} + \frac{n-1}{r}\partial_r) P_{j,m} + \sum_{j=1}^{m+1} \Delta_\theta^j r^{-2} P_{j-1,m} \\
&= \sum_{j=0}^{m+1} \Delta_\theta^j P_{j,m+1}.
\end{aligned}$$

Here we have used Proposition 2.1. Therefore by mathematical induction, one gets (2.4). □

Next we state the following two Theorems on classifying the homogeneous stable solution in [28].

Theorem 2.1. *Let $u \in W^{m,2}_{loc}(\mathbb{R}^n \setminus \{0\}), |u|^{p+1} \in L^1(\mathbb{R}^n \setminus \{0\})$ be a homogeneous, stable solution of equation (1.1). Then the following inequality holds:*

$$\sum_{j=0}^{m} \int_{S^{n-1}} \Big(p J_{j,m}(\frac{2m}{p-1}) - J_{j,m}(\frac{n-2m}{2}) \Big) |\nabla_\theta^j \circ w|^2 \le 0.$$

Here the symmetric function $J_{j,m}(x)$ is defined by (see (2.1))

$$J_{t,m}(x) = \sum_{0 \le i_1 < i_2 < \cdots < i_{m-t} \le m-1} \Pi_{j=1}^{m-t} a(x + 2i_j), \; a(x) = x(n-2-x).$$

Theorem 2.2. *Assume that $n > 2m$. Let $u \in W^{m,2}_{loc}(\mathbb{R}^n \setminus \{0\}), |u|^{p+1} \in L^1(\mathbb{R}^n \setminus \{0\})$ be a homogeneous stable solution of the polyharmonic Lane-Emden equation (1.1). If $\frac{n+2m}{n-2m} < p < p_{JL}(n, m)$, then $u \equiv 0$.*

2.2. **The algebra between $\frac{\partial^j}{\partial r^j}u^\lambda$ and $\frac{\partial^i}{\partial \lambda^i}u^\lambda$.** In this subsection we rewrite the term $\frac{\partial^j}{\partial r^j}u^\lambda$ in terms of $\frac{\partial^i}{\partial \lambda^i}u^\lambda$, i.e., we want to obtain the precise constants q_{ji} such that

$$\frac{\partial^j}{\partial r^j}u^\lambda = \sum_i q_{ji}\frac{\partial^i}{\partial \lambda^i}u^\lambda.$$

For this, let us introduce the notation of the falling and rising factorials:

$$(x)_n := x(x-1)\cdots(x-n+1); \quad (x)^n := x(x+1)\cdots(x+n-1). \tag{2.5}$$

Then

$$\begin{aligned} (x)^{s-j} &= (-1)^{s-j}(-x)_{s-j}, \\ (x+y)_n &= \sum_{j=0}^{n} C_n^j (x)_j (y)_{n-j}. \end{aligned} \tag{2.6}$$

Set the notation $u^\lambda(X) := \lambda^k u(\lambda X)$. We have the following relations:

Lemma 2.1.

$$\lambda^j \frac{d^j u^\lambda(X)}{d\lambda^j} = \sum_{i=0}^{j}(k)_{j-i}C_j^i r^i \frac{d^i u^\lambda(X)}{dr^i}, \quad j = 0, 1, 2, 3, ...$$

Proof. Equivalently, one proves

$$\frac{d^j u^\lambda(X)}{d\lambda^j} = \sum_{i=0}^{j}(k)_{j-i}C_j^i \lambda^{-j} r^i \frac{d^i u^\lambda(X)}{dr^i}, \quad j = 0, 1, 2, 3, ... \tag{2.7}$$

For $j = 0$, (2.7) holds immediately. By induction, suppose that (2.7) holds for $j = N$. Consider $j = N + 1$, differentiating (2.7) for $j = N$ with respect to λ, we have

$$\begin{aligned} \frac{d^{N+1}u^\lambda(X)}{d\lambda^{N+1}} &= \sum_{i=0}^{N}(k)_{N-i}C_N^i(-(N-i))\lambda^{-N-1}r^i\frac{d^i u^\lambda(X)}{dr^i} + \sum_{i=0}^{N}(k)_{N-i}C_N^i\lambda^{-N}r^{i+1}\frac{d^{i+1}u^\lambda(X)}{dr^{i+1}} \\ &= \sum_{i=0}^{N}(k)_{N+1-i}C_N^i\lambda^{-(j+1)}r^i\frac{d^i u^\lambda(X)}{dr^i} + \sum_{i=1}^{N+1}(k)_{N+1-i}C_N^{i-1}\lambda^{-(N+1)}r^i\frac{d^i u^\lambda(X)}{dr^i} \\ &= \sum_{i=0}^{N+1}(k)_{N+1-i}C_{N+1}^i\lambda^{-(N+1)}r^i\frac{d^i u^\lambda(X)}{dr^i}. \end{aligned}$$

Hence (2.7) holds for $j = N + 1$. □

Remark 2.1. *Let A_k^i denote the number of permutations, i.e., $A_k^i = k(k-1)\cdots(k-i+1)$, C_j^i denote the number of combination, i.e., $C_j^i = \frac{j!}{i!(j-i)!}$. To unify the process of the following proofs, we assume that $A_k^i = 0$ if $i < 0$, $C_j^i = 0$ if $i > j$. Notice that $(k)_i = A_k^i$ and $(k)^j = A_{k+j-1}^j$.*

We first construct the identity of falling and rising factorials.

Lemma 2.2.

$$\sum_{j=0}^{p}(-1)^{s-j}C_j^iC_s^j(k)_{j-i}(k)^{s-j} = \delta_{is}, \quad 0 \le i, j, s \le p, \quad p = 1, 2, 3, ...$$

Proof. Let $i \le s$, then we have $C_j^i \cdot C_s^j = C_s^i \cdot C_{s-i}^{s-j}$. When $i = s$, by the assumption of the above remark, we have

$$\sum_{j=0}^{p}(-1)^{s-j}C_j^iC_s^j(k)_{j-i}(k)^{s-j} = C_i^iC_i^i(k)_0(k)^0 = 1.$$

For $i > s$, then $i > j$ or $j > s$ (if else $i \le j \le s$), in any case we have $C_j^i C_s^j = 0$, then $\sum_{j=0}^{p}(-1)^{s-j}C_j^i C_s^j (k)_{j-i}(k)^{s-j} = 0$. It remains to consider $i < s$, it holds

$$\begin{aligned}\sum_{j=0}^{p}(-1)^{s-j}C_j^i C_s^j (k)_{j-i}(k)^{s-j} &= C_s^i \sum_{j=0}^{s}(-1)^{s-j}C_{s-i}^{s-j}(k)_{j-i}(k)^{s-j} \\ &= C_s^i \sum_{j=0}^{s} C_{s-i}^{s-j}(k)_{j-i}(-k)_{s-j} \\ &= C_s^i (k-k)_{s-i} = 0.\end{aligned} \tag{2.8}$$

Here we have used (2.6).

□

We need to express $\frac{d^i u^\lambda(X)}{dr^i}$ in terms of $\frac{d^j u^\lambda(X)}{d\lambda^j}$, then we construct the following

Lemma 2.3.

$$r^j \frac{d^j u^\lambda(X)}{dr^j} = \sum_{i=0}^{j}(-1)^{j-i}(k)^{j-i}C_j^i \lambda^i \frac{d^i u^\lambda(X)}{d\lambda^i}, \quad j = 0, 1, 2, 3, \ldots$$

Proof. Let $M_{i,j} := (k)_{j-i}C_j^i$ and $N_{j,s} := (-1)^{s-j}(k)^{s-j}C_s^j$. From Lemma 2.1, it is equivalent to prove that $\sum_{j=0}^{p} M_{i,j}N_{j,s} = \delta_{is}$, for $0 \le i, j, s \le p$. That is,

$$\sum_{j=0}^{p}(-1)^{s-j}C_j^i C_s^j (k)_{j-i}(k)^{s-j} = \delta_{is}, \quad 0 \le i, j, s \le p, \quad p = 1, 2, 3, \ldots$$

By Lemma 2.2, the proof is complete.

□

2.3. **Tri-harmonic operator: an alternative representation.** We now consider tri-harmonic operator in terms of the radial and sphere parts. Then by the formulas in Lemma 2.3, we can transfer the parts involving derivative with respect to r to derivative with respect to λ. This computation will be used in Sections 4-6.

By the definition and direct computations, we have

$$\Delta u = (\partial_{rr} + \frac{n-1}{r}\partial_r)u + \Delta_\theta(r^{-2}u),$$

and

$$\Delta^3 u := F_0(u) + \Delta_\theta F_1(u) + \Delta_\theta^2 F_2(u) + \Delta_\theta^3 F_3(u). \tag{2.9}$$

where for $r = 1$ and $a = n-1$,

$$\begin{aligned}F_0(u) :=& (\partial_{rr} + \frac{n-1}{r}\partial_r)^3 u = \Big(\partial_{r^6} + 3a\partial_{r^5} + 3a(a-2)\partial_{r^4} + a(a-2)(a-7)\partial_{r^3} \\ & - 3a(a-2)(a-4)\partial_{r^2} + 3a(a-2)(a-4)\partial_r\Big)u \\ :=& \sum_{j=1}^{6} a_j \partial_{r^j} u, \\ F_1(u) :=& \Big(3\partial_{r^4} + (6a-12)\partial_{r^3} + (3a^2 - 24a + 42)\partial_{r^2} + (60a - 9a^2 - 96)\partial_r + \\ & (8a^2 - 64a + 120)\Big)u \\ :=& \sum_{j=0}^{4} b_j \partial_{r^j} u,\end{aligned}$$

and

$$F_3(u) := \Big(3\partial_{r^2} + (3a-12)\partial_r + 26 - 6a\Big)u = \sum_{j=0}^{2} v_j \partial_{r^j},$$

here $\partial_{r^3} := \partial_{rrr}$ and so on.

Recalling Lemma 2.3, we have the following

$$F_0(u) = \sum_{j=0}^{6} k_j \lambda^j \frac{d^j u^\lambda}{d\lambda^j}, F_1(u) = \sum_{j=0}^{4} (-t_j)\lambda^j \frac{d^j u^\lambda}{d\lambda^j}, F_2(u) = \sum_{j=0}^{2} e_j \lambda^j \frac{d^j u^\lambda}{d\lambda^j}. \tag{2.10}$$

For simplicity, we denote

$$a_6 = 1, a_5 = 3a, a_4 = 3a(a-2), a_3 = a(a-2)(a-7), a_2 = -3a(a-2)(a-4),$$
$$a_1 = 3a(a-2)(a-4).$$

Then k_i are determined by

$$k_i = \sum_{j=i}^{6} (-1)^{j-i} C_j^i \cdot (k)^{j-i} \cdot a_j, i = 0, 1 \cdots, 6. \tag{2.11}$$

See (2.5) for the notations. Here thereafter we denote that $a_0 = 0, (k)^0 = 1$ for convenience.

And t_i are determined by

$$t_i = \sum_{j=i}^{4} (-1)^{j-i+1} (k)^j b_j, i = 0, 1 \cdots, 4. \tag{2.12}$$

For $e_j, j = 0, 1 \cdots, 2$, they are given by

$$e_2 = 3, e_1 = -6k + 3a - 12, e_0 = 3k(k+1) - (3a-12)k + 26 - 6a. \tag{2.13}$$

In (2.9), we have the representation of $\Delta^3 u^\lambda$ by $F_j(u)$ which will be useful in calculating the terms $\overline{E}_{d_2}(u^\lambda, 1)$,$\overline{E}_{d_1}(u^\lambda, 1)$ in Section 4.

2.4. **Differentiation by parts formulas.** Finally we state some differentiation by parts formulas.

Denote that $f^{(j)} = \frac{d^j f}{d\lambda^j}$.

Lemma 2.4. *We have the following type-1 (i.e., $\lambda^j f^{(j)} f^{(1)}$) differentiation by parts formulas:*

$$f f^{(1)} = \frac{d}{d\lambda}(\frac{1}{2}f^2), \lambda^2 f^{(2)} f^{(1)} = -\lambda (f^{(1)})^2 + \frac{d}{d\lambda}(\frac{1}{2}\lambda^2 f^{(1)} f^{(1)}),$$
$$\lambda^3 f^{(3)} f^{(1)} = 3\lambda (f^{(1)})^2 - \lambda^3 (f^{(2)})^2 + \frac{d}{d\lambda}(\lambda^3 f^{(2)} f^{(1)}),$$

$$\lambda^4 f^{(4)} f^{(1)} = -12\lambda (f^{(1)})^2 + 6\lambda^3 (f^{(2)})^2 + \frac{d}{d\lambda}\big(\lambda^4 f^{(3)} f^{(1)} - \frac{1}{2}\lambda^4 f^{(2)} f^{(2)} - 4\lambda^3 f^{(2)} f^{(1)} + 6\lambda^2 f^{(1)} f^{(1)}\big),$$

$$\begin{aligned}\lambda^5 f^{(5)} f^{(1)} =& 60\lambda (f^{(1)})^2 - 40\lambda^3 (f^{(2)})^2 + \lambda^5 (f^{(3)})^2 + \frac{d}{d\lambda}\big(\lambda^5 f^{(4)} f^{(1)} - \lambda^5 f^{(3)} f^{(2)} \\ &- 5\lambda^4 f^{(3)} f^{(1)} + 5\lambda^4 f^{(2)} f^{(2)} + 20\lambda^3 f^{(2)} f^{(1)} - 30\lambda^2 f^{(1)} f^{(1)}\big),\end{aligned}$$

$$\begin{aligned}\lambda^6 f^{(6)} f^{(1)} =& -360\lambda (f^{(1)})^2 + 300\lambda^3 (f^{(2)})^2 - 14\lambda^5 (f^{(3)})^2 + \frac{d}{d\lambda}\Big(\lambda^6 f^{(5)} f^{(1)} \\ &- 6\lambda^5 f^{(4)} f^{(1)} + 12\lambda^5 f^{(3)} f^{(2)} + 30\lambda^4 f^{(3)} f^{(1)} - 45\lambda^4 f^{(2)} f^{(2)} \\ &- 120\lambda^3 f^{(2)} f^{(1)} + 180\lambda^2 f^{(1)} f^{(1)} - \lambda^6 f^{(4)} f^{(2)} + \frac{1}{2}\lambda^6 f^{(3)} f^{(3)}\Big),\end{aligned}$$

$$\begin{aligned}\lambda^7 f^{(7)} f^{(1)} =& 2520\lambda(f^{(1)})^2 - 2520\lambda^3(f^2)^2 + 189\lambda^5(f^{(3)})^2 - \lambda^7(f^{(4)})^2 \\ & + \frac{d}{d\lambda}\Big(\lambda^7 f^{(6)} f^{(1)} - 7\lambda^6 f^{(5)} f^{(1)} + 42\lambda^5 f^{(4)} f^{(1)} - 84\lambda^5 f^{(3)} f^{(2)} \\ & - 210\lambda^4 f^{(3)} f^{(1)} + 315\lambda^4 f^{(2)} f^{(2)} + 840\lambda^3 f^{(2)} f^{(1)} \\ & - 1260\lambda^2 f^{(1)} f^{(1)} + 7\lambda^6 f^{(4)} f^{(2)} - 7\lambda^6 f^{(3)} f^{(3)} \\ & - \lambda^7 f^{(5)} f^{(2)} + \lambda^7 f^{(4)} f^{(3)}\Big).\end{aligned}$$

Proof. This follows from straightforward calculations. □

Remark 2.2. *In general, the term $\lambda^j f^{(j)} f^{(1)}$ can be rewritten in two parts, the quadratic form and derivative term, i.e.,*

$$\lambda^j f^{(j)} f^{(1)} = \sum_{s \le \frac{j+1}{2}, s \in N} b_{j,s} \lambda^{2s-1}(f^{(s)})^2 + \frac{d}{d\lambda}\Big(\sum_{i,l} c_{i,l}\lambda^{i+l} f^{(i)} f^{(l)}\Big).$$

The constants $b_{j,s}$ can quantified as in Lemma 2.4.

Lemma 2.5. *We have the following type-2 (i.e., $\lambda^{j+1} f^{(j)} f^{(2)}$) differentiation by parts formulas:*

$$\begin{aligned}&\lambda f f^{(2)} = -\lambda(f^{(1)})^2 + \frac{d}{d\lambda}\big(\lambda f f^{(1)} - \frac{1}{2} f^2\big), \\ &\lambda^2 f^{(1)} f^{(2)} = -\lambda(f^{(1)})^2 + \frac{d}{d\lambda}(\frac{1}{2}\lambda^2 f^{(1)} f^{(1)}), \\ &\lambda^4 f^{(3)} f^{(2)} = -2\lambda^3(f^{(2)})^2 + \frac{d}{d\lambda}(\frac{1}{2}\lambda^4 f^{(2)} f^{(2)}), \\ &\lambda^5 f^{(4)} f^{(2)} = 10\lambda^3(f^{(2)})^2 - \lambda^5(f^{(3)})^2 + \frac{d}{d\lambda}\big(\lambda^5 f^{(3)} f^{(2)} - \frac{5}{2}\lambda^4 f^{(2)} f^{(2)}\big), \\ &\lambda^6 f^{(5)} f^{(2)} = -60\lambda^3(f^{(2)})^2 + 9\lambda^5(f^{(3)})^2 + \frac{d}{d\lambda}\Big(\lambda^6 f^{(4)} f^{(2)} - 6\lambda^5 f^{(3)} f^{(2)} \\ &\quad + 15\lambda^4 f^{(2)} f^{(2)} - \frac{1}{2}\lambda^6 f^{(3)} f^{(3)}\Big), \\ &\lambda^7 f^{(6)} f^{(2)} = 420\lambda^3(f^{(2)})^2 - 84\lambda^5(f^{(3)})^2 + \lambda^7(f^{(4)})^2 + \frac{d}{d\lambda}\Big(\lambda^7 f^{(5)} f^{(2)} \\ &\quad - \lambda^7 f^{(4)} f^{(3)} + \frac{7}{2}\lambda^6 f^{(3)} f^{(3)} - 7\lambda^6 f^{(4)} f^{(2)} + 42\lambda^5 f^{(3)} f^{(2)} \\ &\quad - 105\lambda^4 f^{(2)} f^{(2)} + \frac{7}{2}\lambda^6 f^{(3)} f^{(3)}\Big).\end{aligned}$$

Proof. This follows by straightforward computations. □

Remark 2.3. *We note that term $\lambda^{j+1} f^{(j)} f^{(2)}$ can be decomposed into two parts, i.e., the quadratic form and derivative term respectively, i.e.,*

$$\lambda^{j+1} f^{(j)} f^{(2)} = \sum_{s \le \frac{j+2}{2}, s \in N} a_{j,s}\lambda^{2s-1}(f^{(s)})^2 + \frac{d}{d\lambda}\Big(\sum_{i,l} c_{i,l}\lambda^{i+l} f^{(i)} f^{(l)}\Big),$$

with the constants $a_{j,s}$ can be quantified by Lemma 2.5.

3. Proof of Monotonicity Formula in Theorem 1.2 in the cases $m \ge 5$

In this section we aim to establish the monotonicity formula in Theorem 1.2 for the general cases $m \ge 5$, provided that n is large enough. The proof in this section depends on some asymptotic estimates of leading order coefficients and hence we are not able to cover all dimensions. In the subsequent sections, for the special cases $m = 3$ or 4, we can obtain more precise estimates and cover all dimensions. The proof of Theorem 1.2 in the cases $m \ge 5$ starts from Lemma 3.1

which presents some symmetric structures inside the energy functional. With Lemma 3.4, the inner symmetric structures are boiled down to a series coercive integral inequalities as the one in Proposition 3.1. The proof of Proposition 3.1 can be reduced to prove a type of coercive differential inequalities as in Proposition 3.2. The Proposition 3.2 plays a key role in the proof Theorem 1.2 in the cases $m \geq 5$. To finish the proof of Proposition 3.2 we use the Emden-Fowler transform to uncover the inner structures and two general key identities (3.6),(3.7) to achieve the desired result.

In the following, we let $m \geq 3$ be fixed and $n \geq n(m)$ large enough. All the constants in this section depend only on m only. Similarly we mean $A \sim B$ if there are two constants C_1, C_2 depending on m only such that $C_1 A \leq B \leq C_2 B$. We use $\mathcal{O}(A)$ to denote $|\mathcal{O}(A)| \leq C|A|$.

Let the initial energy functional corresponding to equation (1.1) be

$$\mathcal{E}_0(u^\lambda; m) = \int_{B_1} \frac{1}{2}|\nabla^m \circ u^\lambda|^2 - \frac{1}{p+1}|u^\lambda|^{p+1} \tag{3.1}$$

where u^λ be defined at (1.16).

Throughout this section we assume that $p < p_{JL}(n,m)$ and let $k = \frac{2m}{p-1}$. By Proposition 1.1, we have that

$$2m < n - 2k < 2m + 2 + 2a_{n,m}\sqrt{n} < 2m + 2 + 2\sqrt{n}. \tag{3.2}$$

Thus for fixed m, $n \sim k$.

We begin with the formula for the derivative of the energy functional.

Lemma 3.1. *Let u be a weak solution of (1.1). Then*

$$\frac{d}{d\lambda}\mathcal{E}_0(u^\lambda; m) \begin{cases} = \int_{\partial B_1} \sum_{j=0, i+j=m-1}^{\frac{m}{2}-1} \Delta^i u^\lambda \frac{\partial}{\partial r}\Delta^j \frac{du^\lambda}{d\lambda} - \frac{\partial}{\partial r}\Delta^i u^\lambda \Delta^j \frac{du^\lambda}{d\lambda} & \text{if} \quad m \text{ is even,} \\ = \int_{\partial B_1} \sum_{j=0, i+j=m-1}^{\frac{m-3}{2}} \frac{\partial}{\partial r}\Delta^i u^\lambda \Delta^j \frac{du^\lambda}{d\lambda} - \Delta^i u^\lambda \frac{\partial}{\partial r}\Delta^j \frac{du^\lambda}{d\lambda} & \\ + \int_{\partial B_1} \frac{\partial}{\partial r}\Delta^{\frac{m-1}{2}} u^\lambda \Delta^{\frac{m-1}{2}} u^\lambda, & \text{if} \quad m \text{ is odd.} \end{cases}$$

Proof. The proof follows by integration by parts. We only prove the odd integer case, as the proof for the even integer case is similar.

Suppose m is odd. In this case

$$\mathcal{E}_0(u^\lambda; m) = \int_{B_1} \frac{1}{2}|\nabla\Delta^{\frac{m-1}{2}} u^\lambda|^2 - \frac{1}{p+1}|u^\lambda|^{p+1} \tag{3.3}$$

Taking derivative of the energy with respect to λ and integrating by part, we have

$$\begin{aligned} \frac{d}{d\lambda}\mathcal{E}_0(u^\lambda; m) &= \int_{B_1} \nabla\Delta^{\frac{m-1}{2}} u^\lambda \nabla\Delta^{\frac{m-1}{2}} \frac{du^\lambda}{d\lambda} - |u^\lambda|^{p-1}u^\lambda \frac{du^\lambda}{d\lambda} \\ &= \int_{\partial B_1} \frac{\partial}{\partial r}\Delta^{\frac{m-1}{2}} u^\lambda \Delta^{\frac{m-1}{2}} u^\lambda - \int_{B_1} \Delta^{\frac{m+1}{2}} u^\lambda \Delta^{\frac{m-1}{2}} \frac{du^\lambda}{d\lambda} - |u^\lambda|^{p-1}u^\lambda \frac{du^\lambda}{d\lambda} \\ &= \int_{\partial B_1} \frac{\partial}{\partial r}\Delta^{\frac{m-1}{2}} u^\lambda \Delta^{\frac{m-1}{2}} u^\lambda + \int_{\partial B_1} \frac{\partial}{\partial r}\Delta^{\frac{m+1}{2}} u^\lambda \Delta^{\frac{m-3}{2}} \frac{du^\lambda}{d\lambda} \\ &\quad - \Delta^{\frac{m+1}{2}} u^\lambda \frac{\partial}{\partial r}\Delta^{\frac{m-3}{2}} \frac{du^\lambda}{d\lambda} - \int_{B_1} \Delta^{\frac{m+3}{2}} u^\lambda \Delta^{\frac{m-3}{2}} \frac{du^\lambda}{d\lambda} - |u^\lambda|^{p-1}u^\lambda \frac{du^\lambda}{d\lambda}. \end{aligned}$$

Thus we get

$$\begin{aligned}\frac{d}{d\lambda}\mathcal{E}_0(u^\lambda;m) &= \int_{\partial B_1}\sum_{j=0,i+j=m-1}^{\frac{m-3}{2}}\frac{\partial}{\partial r}\Delta^i u^\lambda\Delta^j\frac{du^\lambda}{d\lambda}-\Delta^i u^\lambda\frac{\partial}{\partial r}\Delta^j\frac{du^\lambda}{d\lambda}\\ &\quad+\int_{B_1}(-\Delta)^m u^\lambda\frac{du^\lambda}{d\lambda}-|u^\lambda|^{p-1}u^\lambda\frac{du^\lambda}{d\lambda}+\int_{\partial B_1}\frac{\partial}{\partial r}\Delta^{\frac{m-1}{2}}u^\lambda\Delta^{\frac{m-1}{2}}u^\lambda\\ &= \int_{\partial B_1}\sum_{j=0,i+j=m-1}^{\frac{m-3}{2}}\frac{\partial}{\partial r}\Delta^i u^\lambda\Delta^j\frac{du^\lambda}{d\lambda}-\Delta^i u^\lambda\frac{\partial}{\partial r}\Delta^j\frac{du^\lambda}{d\lambda}\\ &\quad+\int_{\partial B_1}\frac{\partial}{\partial r}\Delta^{\frac{m-1}{2}}u^\lambda\Delta^{\frac{m-1}{2}}u^\lambda.\end{aligned}\tag{3.4}$$

□

The next two lemmas follow from direct calculations. The proofs are thus omitted. First we have

Lemma 3.2. *For $j\geq 0$, it holds*

$$(\partial_{rr}+\frac{n-1}{r}\partial_r)\frac{d^j u^\lambda}{d\lambda^j}\equiv\Big(-k^2\frac{d^j u^\lambda}{d\lambda^j}+\mathcal{O}(n-2k)\lambda\frac{d^{j+1}u^\lambda}{d\lambda^{j+1}}+\lambda^2\frac{d^{j+2}u^\lambda}{d\lambda^{j+2}}\Big).$$

Lemma 3.3. *For any integer $q\geq 1$, it holds*

$$\begin{aligned}(\partial_{rr}+\frac{n-1}{r}\partial_r)^q u^\lambda &= r^{-2q}(-k^2+\mathcal{O}(n-2k)\lambda\frac{d}{d\lambda}+\lambda^2\frac{d^2}{d\lambda^2})^q u^\lambda\\ &= r^{-2q}\sum_{i=0}^{2q}a_{i,2q}\lambda^i\frac{d^i u^\lambda}{d\lambda^i}\end{aligned}$$

where $a_{i,2q}$ behaviors different when i is even or odd, as follows

$$a_{2j,2q}=C_q^j(-k^2)^{q-j};\ \ a_{2j+1,2q}=(-k^2)^{q-j-1}\mathcal{O}(n-2k).\tag{3.5}$$

(Here we denote that $a_{i,2q}=0$ if $i<0$ or $i>2q$.)

With Lemma 3.3, we can express the polyharmonic operator in terms of the combination of $\lambda^j\Delta_\theta^i\frac{d^j u^\lambda}{d\lambda^j}$.

Lemma 3.4. *For any integer $l\geq 1$, it holds*

$$\begin{aligned}\Delta^l u^\lambda &\equiv \sum_{j=0}^{j}C_l^j\Delta_\theta^{l-j}(-k^2+\mathcal{O}(n-2k)\lambda\frac{d}{d\lambda}+\lambda^2\frac{d^2}{d\lambda^2})^j u^\lambda\\ &\equiv\sum_{j=0}^{l}\sum_{i=0}^{j}r^{-2j}C_l^jC_j^i a_{i,2j}\lambda^i\Delta_\theta^{l-j}\frac{d^i u^\lambda}{d\lambda^i}\end{aligned}$$

where $a_{i,2j}$ defined as in (3.5).

Proof. We prove it by induction on q. The case $q=1$ follows from Lemma 3.2. Suppose that the conclusion holds for q. Let us consider the case $q+1$. By Lemma 3.3

$$\begin{aligned}(\partial_{rr}+\frac{n-1}{r}\partial_r)^{q+1}u^\lambda &= (\partial_{rr}+\frac{n-1}{r}\partial_r)(\partial_{rr}+\frac{n-1}{r}\partial_r)^q u^\lambda \\ &= (\partial_{rr}+\frac{n-1}{r}\partial_r)\Big(r^{-2q}\sum_{i=0}^{2q}a_{i,2q}\lambda^i\frac{d^iu^\lambda}{d\lambda^i}\Big) \\ &= r^{-(2q+2)}\sum_{t=0}^{2q+2}\Big(\big(k^2-k(n-4q-1)-2q(2q+3-2n)\big)a_{t,2q} \\ &\quad +(n-2k-4q-1)a_{t-1,2q}+a_{t-2,2q}\Big)\lambda^t\frac{d^tu^\lambda}{d\lambda^t}.\end{aligned}$$

This gives the following recursive relation

$$\begin{aligned}a_{t,2(q+1)} =& \big(k^2-k(n-4q-1)-2q(2q+3-2n)\big)a_{t,2q} \\ &+(n-2k-4q-1)a_{t-1,2q}+a_{t-2,2q} \\ =& -k^2a_{t,2q}+\mathcal{O}(n-2k)a_{t-1,2q}+a_{t-2,2q}.\end{aligned}$$

Since $C_q^t+C_q^{t-1}=C_{q+1}^t$, by the inductive assumption and a direct calculation we have

$$\begin{aligned}a_{2j+1,2(q+1)} &\sim (k^2)^{q-j}\mathcal{O}(n-2k), \\ a_{2j,2(q+1)} &= C_{q+1}^j(-k^2)^{q+1-j}+C_q^{i-1}(-k^2)^{q-j}(\mathcal{O}(n-2k)^2+\mathcal{O}(1)) \\ &\sim C_{q+1}^j(-k^2)^{q+1-j}\end{aligned}$$

In the last step we have used the sharp estimate of $n-2k$ in (3.2). Therefore the conclusion holds for the $q+1$ case. This proves the lemma. □

Proposition 3.1. *Assume that $\frac{n+2m}{n-2m}<p<p_{JL}(n,m)$. For any integer $l\geq 1$, it holds*

$$\begin{aligned}&(-1)^{l+1}\int_{\partial B_1}\Delta^l u^\lambda\cdot(\lambda\frac{d^2u^\lambda}{d\lambda^2}+(l+1)\frac{du^\lambda}{d\lambda}) \\ &\geq Ck^{2l}\int_{\partial B_1}\lambda(\frac{du^\lambda}{d\lambda})^2+\frac{d}{d\lambda}\int_{\partial B_1}\sum_{i+j+2s\leq 2l+1}c_{i,j,s}\nabla_\theta^s\circ\frac{d^iu^\lambda}{d\lambda^i}\nabla_\theta^s\circ\frac{d^ju^\lambda}{d\lambda^j}.\end{aligned}$$

Remark 3.1. *The inequality is called coercive integral equality since in the estimates, the derivative term $\int_{\partial B_1}\nabla_\theta^s\circ\frac{d^iu^\lambda}{d\lambda^i}\nabla_\theta^s\circ\frac{d^ju^\lambda}{d\lambda^j}$ can be controlled by the positive term $\int_{\partial B_1}\lambda(\frac{du^\lambda}{d\lambda})^2$.*

Proof. With Lemma 3.4 and Proposition 3.2, we have

$$\begin{aligned}&(-1)^{l+1}\int_{\partial B_1}\Delta^l u^\lambda\cdot(\lambda\frac{d^2u^\lambda}{d\lambda^2}+(l+1)\frac{du^\lambda}{d\lambda}) \\ &=(-1)^{l+1}\int_{\partial B_1}C_l^j\Delta_\theta^{l-j}(-k^2+\mathcal{O}(n-2k)\lambda\frac{d}{d\lambda}+\lambda^2\frac{d^2}{d\lambda^2})^ju^\lambda\cdot(\lambda\frac{d^2u^\lambda}{d\lambda^2}+(l+1)\frac{du^\lambda}{d\lambda}) \\ &=(-1)^{l+1}\int_{\partial B_1}C_l^j(-1)^{l-j}\nabla_\theta^{l-j}\circ(-k^2+\mathcal{O}(n-2k)\lambda\frac{d}{d\lambda}+\lambda^2\frac{d^2}{d\lambda^2})^ju^\lambda \\ &\quad\cdot(\lambda\nabla_\theta^{l-j}\circ\frac{d^2u^\lambda}{d\lambda^2}+(l+1)\nabla_\theta^{l-j}\circ\frac{du^\lambda}{d\lambda}) \\ &=\sum_{j=0}^{l}\int_{\partial B_1}C_l^jk^{2(l-j)}(\nabla_\theta^{l-j}\circ\frac{d^{j+1}u^\lambda}{d\lambda^{j+1}})^2+\frac{d}{d\lambda}\int_{\partial B_1}\sum_{i+j+2s\leq 2l+1}c_{i,j,s}\nabla_\theta^s\circ\frac{d^iu^\lambda}{d\lambda^i}\nabla_\theta^s\circ\frac{d^ju^\lambda}{d\lambda^j}.\end{aligned}$$

Therefore the conclusion follows.

□

Proposition 3.2. *For any integer $l \geq 1$, $l \leq s$, we have*

$$(-1)^{l+1}\Big(-k^2+\mathcal{O}(n-2k)\lambda\frac{d}{d\lambda}+\lambda^2\frac{d^2}{d\lambda^2}\Big)^l u^\lambda\cdot(\lambda\frac{d^2u^\lambda}{d\lambda^2}+(s+1)\frac{du^\lambda}{d\lambda})$$
$$\geq C\lambda(\frac{du^\lambda}{d\lambda})^2+\frac{d}{d\lambda}(\sum_{0\leq i+j\leq l+1}c_{i,j}\lambda^{i+j}\frac{d^iu^\lambda}{d\lambda^i}\frac{d^ju^\lambda}{d\lambda^j})$$

Proof. It is more convenient to use Emden-Fowler variable. Let $\lambda = e^t$. Then we have

$$\lambda\frac{d}{d\lambda}=\frac{d}{dt};\ \ \lambda^2\frac{d^2}{d\lambda^2}=\frac{d^2}{dt^2}-\frac{d}{dt}.$$

In fact, more generally, denoting Id as the identity operator, we have

$$\lambda^j\frac{d^j}{d\lambda^j}=\frac{d}{dt}(\frac{d}{dt}-Id)(\frac{d}{dt}-2Id)\cdots(\frac{d}{dt}-(j-1)Id), j=1,2,\cdots,$$

and

$$\frac{d^j}{dt^j}=\sum_{i=1}^{j}c_i\lambda^i\frac{d^i}{d\lambda^i},$$

where c_i is identical to the coefficient of x^i in the expand expression of $x(x+1)\cdots(x+j-1)$. Thus we have

$$(-k^2+O(n-2k)\lambda\frac{d}{d\lambda}+\lambda^2\frac{d^2}{d\lambda^2})^j=(-k^2+\mathcal{O}(n-2k)\frac{d}{dt}+\frac{d^2}{dt^2})^j, j=1,2,3\cdots,$$
$$\lambda\frac{d^2u^\lambda}{d\lambda^2}+(s+1)\frac{du^\lambda}{d\lambda}=\frac{d^2u^\lambda}{dt^2}+s\frac{du^\lambda}{dt}.$$

We make two observations. First when $b-a$ is an odd integer, it holds

$$\frac{d^aw}{dt^a}\cdot\frac{d^bw}{dt^b}=\frac{d}{dt}\Big((\sum_{a\leq i\leq\frac{b-a-1}{2}}^{i+j=a+b-1}(-1)^{i-a}\frac{d^iw}{dt^i}\frac{d^jw}{dt^j})+\frac{1}{2}(-1)^{\frac{b-a-1}{2}}(\frac{d^{\frac{b-a+1}{2}}w}{dt^{\frac{b-a+1}{2}}})^2\Big) \tag{3.6}$$

all the term can be expressed by derivative terms.

Secondly, when $b-a$ is an even integer, we then have

$$\frac{d^aw}{dt^a}\cdot\frac{d^bw}{dt^b}=\frac{d}{dt}(\sum_{a\leq i\leq\frac{b-a-2}{2}}^{i+j=a+b-1}(-1)^{i-a}\frac{d^iw}{dt^i}\frac{d^jw}{dt^j})+(-1)^{\frac{a+b}{2}}(\frac{d^{\frac{a+b}{2}}w}{dt^{\frac{a+b}{2}}})^2. \tag{3.7}$$

Thus we have

$$(-1)^{l+1}\Big(-k^2+\mathcal{O}(n-2k)\lambda\frac{d}{d\lambda}+\lambda^2\frac{d^2}{d\lambda^2}\Big)^l u^\lambda\cdot(\lambda\frac{d^2u^\lambda}{d\lambda^2}+(s+1)\frac{du^\lambda}{d\lambda})$$
$$(-1)^{l+1}(-k^2+O(n-2k)\frac{d}{dt}+\frac{d^2}{dt^2})^ju^\lambda\cdot(\frac{d^2u^\lambda}{dt^2}+s\frac{du^\lambda}{dt})\equiv(-1)^{l+1}\sum_{i=0}^{l}a_{i,2l}\frac{d^iu^\lambda}{dt^i}(\frac{d^2u^\lambda}{dt^2}+s\frac{du^\lambda}{dt})$$
$$=\sum_{j\geq0}\big(a_{2j,2l}\frac{d^{2j}u^\lambda}{dt^{2j}}+a_{2j+1,2l}\frac{d^{2j+1}u^\lambda}{dt^{2j+1}}\big)(\frac{d^2u^\lambda}{dt^2}+s\frac{du^\lambda}{dt})$$
$$=(-1)^{l+1}\sum_{j\geq0}\big((-1)^{j-1}a_{2j,2l}+s(-1)^ja_{2j+1,2l}\big)(\frac{d^{j+1}u^\lambda}{dt^{j+1}})^2\quad+\frac{d}{dt}\Big(\sum_{i+j\leq2l+1}c_{i,j}\frac{d^iu^\lambda}{dt^i}\frac{d^ju^\lambda}{dt^j}\Big)$$
$$\equiv\sum_{j=0}^{l}C_l^jk^{2(l-j)}(\frac{d^{j+1}u^\lambda}{dt^{j+1}})^2+\frac{d}{dt}\Big(\sum_{i+j\leq2l+1}c_{i,j}\frac{d^iu^\lambda}{dt^i}\frac{d^ju^\lambda}{dt^j}\Big)\geq C(\frac{du^\lambda}{dt})^2+\frac{d}{dt}(\sum_{0\leq i+j\leq2l+1}c_{i,j}\frac{d^iu^\lambda}{dt^i}\frac{d^iu^\lambda}{dt^i})$$

Here we have used $a_{i,2l}$ defined at (3.5) and differentiation by parts formulas, i.e., (3.6) and (3.7).

□

Finally we give the proof of Theorem 1.2 in the case of $m \geq 5$.

Proof. As in Lemma 3.1, for m is odd or even, there are some common terms like

$$\int_{\partial B_1} \Delta^i u^\lambda \frac{\partial}{\partial r} \Delta^j \frac{du^\lambda}{d\lambda} - \frac{\partial}{\partial r} \Delta^i u^\lambda \Delta^j \frac{du^\lambda}{d\lambda}$$

appear. We now decompose these terms.

From the definition of u^λ, recalling that $k = \frac{2m}{p-1}$, by a direct calculation we have

$$\begin{aligned} \frac{\partial}{\partial r} \Delta^j u^\lambda &= \lambda \frac{d\Delta^j u^\lambda}{d\lambda} - (k+2j)\Delta^j u^\lambda \\ \frac{\partial}{\partial r} \Delta^j \frac{du^\lambda}{d\lambda} &= \lambda \frac{d^2 \Delta^j u^\lambda}{d\lambda^2} - (k+2j-1)\frac{d\Delta^j u^\lambda}{d\lambda} \end{aligned}$$

Let $w_j^\lambda = \Delta^j u^\lambda$ and $i + j = m - 1$. We obtain that

$$\begin{aligned} &\int_{\partial B_1} \Delta^i u^\lambda \frac{\partial}{\partial r} \Delta^j \frac{du^\lambda}{d\lambda} - \frac{\partial}{\partial r} \Delta^i u^\lambda \Delta^j \frac{du^\lambda}{d\lambda} \\ &= \int_{\partial B_1} \lambda \Delta^i u^\lambda \frac{d^2 \Delta^j u^\lambda}{d\lambda^2} + (2i - 2j + 1)\Delta^i u^\lambda \frac{d\Delta^j u^\lambda}{d\lambda} - \lambda \frac{d\Delta^i u^\lambda}{d\lambda}\frac{d\Delta^j u^\lambda}{d\lambda} \\ &= \int_{\partial B_1} 2\lambda \Delta^i u^\lambda \frac{d^2 \Delta^j u^\lambda}{d\lambda^2} + (2i - 2j + 2)\Delta^i u^\lambda \frac{d\Delta^j u^\lambda}{d\lambda} - \frac{d}{d\lambda}\big(\lambda \Delta^i u^\lambda \frac{d\Delta^j u^\lambda}{d\lambda}\big) \\ &= \int_{\partial B_1} 2\Big(\lambda \Delta^{m-2j-1} w_j^\lambda \frac{d^2 w_j^\lambda}{d\lambda^2} + (m-2j)\Delta^{m-2j-1} w_j^\lambda \frac{dw_j^\lambda}{d\lambda}\Big) - \frac{d}{d\lambda}\big(\lambda \Delta^i u^\lambda \frac{d\Delta^j u^\lambda}{d\lambda}\big) \\ &= \int_{\partial B_1} 2\Delta^{m-2j-1} w_j^\lambda \Big(\lambda \frac{d^2 w_j^\lambda}{d\lambda^2} + (m-2j)\frac{dw_j^\lambda}{d\lambda}\Big) - \frac{d}{d\lambda}\big(\lambda \Delta^i u^\lambda \frac{d\Delta^j u^\lambda}{d\lambda}\big). \end{aligned} \tag{3.8}$$

From Proposition 1.1, we see that the term $\int_{\partial B_1} \Delta^{m-2j-1} w_j^\lambda \Big(\lambda \frac{d^2 w_j^\lambda}{d\lambda^2} + (m-2j)\frac{dw_j^\lambda}{d\lambda}\Big)$ is coercive, i.e.

$$\begin{aligned} &\int_{\partial B_1} \Delta^{m-2j-1} w_j^\lambda \Big(\lambda \frac{d^2 w_j^\lambda}{d\lambda^2} + (m-2j)\frac{dw_j^\lambda}{d\lambda}\Big) \geq Ck^{2(m-2j-1)} \int_{\partial B_1} \lambda (\frac{dw_j^\lambda}{d\lambda})^2 \\ &+ \frac{d}{d\lambda} \int_{\partial B_1} \sum_{i+t+2s \leq 2(m-2j-1)+1} c_{i,t,s} \nabla_\theta^s \circ \frac{d^i w_j^\lambda}{d\lambda^i} \nabla_\theta^s \circ \frac{d^t w_j^\lambda}{d\lambda^t}. \end{aligned}$$

Combining (3.4) and (3.8), collecting the boundary terms, we obtain Theorem 1.2 for the cases $m \geq 5$. □

4. Monotonicity formula in the case $m = 4$: Part One

In this and next section, we aim to prove Theorem 1.2 for the case $m = 4$.

Throughout this section, $k = \frac{2m}{p-1} = \frac{8}{p-1}$ since $m = 4$ here. $\delta_j, a, b, \alpha, \beta$ are constants which are different from previous Section.

We first establish the following monotonicity formula using Lemma 3.1 and the alternative representation of tri-harmonic operator in Section 2.

Theorem 4.1. *Suppose that u is a solution of (1.1) for $m = 4$, then we have the following monotonicity formula*

$$\begin{aligned}\frac{d}{d\lambda}E(\lambda,x,u) &= \int_{\partial B_1}\sum_{j=1}^{4}(A_j+a_j)\lambda^{2j-1}(\frac{d^ju^\lambda}{d\lambda^j})^2 + 2\lambda\int_{\partial B_1}|\nabla_\theta\Delta_\theta\frac{du^\lambda}{d\lambda}|^2 \\ &+ \int_{\partial B_1}\sum_{s=1}^{3}(B_s+b_s)\lambda^{2s-1}|\nabla_\theta\frac{d^su^\lambda}{d\lambda^s}|^2 + \int_{\partial B_1}\sum_{l=1}^{2}(C_l+c_l)\lambda^{2l-1}(\Delta_\theta\frac{d^lu^\lambda}{d\lambda^l})^2 \\ &+ 2\int_{\partial B_1}\lambda|\nabla_\theta\frac{dv^\lambda}{d\lambda}|^2,\end{aligned} \tag{4.1}$$

where the constants are given as follows:

$$\begin{aligned}A_1+a_1 &= -4k^6+(-112+12n)k^5+(-12n^2+268n-1140)k^4 \\ &+(4n^3-200n^2+2056n-5600)k^3+(44n^3-1040n^2+7016n-14156)k^2 \\ &+(124n^3-2040n^2+10604n-17328)k+84n^3-1188n^2+5388n-7740,\end{aligned}$$

$$\begin{aligned}A_2+a_2 &= 28k^4+(448-56n)k^3+(32n^2-632n+2524)k^2 \\ &+(-4n^3+216n^2-2204n+6000)k-12n^3+324n^2-2364n+5100,\end{aligned} \tag{4.2}$$

$$A_3+a_3 = -28k^2+(-208+28n)k-4n^2+92n-400,$$

$$A_4+a_4 = 4,$$

$$\begin{aligned}B_1+b_1 &= 6k^4+(144-12n)k^3+(6n^2-204n+994)k^2+(60n^2-850n+2724)k+94n^2-1008n+2628, \\ B_2+b_2 &= -38k^2+(-300+38n)k-6n^2+136n-560, \\ B_3+b_3 &= 8,\end{aligned}$$

$$C_1+c_1 = -8k^2+(-86+8n)k-190+36n,$$

$$C_2+c_2 = 8.$$

In the next step, we need to obtain the coercive estimates up the some derivative terms on the boundary for three kinds of integrals in Theorem 4.1.

Theorem 4.2. *Suppose that u is a solution of (1.1) for $m = 4$. Then for $\frac{n+8}{n-8} < p < p_{JL}(n,4)$ we have the following monotonicity inequality: there exist positive constants K_1, K_2 and K_3 such that*

$$\begin{aligned}\int_{\partial B_1}\sum_{j=1}^{4}(A_j+a_j)\lambda^{2j-1}(\frac{d^ju^\lambda}{d\lambda^j})^2 &\geq K_1\int_{\partial B_1}\lambda(\frac{du^\lambda}{d\lambda})^2+\frac{d}{d\lambda}\int_{\partial B_1}I_1 \\ \int_{\partial B_1}\sum_{s=1}^{3}(B_s+b_s)\lambda^{2s-1}(\nabla_\theta\frac{d^su^\lambda}{d\lambda^s})^2 &\geq K_2\int_{\partial B_1}\lambda|\nabla_\theta\frac{du^\lambda}{d\lambda}|^2+\frac{d}{d\lambda}\int_{\partial B_1}I_2 \\ \int_{\partial B_1}\sum_{l=1}^{2}(C_l+c_l)\lambda^{2l-1}(\Delta_\theta\frac{d^lu^\lambda}{d\lambda^l})^2 &\geq K_3\int_{\partial B_1}\lambda(\Delta_\theta\frac{du^\lambda}{d\lambda})^2+\frac{d}{d\lambda}\int_{\partial B_1}I_3\end{aligned} \tag{4.3}$$

where I_1, I_2 and I_3 are some boundary integrals of the type $\sum_{i+j+2s\leq 7}C_{i,j,s}\nabla_\theta^s\frac{d^iu^\lambda}{d\lambda^i}\nabla_\theta^s\frac{d^ju^\lambda}{d\lambda^j}$.

Theorem 1.2 for the case $m = 4$ follows from Theorems 4.1 and 4.2.

In this section we give the proof of Theorem 4.1 and we leave the proof of Theorem 4.2 to the next section.

To prove Theorem 4.1, we start from Lemma 3.1,

$$\begin{aligned}\frac{d\mathcal{E}(u^\lambda;4)}{d\lambda} &= \underbrace{\int_{\partial B_1} \lambda z^\lambda \frac{d^2u^\lambda}{d\lambda^2} + 7z^\lambda\frac{du^\lambda}{d\lambda} - \lambda\frac{dz^\lambda}{d\lambda}\frac{du^\lambda}{d\lambda}} \\ &+ \underbrace{\int_{\partial B_1} \lambda w^\lambda\frac{d^2v^\lambda}{d\lambda^2} + 3w^\lambda\frac{dv^\lambda}{d\lambda} - \lambda\frac{dw^\lambda}{d\lambda}\frac{dv^\lambda}{d\lambda}} \\ &:= \overline{E}_{d_1}(u^\lambda,1) + \overline{E}_{d_2}(u^\lambda,1)\end{aligned} \tag{4.4}$$

where $\overline{E}_{d_1}(u^\lambda,1)$ and $\overline{E}_{d_2}(u^\lambda,1)$ are defined at the last equality. Here

$$\Delta u^\lambda = v^\lambda, \Delta v^\lambda = w^\lambda, \Delta w^\lambda = z^\lambda. \tag{4.5}$$

In the next two subsections, we analyze the precise terms in $\overline{E}_{d_1}(u^\lambda,1)$ and $\overline{E}_{d_2}(u^\lambda,1)$.

4.1. **The term $\overline{E}_{d_2}(u^\lambda,1)$ in (4.4).** By (4.4)

$$\overline{E}_{d_2}(u^\lambda,1) = \int_{\partial B_1} (\lambda w^\lambda\frac{d^2v^\lambda}{d\lambda^2} + 3w^\lambda\frac{dv^\lambda}{d\lambda} - \lambda\frac{dw^\lambda}{d\lambda}\frac{dv^\lambda}{d\lambda}). \tag{4.6}$$

Since $w^\lambda = \Delta v^\lambda = \partial_{rr}v^\lambda + \frac{n-1}{r}\partial_r v^\lambda + r^{-2}\mathbf{div}_\theta(\nabla_\theta v^\lambda)$, in view of (2.3), on the boundary ∂B_1, we have

$$\begin{aligned}w^\lambda &= \lambda^2\frac{d^2v^\lambda}{d\lambda^2} + \lambda\frac{dv^\lambda}{d\lambda}(n-1-\frac{16}{p-1}) + u^\lambda\frac{8}{p-1}(2+\frac{8}{p-1}-n) + \mathbf{div}_\theta(\nabla_\theta v^\lambda) \\ &:= \lambda^2\frac{d^2v^\lambda}{d\lambda^2} + \alpha\lambda\frac{dv^\lambda}{d\lambda} + \beta v^\lambda + \mathbf{div}_\theta(\nabla_\theta v^\lambda),\end{aligned}$$

where

$$\alpha = n-1-\frac{16}{p-1}, \beta = 2+\frac{8}{p-1}-n. \tag{4.7}$$

Integrating by parts several times, one deduces that

$$\begin{aligned}\overline{E}_{d_2}(u^\lambda,1) &= \int_{\partial B_1} \left[2\lambda^3(\frac{d^2v^\lambda}{d\lambda^2})^2 + (2\alpha-2\beta-4)\lambda(\frac{dv^\lambda}{d\lambda})^2\right] \\ &+ \quad \frac{d}{d\lambda}\int_{\partial B_1}\Big[\frac{\beta}{2}\frac{d}{d\lambda}(\lambda(v^\lambda)^2)^2 - \frac{1}{2}\lambda^3\frac{d}{d\lambda}(\frac{dv^\lambda}{d\lambda})^2 + (\frac{\beta}{2}+2)(v^\lambda)^2 \\ &\quad -\frac{1}{2}\frac{d}{d\lambda}(\lambda|\nabla_\theta v^\lambda|^2) - \frac{1}{2}|\nabla_\theta v^\lambda|^2\Big] + 2\int_{\partial B_1}\lambda|\nabla_\theta\frac{dv^\lambda}{d\lambda}|^2.\end{aligned} \tag{4.8}$$

Let us further investigate the inner structure of $\overline{E}_{d_2}(u^\lambda,1)$, which allows us to obtain precise information for our construction of the monotonicity formula under the desired condition. Since $v^\lambda = \Delta u^\lambda$, on the boundary ∂B_1, by a direct calculation, one has

$$\begin{aligned}\frac{dv^\lambda}{d\lambda} &= \lambda^2\frac{d^3u^\lambda}{d\lambda^3} + (\alpha+2)\lambda\frac{d^2u^\lambda}{d\lambda^2} + (\alpha+\beta)\frac{du^\lambda}{d\lambda} + \Delta_\theta\frac{du^\lambda}{d\lambda}, \\ \frac{d^2v^\lambda}{d\lambda^2} &= \lambda^2\frac{d^4u^\lambda}{d\lambda^4} + (\alpha+4)\lambda\frac{d^3u^\lambda}{d\lambda^3} + (2\alpha+\beta+2)\frac{d^2u^\lambda}{d\lambda^2} + \Delta_\theta\frac{d^2u^\lambda}{d\lambda^2}.\end{aligned}$$

Differentiating by parts, one deduces that

$$\begin{aligned}&\int_{\partial B_1}(2\alpha-2\beta-4)\lambda(\frac{dv^\lambda}{d\lambda})^2 + 2\lambda^3(\frac{d^2v^\lambda}{d\lambda^2})^2 \\ &= \int_{\partial B_1}\sum_{j=1}^4 a_j\lambda^{2j-1}(\frac{d^ju^\lambda}{d\lambda^j})^2 + \Big(\sum_{s=1}^3 b_s\lambda^{2s-1}(\nabla_\theta\frac{d^su^\lambda}{d\lambda^s})^2\Big) + \sum_{l=1}^2 c_l\lambda^{2l-1}(\Delta_\theta\frac{d^lu^\lambda}{d\lambda^l})^2 \\ &+ \frac{d}{d\lambda}\Big(\sum_{i,j}c_{i,j}\lambda^{i+j}\frac{d^iu^\lambda}{d\lambda^i}\frac{d^ju^\lambda}{d\lambda^j}\Big) + \frac{d}{d\lambda}\Big(\sum_{i,j}e_{i,j}\lambda^{i+j}\nabla_\theta\frac{d^iu^\lambda}{d\lambda^i}\nabla_\theta\frac{d^ju^\lambda}{d\lambda^j}\Big),\end{aligned} \tag{4.9}$$

where $c_{i,j}$ and $e_{i,j}$ are some coefficients.

Here

$$\begin{aligned} c_1 = 2\alpha - 2\beta - 4, c_2 = 2; b_1 = -2 - 2\beta, b_2 = 14 - 2\beta, b_3 = 2; \\ a_1 = -2\alpha^3 + (2\beta + 8)\alpha^2 + (2\beta^2 - 8)\alpha - 2\beta^3 - 8\beta^2 - 8\beta, \\ a_2 = 2\alpha^3 + (-16 - 2\beta)\alpha^2 + 16\alpha + 6\beta^2 + 32\beta + 40, \\ a_3 = 2\alpha^2 - 2\alpha - 6\beta - 28, a_4 = 2 \end{aligned} \tag{4.10}$$

Remark 4.1. *By (4.8), the first term of the above integral is positive. Recall that $v^\lambda = \Delta u^\lambda$. Then we have*

$$\begin{aligned} \overline{E}_{d_2}(u^\lambda, 1) \geq \frac{d}{d\lambda} \int_{\partial B_1} \Big[\frac{\beta}{2} \frac{d}{d\lambda}(\lambda(\Delta u^\lambda)^2)^2 - \frac{1}{2}\lambda^3 \frac{d}{d\lambda}(\frac{d\Delta u^\lambda}{d\lambda})^2 + \frac{\beta}{2}(\Delta u^\lambda)^2 \\ - \frac{1}{2}\frac{d}{d\lambda}(\lambda|\nabla_\theta \Delta u^\lambda|^2) - \frac{1}{2}|\nabla_\theta \Delta u^\lambda|^2 \Big]. \end{aligned} \tag{4.11}$$

If we use this estimate alone, we can not construct the desired monotonicity formula for all n with $\frac{n+8}{n-8} < p < p_{JL}(n, 4)$. More precisely, when n is small, it seems that, under the condition $\frac{n+8}{n-8} < p < p_{JL}(n, 4)$, the desired monotonicity formula can not hold.

4.2. **The term $\overline{E}_{d_1}(u^\lambda, 1)$ in (4.4).** We now analyze the term $\overline{E}_{d_1}(u^\lambda, 1)$ in (4.4):

$$\begin{aligned} \overline{E}_{d_1}(u^\lambda, 1) &= \int_{\partial B_1} \big(\lambda z^\lambda \frac{d^2u^\lambda}{d\lambda^2} + 7z^\lambda \frac{du^\lambda}{d\lambda} - \lambda \frac{dz^\lambda}{d\lambda}\frac{du^\lambda}{d\lambda}\big) \\ &= \int_{\partial B_1} \big(\lambda \Delta^3 u^\lambda \frac{d^2u^\lambda}{d\lambda^2} + 7\Delta^3 u^\lambda \frac{du^\lambda}{d\lambda} - \lambda \frac{d\Delta^3 u^\lambda}{d\lambda}\frac{du^\lambda}{d\lambda}\big). \end{aligned}$$

By the computations in Section 2, we have the following representations of $\Delta^3 u^\lambda$ in terms of F_0, F_1, F_2 in (2.9), i.e.,

$$\Delta^3 u^\lambda = F_0(u) + \Delta_\theta F_1(u) + \Delta_\theta^2 F_2(u) + \Delta_\theta^3 F_3(u).$$

We split the calculations into three subsections corresponding to F_0, F_1, F_2 respectively.

4.2.1. *The integral corresponding to the operator F_0.* With F_0 defined in (2.10), we divide it into two parts, given by

$$\begin{aligned} F_{01} &:= \int_{\partial B_1} \big(\lambda F_0(u^\lambda)\frac{d^2u^\lambda}{d\lambda^2}\big), \\ F_{02} &:= \int_{\partial B_1} \big(7F_0(u^\lambda) - \lambda \frac{dF_0(u^\lambda)}{d\lambda}\big)\frac{du^\lambda}{d\lambda}. \end{aligned}$$

Recall that

$$F_0(u^\lambda) = \sum_{j=0}^{6} k_j \lambda^j \frac{d^j u^\lambda}{d\lambda^j}.$$

Hence,

$$\begin{aligned} 7F_0(u^\lambda) - \lambda \frac{dF_0(u^\lambda)}{d\lambda} = -k_6\lambda^7 \frac{d^7u^\lambda}{d\lambda^7} + (k_6 - k_5)\lambda^6 \frac{d^6u^\lambda}{d\lambda^6} + (2k_5 - k_4)\lambda^5 \frac{d^5u^\lambda}{d\lambda^5} \\ + (3k_4 - k_3)\lambda^4 \frac{d^4u^\lambda}{d\lambda^4} + (4k_3 - k_2)\lambda^3 \frac{d^3u^\lambda}{d\lambda^3} + (5k_2 - k_1)\lambda^2 \frac{d^2u^\lambda}{d\lambda^2} + (6k_1 - k_0)\lambda \frac{du^\lambda}{d\lambda} + 7k_0 u^\lambda. \end{aligned}$$

Let $f = u^\lambda$. We use the following two identities from Section 2: the first one is

$$\begin{aligned} \sum_{j=0}^{6} \lambda^{j+1} f^{(j)} f^{(2)} &= \lambda^7 (f^{(4)})^2 + (9k_5 - k_4 - 84)\lambda^5 (f^{(3)})^2 \\ &+ (-60k_5 + 10k_4 - 2k_3 + k_2 + 420)\lambda^3 (f^{(2)})^2 - k_0\lambda (f^{(1)})^2 + +k_1\lambda^2 f^{(2)} f^{(1)} + \Big(R_1\Big)', \end{aligned}$$

where the term R_1 is not important but can be determined by

$$\begin{aligned}R_1 =& \lambda^7 f^{(5)} f^{(2)} - \lambda^7 f^{(4)} f^{(3)} + (k_5-7)\lambda^6 f^{(4)} f^{(2)} + (7-\frac{k_5}{2})\lambda^6 (f^{(3)})^2 \\ &+ (-6k_5+k_4+42)\lambda^5 f^{(3)} f^{(2)} + (15k_5 - \frac{5}{2}k_4 + \frac{1}{2}k_3 - 105)\lambda^4 (f^{(2)})^2 \\ &+ k_0 \lambda f f^{(1)} - \frac{1}{2}k_0 f^2 + \frac{k_1}{2}\lambda^2 (f^{(1)})^2\end{aligned}$$

The second one is

$$\begin{aligned}&- k_6\lambda^7 f^{(7)} f^{(1)} + (k_6-k_5)\lambda^6 f^{(6)} f^{(1)} + (2k_5-k_4)\lambda^5 f^{(5)} f^{(1)} + (3k_4-k_3)\lambda^4 f^{(4)} f^{(1)} \\ &+ (4k_3-k_2)\lambda^3 f^{(3)} f^{(1)} + (5k_2-k_1)\lambda^2 f^{(2)} f^{(1)} + (6k_1-k_0)\lambda (f^{(1)})^2 + 7k_0 f f^{(1)} \\ =& \lambda^7 (f^{(4)})^2 + (14k_5 - k_4 - 138)\lambda^5 (f^{(3)})^2 + (22-k_5)\lambda^6 f^{(4)} f^{(3)} \\ &+ (-380k_5 + 58k_4 - 10k_3 + k_2 + 2820)\lambda^3 (f^{(2)})^2 + (6k_1 - k_0)\lambda (f^{(1)})^2 \\ &+ (-480k_5 + 96k_4 - 24k_3 + 8k_2 - k_1 + 2880)\lambda^2 f^{(2)} f^{(1)} + \Big(R_2\Big)'.\end{aligned}$$

Here the term R_2 is not important and can be determined by

$$\begin{aligned}R_2 = \Big[&- \lambda^7 f^{(6)} f^{(1)} + \lambda^7 f^{(5)} f^{(2)} - \lambda^7 f^{(4)} f^{(3)} + (8-k_5)\lambda^6 f^{(5)} f^{(1)} + (k_5-15)\lambda^6 f^{(4)} f^{(2)} \\ &- (14k_5 - k_4 - 138)\lambda^5 f^{(3)} f^{(2)} + (55k_5 - \frac{13}{2}k_4 + \frac{1}{2}k_3 - 480)\lambda^4 (f^{(2)})^2 \\ &+ (8k_5 - k_4 - 48)\lambda^5 f^{(4)} f^{(1)} + (-40k_5 + 8k_4 - k_3 + 240)\lambda^4 f^{(3)} f^{(1)} \\ &+ (160k_5 - 32k_4 + 8k_3 - k_2 - 960)\lambda^3 f^{(2)} f^{(1)} + \frac{7}{2}k_0 f^2 \Big]'.\end{aligned}$$

Combining these identities with the differentiation by parts formulas in Section 2, we obtain the integral corresponding to the operator A:

$$\begin{aligned}\mathcal{F}_0 =& \int_{\partial B_1} \Big(\lambda F_0(u^\lambda)\frac{d^2u^\lambda}{d\lambda^2} + 7F_0(u^\lambda)\frac{du^\lambda}{d\lambda} - \lambda \frac{dF_0(u^\lambda)}{d\lambda}\frac{du^\lambda}{d\lambda}\Big) \\ =& \mathcal{F}_{01} + \int_{\partial B_1} \Big[A_4\lambda^7 (\frac{d^4u^\lambda}{d\lambda^4})^2 + A_3\lambda^5 (\frac{d^3u^\lambda}{d\lambda^3})^2 + A_2\lambda^3 (\frac{d^2u^\lambda}{d\lambda^2})^2 + A_1\lambda (\frac{du^\lambda}{d\lambda})^2\Big],\end{aligned}$$

where

$$\begin{aligned}A_4 = 2, A_1 = 480k_5 - 96k_4 + 24k_3 - 8k_2 + 6k_1 - 2k_0 - 2280k_6, \\ A_3 = 26k_5 - 2k_4 - 288k_6, A_2 = -440k_5 + 68k_4 - 12k_3 + 2k_2 + 3240k_6\end{aligned} \tag{4.12}$$

and the part $\mathcal{F}_{01}$ is of the form $\frac{d}{d\lambda}$ for some function. Precisely we have

$$\begin{aligned}\mathcal{F}_{01} := \frac{d}{d\lambda}\int_{\partial B_1} \Big[&- \lambda^7 \frac{d^6u^\lambda}{d\lambda^6}\frac{du^\lambda}{d\lambda} + 2\lambda^7 \frac{d^5u^\lambda}{d\lambda^5}\frac{d^2u^\lambda}{d\lambda^2} - 2\lambda^7 \frac{d^4u^\lambda}{d\lambda^4}\frac{d^3u^\lambda}{d\lambda^3} \\ &+ (8-k_5)\lambda^6 \frac{d^5u^\lambda}{d\lambda^5}\frac{du^\lambda}{d\lambda} + (-20k_5 + 2k_4 + 180)\lambda^5 \frac{d^3u^\lambda}{d\lambda^3}\frac{d^2u^\lambda}{d\lambda^2} \\ &+ (70k_5 - 9k_4 + k_3 - 585)\lambda^4 (\frac{d^2u^\lambda}{d\lambda^2})^2 + (8k_5 - k_4 - 48)\lambda^5 \frac{d^4u^\lambda}{d\lambda^4}\frac{du^\lambda}{d\lambda} \\ &+ (-40k_5 + 8k_4 - k_3 + 240)\lambda^4 \frac{d^3u^\lambda}{d\lambda^3}\frac{du^\lambda}{d\lambda} \\ &+ (160k_5 - 32k_4 + 8k_3 - k_2 - 960)\lambda^3 \frac{d^2u^\lambda}{d\lambda^2}\frac{du^\lambda}{d\lambda} + 3k_0 (u^\lambda)^2 \\ &+ (2k_5 - 22)\lambda^6 \frac{d^4u^\lambda}{d\lambda^4}\frac{d^2u^\lambda}{d\lambda^2} + (18-k_5)\lambda^6 (\frac{d^3u^\lambda}{d\lambda^3})^2 + k_0\lambda u_e^\lambda \frac{du^\lambda}{d\lambda} \\ &+ (-240k_5 + 48k_4 - 12k_3 + 4k_2 + \frac{1}{2}k_1 + 1440)\lambda^2 (\frac{du^\lambda}{d\lambda})^2\Big]\end{aligned}$$

where $k_j, j = 1, ..., 4$ are defined at (2.11).

4.2.2. *The integral corresponding to the operator $\Delta_\theta F_1$.* Let us define

$$I_1(F_1) := \int_{\partial B_1} \lambda^1 \Delta_\theta F_1(u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} = \int_{\partial B_1} \lambda^{j+1} \sum_{j=0}^{4} (-t_j) \Delta_\theta \frac{d^j u^\lambda}{d\lambda^j} \frac{d^2 u^\lambda}{d\lambda^2}$$
$$= \int_{\partial B_1} \sum_{j=0}^{4} t_j \lambda^{j+1} \nabla_\theta \frac{d^j u^\lambda}{d\lambda^j} \nabla_\theta \frac{d^2 u^\lambda}{d\lambda^2},$$

and

$$I_2(F_1) : = \int_{\partial B_1} \Big(7\Delta_\theta F_1(u^\lambda) - \lambda \frac{d}{d\lambda} \Delta_\theta F_1(u)\Big) \frac{du^\lambda}{d\lambda}$$
$$= \int_{\partial B_1} \Big(7\Delta_\theta \big(\sum_{j=0}^{4} (-t_j) \lambda^j \frac{d^j u^\lambda}{d\lambda^j}\big) - \lambda \frac{d}{d\lambda} \Delta_\theta \big(\sum_{j=0}^{4} (-t_j) \lambda^j \frac{d^j u^\lambda}{d\lambda^j}\big)\Big) \frac{du^\lambda}{d\lambda}$$
$$= \int_{\partial B_1} \sum_{j=1}^{5} t_{0j} \lambda^j \nabla_\theta \frac{d^j u^\lambda}{d\lambda^j} \nabla_\theta \frac{du^\lambda}{d\lambda},$$

where

$$t_{05} = -t_4, t_{04} = 3t_4 - t_3, t_{03} = 4t_3 - t_2, t_{02} = 5t_2 - t_1, t_{01} = 6t_1 - t_0,$$

and t_j is defined in (2.12). By the formulas in Lemmas 2.4-2.5, we obtain that

$$I_1(F_1) + I_2(F_1) = \int_{\partial B_1} B_1 \lambda (\nabla_\theta \frac{du^\lambda}{d\lambda})^2 + B_2 \lambda^3 (\nabla_\theta \frac{d^2 u^\lambda}{d\lambda^2})^2 + B_3 \lambda^5 (\nabla_\theta \frac{d^3 u^\lambda}{d\lambda^3})^2$$
$$+ \frac{d}{d\lambda} \int_{\partial B_1} \Big(\sum_{0 \le i,j \le 2} b_{i,j} \lambda^{i+j} \nabla_\theta \frac{d^i u^\lambda}{d\lambda^i} \nabla_\theta \frac{d^j u^\lambda}{d\lambda^j} \Big),$$

where

$$B_1 = -2t_0 + 6t_1 - 8t_2 + 24t_3 - 96t_4, B_2 = 2t_2 - 12t_3 + 68t_4, B_3 = -2t_4 = 6. \tag{4.13}$$

The coefficients $b_{i,j}$ are determined by t_j but not given here since they are not the coefficients of the key terms.

4.2.3. *The integral corresponding to the operator $\Delta_\theta^2 F_2$.* Recall that the sphere representation of triple-harmonic operator, i.e., (2.9) and (2.10). Let us define the following

$$I_1(F_2) : = \int_{\partial B_1} \lambda^{j+1} \Delta_\theta^2 F_2(u^\lambda) \frac{d^2 u^\lambda}{d\lambda^2} = \int_{\partial B_1} \lambda^1 \sum_{j=0}^{2} e_j \Delta_\theta^2 \frac{d^2 u^\lambda}{d\lambda^2} \frac{d^2 u^\lambda}{d\lambda^2}$$
$$= \int_{\partial B_1} \lambda^{j+1} \Delta_\theta \frac{d^j u^\lambda}{d\lambda^j} \Delta_\theta \frac{d^2 u^\lambda}{d\lambda^2}$$

and

$$I_2(F_2) : = \int_{\partial B_1} \Big(7\Delta_\theta F_2(u^\lambda) - \lambda \frac{d}{d\lambda} \Delta_\theta^2 F_2(u^\lambda)\Big) \frac{du^\lambda}{d\lambda}$$
$$= \int_{\partial B_1} \Big(7\Delta_\theta \big(\sum_{j=0}^{2} e_j \lambda^j \Delta_\theta^2 \frac{d^j u^\lambda}{d\lambda^j}\big) - \lambda \frac{d}{d\lambda} \Delta_\theta^2 \big(\sum_{j=0}^{2} e_j \lambda^j \Delta_\theta^2 \frac{d^j u^\lambda}{d\lambda^j}\big)\Big) \frac{du^\lambda}{d\lambda}$$
$$= \sum_{j=1}^{3} \int_{\partial B_1} e_{0j} \Delta_\theta \frac{d^j u^\lambda}{d\lambda^j} \Delta_\theta \frac{du^\lambda}{d\lambda},$$

where

$$e_{03} = -e_2, e_{02} = 5e_2 - e_1, e_{01} = 6e_1 - e_0$$

and e_j is defined in (2.13).

Again by formulas in Lemmas 2.4-2.5, we obtain that

$$I_1(F_2)+I_2(F_2)=\int_{\partial B_1} C_1\lambda(\Delta_\theta\frac{du^\lambda}{d\lambda})^2+C_2\lambda^3(\Delta_\theta\frac{d^2u^\lambda}{d\lambda^2})^2 \\ +\frac{d}{d\lambda}\int_{\partial B_1}\Big(\sum_{0\le i,j\le 1} C_{i,j}\lambda^{i+j}\Delta_\theta\frac{d^iu^\lambda}{d\lambda^i}\Delta_\theta\frac{d^ju^\lambda}{d\lambda^j}\Big),$$

where

$$C_1=-2e_0+6e_1-8e_2, C_2=2e_2. \tag{4.14}$$

4.2.4. *The integral corresponding to the operator $\Delta_\theta^3 F_3$.* Using the formulas in Lemmas 2.4-2.5, we obtain that

$$\begin{aligned} I(F_3):=&\int_{\partial B_1}\Big(\lambda\Delta_\theta^3F_3(u^\lambda)\frac{d^2u^\lambda}{d\lambda^2}+7\Delta_\theta^3F_3(u^\lambda)\frac{du^\lambda}{d\lambda}-\lambda\frac{d\Delta_\theta^3F_3(u^\lambda)}{d\lambda}\frac{du^\lambda}{d\lambda}\Big)\\ =&\int_{\partial B_1}-\lambda\nabla_\theta\Delta_\theta u^\lambda\nabla_\theta\Delta_\theta\frac{d^2u^\lambda}{d\lambda^2}-7\nabla_\theta\Delta_\theta u^\lambda\nabla_\theta\Delta_\theta\frac{du^\lambda}{d\lambda}+\lambda|\nabla_\theta\Delta_\theta u^\lambda|^2\\ =&\frac{d}{d\lambda}\Big[\int_{\partial B_1}-\lambda\nabla_\theta\Delta_\theta u^\lambda\nabla_\theta\Delta_\theta\frac{du^\lambda}{d\lambda}-3\big(\nabla_\theta\Delta_\theta u^\lambda\big)^2+2\lambda\int_{\partial B_1}\big(\nabla_\theta\Delta_\theta\frac{du^\lambda}{d\lambda}\big)^2\Big].\end{aligned}$$

4.2.5. *The monotonicity formula and the proof of Theorem 4.1 for $m=4$.* From (4.4), we sum up the terms $\overline{E}_{d_2}$ (calculated in Sections 4.3.1, 4.3.2, 4.3.3, 4.3.4) and $\overline{E}_{d_1}$ calculated in Section 4.1.

Therefore, we obtain that for the revised energy functional

$$\begin{aligned}\mathcal{E}(u^\lambda;4):=&\int_{B_1}\frac12|\Delta^2u^\lambda|^2-\frac{1}{p+1}|u^\lambda|^{p+1}\\ &+\int_{\partial B_1}\Big(\sum_{i,j\ge0,i+j\le7}C^0_{i,j}\lambda^{i+j}\frac{d^iu^\lambda}{d\lambda^i}\frac{d^ju^\lambda}{d\lambda^j}+\sum_{i,j\ge0,i+j\le5}C^1_{i,j}\lambda^{i+j}\nabla_\theta\frac{d^iu^\lambda}{d\lambda^i}\nabla_\theta\frac{d^ju^\lambda}{d\lambda^j}\\ &+\sum_{i,j\ge0,i+j\le3}C^2_{i,j}\lambda^{i+j}\Delta_\theta\frac{d^iu^\lambda}{d\lambda^i}\Delta_\theta\frac{d^ju^\lambda}{d\lambda^j}+\sum_{i,j\ge0,i+j\le1}C^3_{i,j}\lambda^{i+j}\nabla_\theta\Delta_\theta\frac{d^iu^\lambda}{d\lambda^i}\nabla_\theta\Delta_\theta\frac{d^ju^\lambda}{d\lambda^j}\Big),\end{aligned}$$

where the constants $C^k_{i,j}$ determined in calculation of $\overline{E}_{d_1}$ and $\overline{E}_{d_2}$ of the above three subsections, we have the identity (4.1). Thus we obtain Theorem 4.1.

5. Monotonicity formula in the case $m=4$: Part Two

In this section we prove Theorem 4.2. We assume that $m=4, k=\frac{8}{p-1}$ and $\frac{n+8}{n-8}<p<p_{JL}(n)$. First, we have

Lemma 5.1. *If $p>\frac{n+8}{n-8}$, then*

$$\sum_{l=1}^{2}(C_l+c_l)\lambda^{2l-1}(\Delta_\theta\frac{d^lu^\lambda}{d\lambda^l})^2\ge0.$$

Proof. We known from (4.14), (2.13) and (4.9) that

$$C_1=-6k^2+(-72+6n)k-178+30n;\ \ C_2+c_2=8,$$

where $k=:\frac{8}{p-1}$. By this one has $p>\frac{n+8}{n-8}$ is equivalent to $0<k<\frac{n-8}{2}$. By finding the roots (denoted by $r_1(n), r_2(n)$) of the equation

$$-6k^2+(-72+6n)k-178+30n=0$$

about variable k, we get that

$$\begin{aligned} r_1(n) : &= \frac{1}{2}n - 6 - \frac{1}{6}\sqrt{9n^2 - 36n + 228} \\ &= \frac{1}{6}(3n - 36 - \sqrt{9n^2 - 36n + 228}) \\ &= \frac{1}{6}\frac{(3n-36)^2 - (9n^2 - 36n + 228)}{3n - 36 + \sqrt{9n^2 - 36n + 228}} \\ &= \frac{-30n + 178}{3n - 36 + \sqrt{9n^2 - 36n + 228}} < 0 \ \text{ for } \ n \geq 6 \end{aligned}$$

and

$$r_2(n) := \frac{1}{2}n - 6 + \frac{1}{6}\sqrt{9n^2 - 36n + 228} > \frac{1}{2}(n-8),$$

therefore we obtain that $C_1 > 0$ if $0 < k < \frac{n-8}{2}$. Recalling that $c_1 = 2\alpha - 2\beta - 4 > 0$, then the conclusion follows. □

Lemma 5.2. *If $\frac{n+8}{n-8} < p < p_{JL}(n,4)$, then*

$$\sum_{s=1}^{3}(B_s + b_s)\lambda^{2s-1}(\nabla_\theta \frac{d^s u^\lambda}{d\lambda^s})^2 \geq 0.$$

Proof. To see this, from (4.13), (2.12) and (4.9), we get that

$$\begin{aligned} B_1 + b_1 =&6k^4 + (144 - 12n)k^3 + (6n^2 - 204n + 994)k^2 \\ &+ (60n^2 - 850n + 2724)k + 94n^2 - 1008n + 2628, \end{aligned}$$

and

$$\begin{aligned} &B_2 + b_2 = -38k^2 + (-300 + 38n)k - 6n^2 + 136n - 560, \\ &B_3 + b_3 = 8. \end{aligned}$$

Claim 1 If $p > \frac{n+8}{n-8}$, then we have $B_1 + b_1 > 0$. Introducing the transform $k = \frac{n-8}{2}a, n = t^2$, hence $0 < a < 1$, we have

$$\begin{aligned} B_1 + b_1 =&6t^4a^4 - 48t^3a^3 + (-3t^6 + 12t^4 - 158t^2)a^2 + (12t^5 - 48t^3 - 148t)a \\ &+ \frac{3}{8}t^8 - 3t^6 + \frac{27}{2}t^4 - 30t^2 - 44 \end{aligned}$$

By a direct calculation we can show that $B_1 + b_1 > 0$, we omit the details here.

Claim 2: Assume that $\frac{n+8}{n-8} < p < p_{JL}(n,4)$ and $n \geq 9$, then $B_2 + b_2 > 0$. Dividing n in large and small, we can get the positiveness of $B_2 + b_2$ by solving the equation analytically and numerically.

Combining Claim 1 and Claim 2, we obtain the proof of the lemma. □

Lemma 5.3. *If $\frac{n+8}{n-8} < p < p_{JL}(n,4)$ and $n \geq 9$, then there exist constants $a_{i,j}$ such that*

$$\sum_{j=1}^{4}(A_j + a_j)\lambda^{2j-1}(\frac{d^j u^\lambda}{d\lambda^j})^2 \geq \frac{d}{d\lambda}\Big(\sum_{0\leq i,j\leq 2, i+j\leq 5} a_{i,j}\frac{d^i u^\lambda}{d\lambda^i}\frac{d^j u^\lambda}{d\lambda^j}\Big). \tag{5.15}$$

Remark 5.1. *Slightly refining our proof, replacing $A_1 + a_1$ by $A_1 + a_1 - \epsilon$, for $\epsilon = \epsilon(p,n) > 0$ small enough, we can also get*

$$\sum_{j=1}^{4}(A_j + a_j)\lambda^{2j-1}(\frac{d^j u^\lambda}{d\lambda^j})^2 \geq \epsilon\lambda(\frac{du^\lambda}{d\lambda})^2 + \frac{d}{d\lambda}\Big(\sum_{0\leq i,j\leq 2, i+j\leq 5} a_{i,j}\frac{d^i u^\lambda}{d\lambda^i}\frac{d^j u^\lambda}{d\lambda^j}\Big).$$

Proof. Recall the definitions of $A_j + a_j$ in (4.2). We divide the proof into the following Claims.

Claim 1. If $p > \frac{n+8}{n-8}$ and $n \geq 9$, then $A_1 + a_1 > 0$.

In fact, we see that

$$A_1 + a_1 = (k-3)(k-1)\big(k-(n-5)\big)\big(k-(n-3)\big)\big(k-(\frac{1}{2}n-8-\frac{1}{2}\sqrt{n^2-4n+84})\big)$$
$$\big(k-(\frac{1}{2}n-8+\frac{1}{2}\sqrt{n^2-4n+84})\big) > 0.$$

Claim 2. If $\frac{n+8}{n-8} < p < p_{JL}(n,4)$ and $n \geq 9$, then $A_3 + a_3 > 0$.

Solving the roots for the equation $A_3 + a_3 = 0$, we obtain

$$r_{10}(n) := \frac{1}{2}n - \frac{26}{7} - \frac{1}{14}\sqrt{21n^2-84n-96},\ r_{20}(n) := \frac{1}{2}n - \frac{26}{7} + \frac{1}{14}\sqrt{21n^2-84n-96}.$$

The claim follows the fact that $p < p_{JL}(n,4)$ implies that $k > \max\{R_1(n), 0\}$. (Recall that

$$R_1(n) = \frac{n-10}{2} - d(n), \tag{5.16}$$

where $d(n) = a_{n,4} \cdot \sqrt{n}$, $a_{n,4}$ is defined in the Appendix 1.)

Claim 3. If $\frac{n+8}{n-8} < p < p_{JL}(n,4)$, we have that $A_2 + a_2 > 0$ except $n = 17$.

This can be proved similar to that of Claim 2. We omit the details here.

It remains to consider the borderline dimension $n = 17$ in Lemma 5.3. This is the most delicate case. This is not surprising as the dimension $n = 17$ is borderline dimension when we define the Joseph-Lundgren exponent in (1.4). See Lemma 5.3.

Assume that $n = 17$. We consider the following two cases.

Case 1. $k \geq 0.04$**.** We start with the following differential identity (denote that $f' = \frac{du^\lambda}{d\lambda}$):

$$\begin{aligned}&\sum_{j=1}^{4}(A_j+a_j)\lambda^{2j-1}(\frac{d^ju^\lambda}{d\lambda^j})^2 := \sum_{j=1}^{4} d_j\lambda^{2j-1}(\frac{d^ju^\lambda}{d\lambda^j})^2\\ &= d_1\lambda(f')^2 + (d_2+4d_3)\lambda^3(f'')^2 + d_3\lambda(\lambda^2 f''' + 2\lambda f'')^2\\ &\quad + d_4\lambda^7(f'''')^2 + \frac{d}{d\lambda}(-2d_3\lambda^4(f'')^2)\end{aligned} \tag{5.17}$$

as in Lemma 2.5. The term $d_2 + 4d_3$ can be negative. Using the mean value inequality with parameter $x \in [0,1]$ to be determined later, we have that

$$\begin{aligned}&d_1\lambda(f')^2 + d_4\lambda^7(f'''')^2 = x\cdot d_1\lambda(f')^2 + d_4\lambda^7(f'''')^2 + (1-x)\cdot d_1\lambda(f')^2\\ &\geq 2\sqrt{x\cdot d_1d_4}\lambda^4 f''''f' + (1-x)d_1\lambda(f')^2\\ &= 2\sqrt{x\cdot d_1d_4}\Big(-12\lambda(f')^2 + 6\lambda^3(f'')^2\Big) + (1-x)d_1\lambda(f')^2\\ &\quad + \frac{d}{d\lambda}\Big(2\sqrt{x\cdot d_1d_4}\big(\lambda^4f'''f' - \frac{1}{2}\lambda^4(f'')^2 - 4\lambda^3f''f' + 6\lambda^2(f')^2\big)\Big)\\ &= \Big((1-x)d_1 - 24\sqrt{x\cdot d_1\cdot d_4}\Big)\lambda(f')^2 + 12\sqrt{x\cdot d_1\cdot d_4}\lambda^3(f'')^2\\ &\quad + \frac{d}{d\lambda}\Big(\sum_{1\leq i,j\leq 3, i+j\leq 4} d_{i,j}\lambda^{i+j}f^{(i)}f^{(j)}\Big).\end{aligned} \tag{5.18}$$

The constants in the derivative terms, namely, $d_{i,j}$ can be determined but we do not need the exactly expressions. In particular, $d_{i,j}$ may be changed in the following derivation, but we still denote as $d_{i,j}$. Combining (5.17) with (5.18), we obtain that

$$\begin{aligned}
&\sum_{j=1}^{4}(A_j+a_j)\lambda^{2j-1}(\frac{d^ju^\lambda}{d\lambda^j})^2 := \sum_{j=1}^{4} d_j\lambda^{2j-1}(\frac{d^ju^\lambda}{d\lambda^j})^2\\
&= d_1\lambda(f')^2+(d_2+4d_3)\lambda^3(f'')^2+d_3\lambda(\lambda^2 f'''+2\lambda f'')^2+d_4\lambda^7(f'''')^2+\frac{d}{d\lambda}(-2d_3\lambda^4(f'')^2),\\
&\geq \Big((1-x)d_1-24\sqrt{x\cdot d_1\cdot d_4}\Big)\lambda(f')^2+(d_2+4d_3+12\sqrt{x\cdot d_1\cdot d_4})\lambda^3(f'')^2\\
&\quad + d_3\lambda(\lambda^2 f'''+2\lambda f'')^2+\frac{d}{d\lambda}\Big(\sum_{1\leq i,j\leq 3, i+j\leq 4} d_{i,j}\lambda^{i+j}f^{(i)}f^{(j)}\Big)\\
&\geq \Big((1-x)d_1-24\sqrt{x\cdot d_1\cdot d_4}\Big)\lambda(f')^2+(d_2+4d_3+12\sqrt{x\cdot d_1\cdot d_4})\lambda^3(f'')^2\\
&\quad +\frac{d}{d\lambda}\Big(\sum_{1\leq i,j\leq 3, i+j\leq 4} d_{i,j}\lambda^{i+j}f^{(i)}f^{(j)}\Big).
\end{aligned} \tag{5.19}$$

If

$$(1-x)d_1-24\sqrt{x\cdot d_1\cdot d_4}>0 \tag{5.20}$$

and

$$d_2+4d_3+12\sqrt{x\cdot d_1\cdot d_4}\geq 0 \tag{5.21}$$

hold simultaneously with some $x\in[0,1]$, then we have Theorem 5.3 for $n=17$.

To find the existence of such constant $x\in[0,1]$, we first have

$$\frac{24^2 d_4\cdot x}{(1-x)^2}<\min_{0\leq k\leq\frac{n-8}{2}} d_1. \tag{5.22}$$

On the other hand, condition (5.21) can be obtained by the following two cases:

$$\begin{aligned}
&\text{If } \ d_2+4d_3\geq 0, \ \text{ then } \ (5.21) \ \text{ holds immediately;}\\
&\text{If } \ d_2+4d_3< 0, \ \text{ then } \ (5.21) \ \text{ is equivalent to } \ 12^2x\cdot d_1\cdot d_4-(d_2+4d_3)^2>0.
\end{aligned} \tag{5.23}$$

For simplicity, we denote that $d:=d(k,n,x)=12^2x\cdot d_1\cdot d_4-(d_2+4d_3)^2$. Now we turn to consider the inequalities (5.22) and (5.23). Recall that $R_1(n)$ is defined at (5.16) and that $p<p_{JL}(n,4)$, then $\max\{R_1(n),0\}<k$.

Remark 5.2. *When $n=17$,*

$$\min_{0\leq k\leq\frac{n-8}{2}|_{n=17}} d_1=110656,$$

then from (5.22) we get that $x\leq 0.8657397553$, thus we select $x=0.8657397553$. It follows that $d_2+4d_3<0$ if $0<k<0.5256119817$ and that

$$d(k,n,x)_{n=17,x=0.8657397553}>0 \text{ if } 0.02175341614<k<1.358050900.$$

Thus, by selecting the parameter $x=0.8657397553$ and $k>0.02175341614$, then the inequalities (5.22) and (5.23) hold simultaneously, hence Lemma 5.3 for $n=17$ holds when $k\geq 0.04$

Case 2. $0<k<0.04$**.** This is the difficult case. In view of Remark 5.2, we only need to consider the case of $0<k<0.04$. In the proof of (5.19), we have dropped the term $d_3\lambda(\lambda^2f'''+2\lambda f'')^2$ (which is nonnegative term hence a "good" term. Now we make full use of this term. To achieve this, let us select parameters $x_1,x_2,y\in[0,1]$ whose exact values are to be determined later. We have

$$\begin{aligned}
&y \cdot d_1 \cdot \lambda (f')^2 + d_3 \cdot \lambda(\lambda^2 f'' + 2\lambda f')^2\\
&= x_1 \cdot y \cdot d_1 \cdot \lambda (f')^2 + d_3 \cdot \lambda(\lambda^2 f'' + 2\lambda f')^2 + (1-x_1)\cdot y \cdot d_1 \cdot \lambda (f')^2\\
&\geq 2\sqrt{x_1 \cdot y \cdot d_1 \cdot d_3}\lambda^3 (f'')^2 + \Big((1-x_1)\cdot y \cdot d_1 - 2\sqrt{x_1 \cdot y \cdot d_1 \cdot d_3}\Big)\lambda(f')^2\\
&\quad + \frac{d}{d\lambda}\Big(2\sqrt{x_1 \cdot y \cdot d_1 \cdot d_3}(\lambda^3 f'' f' - \frac{1}{2}\lambda^2 (f')^2)\Big)
\end{aligned}$$

and

$$\begin{aligned}
&(1-y)\cdot d_1 \lambda (f')^2 + d_4 \lambda^7 (f'''')^2\\
&= x_2 \cdot (1-y)\cdot d_1 \lambda (f')^2 + d_4 \lambda^7 (f'''')^2 + (1-x_2)\cdot(1-y)\cdot d_1 \lambda (f')^2\\
&\geq 2\sqrt{x_2 \cdot (1-y)\cdot d_1 \cdot d_4}\lambda^4 f'''' f' + (1-x_2)(1-y) d_1 \lambda (f')^2\\
&= 2\sqrt{x_2 (1-y) d_1 d_4}\Big(-12\lambda (f')^2 + 6\lambda^3 (f'')^2\Big) + (1-x_2)(1-y) d_1 \lambda (f')^2\\
&\quad + \frac{d}{d\lambda}\Big(2\sqrt{x \cdot d_1 d_4}\big(\lambda^4 f''' f' - \frac{1}{2}\lambda^4 (f'')^2 - 4\lambda^3 f'' f' + 6\lambda^2 (f')^2\big)\Big)\\
&= \Big((1-x_2)(1-y) d_1 - 24\sqrt{x_2(1-y)\cdot d_1 \cdot d_4}\Big)\lambda (f')^2\\
&\quad + 12\sqrt{x_2(1-y)\cdot d_1 \cdot d_4}\lambda^3 (f'')^2 + \frac{d}{d\lambda}\Big(\sum_{1\leq i,j\leq 3, i+j\leq 4} d_{i,j}\lambda^{i+j} f^{(i)} f^{(j)}\Big).
\end{aligned}$$

Therefore from (5.17), combining with the above two inequalities, we get that

$$\begin{aligned}
&\sum_{j=1}^{4}(A_j + a_j)\lambda^{2j-1}(\frac{d^j u^\lambda}{d\lambda^j})^2 := \sum_{j=1}^{4} d_j \lambda^{2j-1}(\frac{d^j u^\lambda}{d\lambda^j})^2\\
&= d_1 \lambda (f')^2 + (d_2 + 4d_3)\lambda^3 (f'')^2 + d_3 \lambda(\lambda^2 f''' + 2\lambda f'')^2\\
&\quad + d_4 \lambda^7 (f'''')^2 + \frac{d}{d\lambda}(-2d_3 \lambda^4 (f'')^2)\\
&\geq (d_2 + 4d_3 + 2\sqrt{x_1 \cdot y \cdot d_1 \cdot d_3} + 12\sqrt{x_2(1-y)\cdot d_1 \cdot d_4})\lambda^3 (f'')^2\\
&\quad + \Big((1-x_1)\cdot y \cdot d_1 + (1-x_2)(1-y) d_1 - 2\sqrt{x_1 \cdot y \cdot d_1 \cdot d_3}\\
&\quad - 24\sqrt{x_2(1-y)\cdot d_1 \cdot d_4}\Big)\lambda (f')^2 + \frac{d}{d\lambda}\Big(\sum_{1\leq i,j\leq 3, i+j\leq 4} d_{i,j}\lambda^{i+j} f^{(i)} f^{(j)}\Big).
\end{aligned}$$

Lemma 5.3 for $n = 17$ follows if

$$(1-x_1) y d_1 + (1-x_2)(1-y) d_1 - 2\sqrt{x_1 y d_1 d_3} - 24\sqrt{x_2(1-y) d_1 d_4} > 0 \tag{5.24}$$

and

$$d_2 + 4d_3 + 2\sqrt{x_1 \cdot y \cdot d_1 \cdot d_3} + 12\sqrt{x_2(1-y)\cdot d_1 \cdot d_4} \geq 0 \tag{5.25}$$

hold simultaneously. We will select proper parameters $x_1, x_2, y \in [0,1]$ to make sure the inequalities (5.24) and (5.25) hold under the condition $0 < k < 0.04$. For simplicity, we denote that

$$\begin{aligned}
f_1 &:= \Big((1-x_1) y + (1-x_2)(1-y)\Big)^2 d_1 - 4x_1 y d_3 - 24^2 x_2 (1-y) d_4,\\
f_2 &:= f_1^2 - (4\times 24)^2 x_1 y x_2 (1-y) d_3 d_4,\\
h_1 &:= (d_2 + 4d_3)^2 - 4x_1 y d_1 d_3 - 12^2 x_2 (1-y) d_1 d_4,\\
h_2 &:= 48^2 x_1 y x_2 (1-y) d_3 d_4 d_1^2 - h_1^2.
\end{aligned}$$

Hence, (5.24) is equivalent to $f_1 > 0$ and $f_2 > 0$. As for (5.25) we observe that

$$\begin{aligned} &\text{if } \ h_1 \leq 0, \ \text{ then } \ (5.25) \ \text{ holds immediately,} \\ &\text{if } \ h_1 > 0, \ \text{ then } \ h_2 > 0. \end{aligned} \tag{5.26}$$

Select the proper parameter x_1, x_2, y by considering the end-point case, namely $y = 0$ and $y = 1$. Then as in (5.22) to determine x_1, x_2. In fact, Let $y = 0.1, x_1 = x_2 = 0.8$, then one has $0 < k < 0.04$, and f_1, f_2, h_1, h_2 all are positive. Therefore, combining with (5.26), we know that (5.24) and (5.25) hold. This completes the proof of Lemma 5.3.

□

6. Proofs of Theorem 1.1

As a result of basic energy estimates and the monotonicity formula established in Theorem 1.2, we give a proof of Theorem 1.1.

By the monotonicity formulas in Theorem 1.2 and the decay estimates in Theorem A(Harrabi [24, 25]), as well as classified the homogeneous stable solution in Theorem 2.2. Now following the scheme in Davila-Dupaigne-Wang-Wei [15], we can give a proof of Theorem 1.1.

Proof. For the stable solutions of (1.1):

- $\frac{n+2m}{n-2m} < p < p_{JL}(n,m)$. With the help of the integral estimates (Theorem A, Harrabi [24, 25]), by the coercive estimates, we deduce that the stable solutions must be homogenous stable solutions then the classification of homogenous stable solutions (Theorem 2.2) gives the solutions must be trivial. The dimension condition in Theorem 1.1 follows from Theorem 1.2 for the case $m \geq 5$.

For solutions which are stable outside a compact set:

- $\frac{n+2m}{n-2m} < p < p_{JL}(n,m)$. By the decay estimates (Theorem A, Harrabi [24, 25]) and monotonicity formulas (Theorem 1.2), applying the dimension reduction and blown-down analysis, we can reduce that stable outside a compact set must be homogeneous stable solutions which must be tribial because of the classification of homogenous stable solutions in Theorem 2.2. The dimension condition in Theorem 1.1 follows from Theorem 1.2 for the case $m \geq 5$.

Combining all cases, we have proved Theorem 1.1.

□

7. Appendix: The explicit expressions of $p_{JL}(n,m)$ for $m = 3, 4$

In this appendix, we obtain give the precise expressions of $p_{JL}(n,3), p_{JL}(n,4)$.

The threshold for triharmonic Lane-Emden equation is

$$p_{JL}(n,3) := \begin{cases} \infty & \text{if } \ n \leq 14, \\ 1 + \frac{12}{n-8-2a_{n,3}\sqrt{n}} & \text{if } \ n \geq 15, \end{cases}$$

where

$$a_{n,3} := \frac{1}{6\sqrt{n}}\Big(9n^2 + 96 - \frac{1536 + 1152n^2}{d_0(n)} - \frac{3}{2}D_0(n)\Big)^{1/2};$$

$$D_0(n) := -(D_1(n) + 36\sqrt{D_2(n)})^{1/3};$$

$$D_1(n) := -94976 + 20736n + 103104n^2 - 10368n^3 + 1296n^5 - 3024n^4 - 108n^6;$$

$$\begin{aligned} D_2(n) : =\ & 6131712 - 16644096n^2 + 6915840n^4 - 690432n^6 - 3039232n \\ &+ 4818944n^3 - 1936384n^5 + 251136n^7 - 30864n^8 - 4320n^9 \\ &+ 1800n^{10} - 216n^{11} + 9n^{12}. \end{aligned}$$

Here and the following, we have

$$\lim_{n\to\infty} a_{n,j} = 1, j = 3, 4. \ \ a_{n,j}, j = 3, 4 \text{ for } \ \forall n \geq 15 \text{ and } 18 \text{ respectively.}$$

The threshold for quadharmonic Lane-Emden equation is

$$p_{JL}(n,4) := \begin{cases} \infty & \text{if } n \le 17, \\ 1 + \frac{16}{n-10-2a_{n,4}\sqrt{n}} & \text{if } n \ge 18, \end{cases}$$

where

$$a_{n,4} := \sqrt{\frac{1}{4}n^2 + 5 + \frac{1}{2}\sqrt{d_6} - \frac{1}{2}\sqrt{d_7 + \frac{d_3}{\sqrt{d_6}}}}/\sqrt{n}$$

and

$$\begin{aligned} d_0 :&= 2097152 - \frac{45}{4}n^{10} + 180n^9 - 396n^8 - 5184n^7 + 36928n^6 + 27648n^5 \\ &\quad - 132096n^4 + 147456n^3 - 1572864n^2; \end{aligned}$$

$$\begin{aligned} d_1 :&= \frac{3}{65536}n^{24} - \frac{9}{4096}n^{23} + \frac{81}{2048}n^{22} - \frac{33}{128}n^{21} - \frac{123}{128}n^{20} + \frac{303}{16}n^{19} + \frac{21}{8}n^{18} \\ &\quad - 1056n^{17} + 3888n^{16} + 25396n^{15} - 279456n^{14} + 947712n^{13} + 1979904n^{12} \\ &\quad - 48427008n^{11} + 135979008n^{10} + 677117952n^9 - 2620588032n^8 \\ &\quad - 3265265664n^7 + 14294188032n^6 + 2415919104n^5 - 16106127360n^4; \end{aligned}$$

$$d_2 := (d_0 + 12\sqrt{d_1})^{\frac{1}{3}}; d_3 := 128n^2;$$

$$d_4 := -\frac{8192}{3} + \frac{1}{32}n^8 - \frac{1}{2}n^7 + n^6 + 16n^5 - \frac{584}{3}n^4 - 128n^3 + \frac{4096}{3}n^2;$$

$$d_5 := \frac{40}{3}n^2 + \frac{128}{3}, \quad d_6 := \frac{1}{2}d_5 + \frac{1}{6}d_2 - \frac{d_4}{d_2}, \quad d_7 := d_5 - \frac{1}{6}d_2 + \frac{d_4}{d_2}.$$

8. Appendix: on the sharp estimate $a_{n,m} < 1$

In this appendix, we aim to prove the sharp estimate $a_{n,m} < 1$ in Proposition 1.1. To achieve this, we first state and prove some propositions on related special functions.

Recall that $\Psi(x) := \frac{d}{dx}(\ln(\Gamma(x))) = \frac{\Gamma'(x)}{\Gamma(x)}$.

Proposition 8.1. *For $n \ge 3$, we have*

$$\Psi(\frac{1}{4}n + \frac{1}{2} - \frac{1}{2}\sqrt{n}) + \Psi(\frac{1}{4}n + \frac{1}{2} + \frac{1}{2}\sqrt{n}) - 2\Psi(\frac{1}{4}n) < 0. \tag{8.1}$$

Remark 8.1. *The inequality (8.1) is very critical since, by the convexity of the function $\Psi(x)$ for $x > 0$ ([27]), we have*

$$\Psi(\frac{1}{4}n + \frac{1}{2} - \frac{1}{2}\sqrt{n}) + \Psi(\frac{1}{4}n + \frac{1}{2} + \frac{1}{2}\sqrt{n}) - 2\Psi(\frac{1}{4}n + \frac{1}{2}) \ge 0.$$

On the other hand, the following inequality

$$\Psi(\frac{1}{4}n + \frac{1}{2} - \frac{1}{2}a\sqrt{n}) + \Psi(\frac{1}{4}n + \frac{1}{2} + \frac{1}{2}a\sqrt{n}) - 2\Psi(\frac{1}{4}n) < 0$$

is no longer true whenever $a < 1$ and n is large properly.

On the other hand, if we let

$$g(a) := g(n,a) := \Psi(\frac{1}{4}n + \frac{1}{2} - \frac{1}{2}a\sqrt{n}) + \Psi(\frac{1}{4}n + \frac{1}{2} + \frac{1}{2}a\sqrt{n}) - 2\Psi(\frac{1}{4}n)$$

then $g'(a) = \frac{1}{2}\sqrt{n}\Big(\Psi'(\frac{1}{4}n + \frac{1}{2} + \frac{1}{2}a\sqrt{n}) - \Psi'(\frac{1}{4}n + \frac{1}{2} - \frac{1}{2}a\sqrt{n})\Big) < 0$ for $a > 0$, hence we see that the inequality in Proposition 8.1 is sharp.

Proof. First, by a direct calculation, we have

$$\lim_{n\to+\infty}\Big(\Psi(\frac{1}{4}n+\frac{1}{2}-\frac{1}{2}\sqrt{n})+\Psi(\frac{1}{4}n+\frac{1}{2}+\frac{1}{2}\sqrt{n})-2\Psi(\frac{1}{4}n)\Big)=0. \tag{8.2}$$

Let $n=t^2$,

$$\begin{aligned}&\Psi(\frac{1}{4}n+\frac{1}{2}-\frac{1}{2}\sqrt{n})+\Psi(\frac{1}{4}n+\frac{1}{2}+\frac{1}{2}\sqrt{n})-2\Psi(\frac{1}{4}n)\\ =&\Psi(\frac{1}{4}t^2+\frac{1}{2}-\frac{1}{2}t)+\Psi(\frac{1}{4}t^2+\frac{1}{2}+\frac{1}{2}t)-2\Psi(\frac{1}{4}t^2)\\ :=&g(t)\end{aligned}$$

where $g(t)$ is defined at the last equality.

Then

$$\frac{d}{dt}g(t)=\Psi'(\frac{1}{4}t^2+\frac{1}{2}-\frac{1}{2}t)(\frac{1}{2}t-\frac{1}{2})+\Psi'(\frac{1}{4}t^2+\frac{1}{2}+\frac{1}{2}t)(\frac{1}{2}t+\frac{1}{2})-\Psi'(\frac{1}{4}t^2)t.$$

Recall the following refined estimates on the derivative Ψ (See e.g. [19]):

$$\frac{1}{x}+\frac{1}{2x^2}<\Psi'(x)<\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3}\quad\text{for}\quad x>0. \tag{8.3}$$

From (8.3), combining with the above identity, we derive that

$$\begin{aligned}\frac{d}{dt}g(t)>&\Big(\frac{1}{\frac{1}{4}t^2+\frac{1}{2}-\frac{1}{2}t}+\frac{1}{2(\frac{1}{4}t^2+\frac{1}{2}-\frac{1}{2}t)^2}\Big)(\frac{1}{2}t-\frac{1}{2})\\ &+\Big(\frac{1}{\frac{1}{4}t^2+\frac{1}{2}+\frac{1}{2}t}+\frac{1}{2(\frac{1}{4}t^2+\frac{1}{2}+\frac{1}{2}t)^2}\Big)(\frac{1}{2}t+\frac{1}{2})\\ &-\Big(\frac{1}{\frac{1}{4}t^2}+\frac{1}{2(\frac{1}{4}t^2)^2}+\frac{1}{6(\frac{1}{4}t^2)^3}\Big)t.\end{aligned}$$

A straightforward calculation shows that

$$\frac{d}{dt}g(t)>\frac{16}{3}\frac{t^8-18t^6-28t^4-24t^2-32}{t^5(t^2-2t+2)^2(t^2+2t+2)^2}.$$

If $t>4.4163$, then $t^8-18t^6-28t^4-24t^2-32>0$, hence $\frac{d}{dt}g(t)>0$. Therefore, when $n=t^2\geq 20$, then $\frac{d}{dt}g(t)>0$. Combining with (8.2), one has

$$\Psi(\frac{1}{4}n+\frac{1}{2}-\frac{1}{2}\sqrt{n})+\Psi(\frac{1}{4}n+\frac{1}{2}+\frac{1}{2}\sqrt{n})-2\Psi(\frac{1}{4}n)<0\quad\text{for}\quad n\geq 20.$$

For the case of $3\leq n\leq 20$, we can also show the desired inequality case by case. □

Proposition 8.2. *If* $\frac{1}{4}t^2-\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}t>0, m>0, t>3$, *we have*

$$\begin{aligned}H(t,m):=&\Psi'(\frac{1}{4}t^2+\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}t)(\frac{1}{2}t-\frac{1}{2})+\Psi'(\frac{1}{4}t^2+\frac{1}{2}m+\frac{1}{2}+\frac{1}{2}t)(\frac{1}{2}t+\frac{1}{2})\\ &-\Psi'(\frac{1}{4}t^2+\frac{1}{2}m)t+\Psi'(\frac{1}{4}t^2-\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}t)(\frac{1}{2}t-\frac{1}{2})\\ &+\Psi'(\frac{1}{4}t^2-\frac{1}{2}m+\frac{1}{2}+\frac{1}{2}t)(\frac{1}{2}t+\frac{1}{2})-\Psi'(\frac{1}{4}t^2-\frac{1}{2}m)t\\ >&0.\end{aligned}$$

Proof. Let

$$B(x):=\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3}, b(x):=\frac{1}{x}+\frac{1}{2x^2}.$$

From (8.3), we have that

$$\begin{aligned}H(t,m):\geq\ & b(\frac{1}{4}t^2+\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}t)(\frac{1}{2}t-\frac{1}{2})+b(\frac{1}{4}t^2+\frac{1}{2}m+\frac{1}{2}+\frac{1}{2}t)(\frac{1}{2}t+\frac{1}{2})\\&-B(\frac{1}{4}t^2+\frac{1}{2}m)t+b(\frac{1}{4}t^2-\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}t)(\frac{1}{2}t-\frac{1}{2})\\&+b(\frac{1}{4}t^2-\frac{1}{2}m+\frac{1}{2}+\frac{1}{2}t)(\frac{1}{2}t+\frac{1}{2})-B(\frac{1}{4}t^2-\frac{1}{2}m)t\\=\ &\frac{32}{3}\frac{H_1(t,m)}{H_2(t,m)},\end{aligned}$$

where

$$\begin{aligned}H_2(t,m)=&(t^2+2m-2t+2)^2(t^2+2m+2t+2)^2(t^2-2m-2t+2)^2\\&(t^2-2m+2t+2)^2(t^2-2m)^3(t^2+2m)^3,\end{aligned}$$

and

$$\begin{aligned}H_1(t,m)&=t^{22}+(36m^2-18)t^{20}+(332m^2-20)t^{18}+(-528m^4+1192m^2-168)t^{16}\\&+(-5472m^4+1344m^2-240)t^{14}+(2688m^6-13376m^4+4352m^2-480)t^{12}\\&+(20864m^6-10112m^4+2880m^2-704)t^{10}+(-4608m^8+256m^6+11008m^4\\&+6784m^2-384)t^8+(9472m^8-142336m^6+13056m^4-2560m^2-512)t^6\\&+(-3072m^{10}+161280m^8-122880m^6+50688m^4)t^4+(-107520m^{10}\\&+125952m^8-82944m^6+21504m^4-6144m^2)t^2+12288m^12-79872m^{10}\\&+79872m^8-18432m^6+6144m^4.\end{aligned}$$

By a straightforward calculation we can prove that under the condition either $\frac{1}{4}t^2-\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}t>0, m\geq\frac{1}{2}$ or $t>4.5, 0<m<\frac{1}{2}$, it holds $H_1(t,m)>0$, hence $H(t,m)>0$. For the remaining cases $0<m<\frac{1}{2}, 3<t<4.5$, we can show that $H(t,m)>0$ directly. These finish the proof of proposition 8.2. □

Proposition 8.3. *If* $\frac{1}{4}n-\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}\sqrt{n}>0, m>0, n>9$*, we have*

$$\begin{aligned}F_0(n,m):=&\ln\Gamma(\frac{1}{4}n+\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}\sqrt{n})+\ln\Gamma(\frac{1}{4}n+\frac{1}{2}m+\frac{1}{2}+\frac{1}{2}\sqrt{n})\\&+2\ln\Gamma(\frac{1}{4}n-\frac{1}{2}m)-\Big(\ln\Gamma(\frac{1}{4}n-\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}\sqrt{n})\\&+\ln\Gamma(\frac{1}{4}n-\frac{1}{2}m+\frac{1}{2}+\frac{1}{2}\sqrt{n})+2\ln\Gamma(\frac{1}{4}n+\frac{1}{2}m)\Big)\\&<0.\end{aligned}$$

Proof. From Proposition 8.2 we know that the function $F_0(n,m)$ is decreasing about the variable m. Combining with $F_0(n,m)\mid_{m=0}=0$, we see that $F_0(n,m)<0$. Hence, Proposition 8.3 is proved. □

Proposition 8.4. *If* $\frac{1}{4}n-\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}\sqrt{n}>0, m>0, n>9$*, we have*

$$\begin{aligned}&F(n,m)\\&:=\frac{1}{2}\Psi(\frac{1}{4}n+\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}\sqrt{n})+\frac{1}{2}\Psi(\frac{1}{4}n+\frac{1}{2}m+\frac{1}{2}+\frac{1}{2}\sqrt{n})-\Psi(\frac{1}{4}n+\frac{1}{2}m)\\&+\frac{1}{2}\Psi(\frac{1}{4}n-\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}\sqrt{n})+\frac{1}{2}\Psi(\frac{1}{4}n-\frac{1}{2}m+\frac{1}{2}+\frac{1}{2}\sqrt{n})-\Psi(\frac{1}{4}n-\frac{1}{2}m)\\&<0.\end{aligned}$$

Proof. Notice that Proposition 8.2 says that the function $F(n,m)$ is increasing about variable n, combining with $\lim_{n\to+\infty}F(n,m)=0$, Proposition 8.4 follows. □

Proof of Proposition 1.1 of $a_{n,m} < 1$. By introducing the transformation:

$$k := \frac{n-(2m+2)}{2} + a\sqrt{n}.$$

we know that $a_{n,m}$ is the solution of $f(n,m,a) = 0$([27]), where

$$\begin{aligned} f(n,m,a) :=& \ln\Gamma(\frac{1}{4}n+\frac{1}{2}m+\frac{1}{2}+\frac{1}{2}a\sqrt{n}) - \ln\Gamma(\frac{1}{4}n+\frac{1}{2}m) \\ &+ \ln\Gamma(\frac{1}{4}n+\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}a\sqrt{n}) - \ln\Gamma(\frac{1}{4}n+\frac{1}{2}m)) \\ &-\Big(\ln\Gamma(\frac{1}{4}n-\frac{1}{2}m+\frac{1}{2}+\frac{1}{2}a\sqrt{n}) - \ln\Gamma(\frac{1}{4}n-\frac{1}{2}m\Big) \\ &-\Big(\ln\Gamma(\frac{1}{4}n-\frac{1}{2}m+\frac{1}{2}-\frac{1}{2}a\sqrt{n}) - \ln\Gamma(\frac{1}{4}n-\frac{1}{2}m)\Big). \end{aligned}$$

In Proposition 8.3, we see that $f(n,m,1) < 0$ whenever $\frac{1}{4}n - \frac{1}{2}m + \frac{1}{2} - \frac{1}{2}\sqrt{n} > 0, m > 0, n > 9$. Since $f(n,m,a)$ is decreasing w.r.t. the variable $a > 0$ and note that $a_{n,m}$ is a positive root of equation $f(n,m,a) = 0$, we know that $a_{n,m} < 1$. This finishes the proof. □

9. Appendix: on the estimate borderline dimension $n_{JL}(m)$

Proof of Proposition 1.1 of (1.7). The estimate of the borderline dimension is mainly for the purely fractional cases, since the when m is integer, the $n_{JL}(m)$ can be calculated directly by the equation (1.5). Therefore, we replace m by s and use the notation $n_{JL}(s)$ here. First we let

$$\frac{2s}{p-1} := k.$$

Since $\Gamma(s+1) = s\Gamma(s)$, the corresponding equality of (1.2) becomes

$$\frac{\Gamma(\frac{n-k}{2})\Gamma(s+\frac{k}{2}+1)}{\Gamma(\frac{k}{2}+1)\Gamma(\frac{n-k-2s}{2})} = \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}. \tag{9.4}$$

From the formula (1.4), we observe that the borderline dimension $n_{JL}(s)$ satisfies the equation by let $p \to +\infty$ (equivalently, $k \to 0$) in (1.2), that is

$$\frac{\Gamma(\frac{n}{2})\Gamma(s+1)}{\Gamma(1)\Gamma(\frac{n-2s}{2})} = \frac{(\Gamma(\frac{n+2s}{4}))^2}{(\Gamma(\frac{n-2s}{4}))^2}.$$

Note that $\Gamma(1) = 1$, take logarithm we have that

$$\ln\Gamma(\frac{n}{2}) + \ln\Gamma(s+1) - \ln\Gamma(\frac{n-2s}{2}) + 2\ln\Gamma(\frac{n-2s}{4}) - 2\ln\Gamma(\frac{n+2s}{4}) = 0.$$

Let

$$G(n,s) := \ln\Gamma(\frac{n}{2}) + \ln\Gamma(s+1) - \ln\Gamma(\frac{n-2s}{2}) + 2\ln\Gamma(\frac{n-2s}{4}) - 2\ln\Gamma(\frac{n+2s}{4})$$

and $\Psi(x) := \frac{d}{dx}(\ln(\Gamma(x))) = \frac{\Gamma'(x)}{\Gamma(x)}$. Since

$$\begin{aligned} \frac{dG}{dn} &= \frac{1}{2}(\Psi(\frac{n}{2}) - \Psi(\frac{n-2s}{2}) + \Psi(\frac{n-2s}{4}) - \Psi(\frac{n+2s}{4})) \\ &= \frac{1}{2}\Big(\Psi(\frac{n+2s}{4} - \Psi(\frac{n-2s}{4})\Big)\Big(\frac{\Psi(\frac{n}{2}) - \Psi(\frac{n-2s}{2})}{\Psi(\frac{n+2s}{4}) - \Psi(\frac{n-2s}{4})} - 1\Big) \\ &= \frac{1}{2}\Big(\frac{\Psi'(\frac{n-2s}{2}+\theta s)}{\Psi'(\frac{n-2s}{4}+\theta s)} - 1\Big)\Big(\Psi(\frac{n+2s}{4} - \Psi(\frac{n-2s}{4}))\Big) < 0 \text{ for } n > 2s, s > 0, \end{aligned} \tag{9.5}$$

hence $G(n,s)$ is a decreasing function with respect to the variable n.

Claim 1: $G(n,s)\mid_{n=2s+4} > 0$. In fact, $G(n,s)\mid_{n=2s+4} = \ln\Gamma(s+2) - \ln\Gamma(s+1) > 0$.

Claim 2: $\lim_{n\to+\infty} G(n,s) = -\infty$ for any fixed $s > 0$. It can be checked directly by using the asymptotical formula of $\lim_{n\to\infty} \frac{\Gamma(n+\alpha)}{\Gamma(n)n^\alpha} = 1$ for $\alpha \in \mathbb{R}$. Therefore, we can see that there exists a unique $n = n(s)$ such that $G(n(s), s) = 0$ for any fixed $s > 0$. The existence and uniqueness of the borderline dimension $n_{JL}(s)$ obtained, combine with Claim 1 and decreasing of $G(n,s)$, we get that $n_{JL}(s) > 2s + 4$.

Taking the derivative in $G(n_0(s), s) = 0$, we get that

$$\frac{dn_0(s)}{ds} - 2 = \frac{2\Psi(\frac{n+2s}{4}) - \Psi(s+1) - \Psi(\frac{n}{2})}{\Psi(\frac{n}{2}) - \Psi(\frac{n-2s}{2}) + \Psi(\frac{n-2s}{4}) - \Psi(\frac{n+2s}{4})} \mid_{n=n_0(s)} . \tag{9.6}$$

From (9.5), we have that

$$\Psi(\frac{n}{2}) - \Psi(\frac{n-2s}{2}) + \Psi(\frac{n-2s}{4}) - \Psi(\frac{n+2s}{4}) < 0.$$

Consider the function

$$h(n,s) := 2\Psi(\frac{n+2s}{4}) - \Psi(s+1) - \Psi(\frac{n}{2}),$$

since

$$\frac{dh(n,s)}{dn} = \frac{1}{2}\Big(\Psi'(\frac{1}{4}n + \frac{1}{2}s) - \Psi'(\frac{1}{2}n)\Big) > 0, \text{ for } n > 2s.$$

and

$$\lim_{n\to+\infty} h(n,s) = 0,$$

one has, $h(n,s) \mid_{n=n_0(s)} < 0$. Combine with (9.6), we derive that $\frac{dn_0(s)}{ds} - 2 > 0$. These complete the proof of (1.7). □

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